Variational Assimilation of Discrete Navier-Stokes Equations

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Incompressible Navier-Stokes Equations

Cauchy problem for **Navier-Stokes**:

$$(NS) \quad \begin{cases} \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathbf{p} = \mathbf{f}, \quad x\in\Omega, \ t\in[0,T], \\ \nabla\cdot\mathbf{v} = 0, \quad x\in\Omega, \quad t\in[0,T], \\ \mathbf{v}(0,x) = \mathbf{v}_0(x), \quad x\in\Omega. \end{cases}$$

Unknowns : velocity $\mathbf{v}(t,x)$ and pressure $\mathbf{p}(t,x)$

Projecting the system (NS) onto $\mathcal{H}_{div}(\Omega)$[1] yields:

$$\partial_t \mathbf{v} = \nu\Delta\mathbf{v} + \mathbb{P}[-(\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{f}] \qquad (NSP)$$

with $\mathbb{P}$ orthogonal projector from $(L^2(\Omega))^d$ to $\mathcal{H}_{div}(\Omega)$.

The pressure $\mathbf{p}$ is recovered through the Helmholtz decomposition:

$$\nabla\mathbf{p} = -(\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{f} - \mathbb{P}[-(\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{f}]$$

[1] $(L^2(\Omega))^d$ divergence-free function space

Helmholtz-Decomposition

The projector $\mathbb{P}$ is explicite in Fourier domain:

$$\forall \mathbf{u} \in (L^2(\Omega))^d, \quad \widehat{\mathbf{u}}(\xi) = \frac{\xi \cdot \xi^T}{|\xi|^2}\widehat{\mathbf{u}}(\xi) + \left(1 - \frac{\xi \cdot \xi^T}{|\xi|^2}\right)\widehat{\mathbf{u}}(\xi)$$

Thus

$$\widehat{\mathbb{P}(\mathbf{u})}(\xi) = \left(1 - \frac{\xi \cdot \xi^T}{|\xi|^2}\right)\widehat{\mathbf{u}}(\xi)$$

For space localization and adaptativity:

$\longrightarrow$ Periodic Anisotropic divergence-free wavelets [`Deriaz,Perrier 08`]

For physical boundary conditions:

$\longrightarrow$ Anisotropic divergence-free wavelets [`Kadri-Harouna,Perrier 10`]

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Temporal discretization

Heat kernel integration problem:

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} = \tilde{\mathbf{f}},$$

with

$$\tilde{\mathbf{f}} = \mathbb{P}[-(\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{f}].$$

Implicite finite difference approximation $\mathbf{v}(x, n\delta t) \approx \mathbf{v}^n$:

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} - \frac{\nu}{2}\Delta(\mathbf{v}^{n+1} + \mathbf{v}^n) = \tilde{\mathbf{f}}^n, \qquad \text{Crank-Nicholson } O(\delta t^2)$$

Heat kernel factorization (ADI method):

$$\left(1 - \alpha\frac{\partial^2}{\partial x^2} - \alpha\frac{\partial^2}{\partial y^2}\right) = \left(1 - \alpha\frac{\partial^2}{\partial x^2}\right)\left(1 - \alpha\frac{\partial^2}{\partial y^2}\right) + O(\alpha^2)$$

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Spatial discretization

Semi implicite treatment for the non-linear term:

$$(\mathbf{v}^{n+1/2} \cdot \nabla)\mathbf{v}^{n+1/2} = \frac{3}{2}(\mathbf{v}^n \cdot \nabla)\mathbf{v}^n - \frac{1}{2}(\mathbf{v}^{n-1} \cdot \nabla)\mathbf{v}^{n-1}$$

CFL condition: $\delta t \leq C\,(\delta x/\mathbf{v}_{max})^{4/3}$.

Scale separation:

$$\mathbf{v}(t,x) = \sum_{\mathbf{j},\mathbf{k}} d^{div}_{\mathbf{j},\mathbf{k}}(t)\ \Psi^{div}_{\mathbf{j},\mathbf{k}}(x)$$

$\longrightarrow$ ODE system on the coefficients $[d^{div}_{\mathbf{j},\mathbf{k}}(t)]$.

Galerkin method in space with $\vec{\mathbf{V}}_j = (V_j^1 \otimes V_j^0) \times (V_j^0 \otimes V_j^1)$.

$\longrightarrow$ At each time step we need to compute the projector $\mathbb{P}$.

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Principle of Variational Assimilation

Measurements (observations) denoted $\mathbf{v}_{ob}^k, \; k = 1, \cdots, N$

Discrete dynamical model equation:

$$L_{-1/2}\mathbf{v}^{n+1} - L_{1/2}\mathbf{v}^n + \frac{3}{2}B^n \circ \mathbf{v}^n - \frac{1}{2}B^{n-1} \circ \mathbf{v}^{n-1} = 0,$$

with

$$L_{-1/2} := 1 - \frac{\delta t}{2}\Delta, \quad L_{1/2} := 1 + \frac{\delta t}{2}\Delta, \quad B^n \circ \mathbf{v}^n := \mathbb{P}(\mathbf{v}^n \cdot \nabla)\mathbf{v}^n$$

Objective: find the most probable state defined both by the measurements and dynamical equations.

Cost function minimization:

$$J(\mathbf{v}_0) = \frac{1}{2}\sum_{k=1}^{N} \|\mathbb{H} \circ \mathbf{v}^k - \mathbf{v}_{ob}^k\|^2 \delta t + \frac{\alpha}{2}\|\mathbf{v}_0\|^2,$$

Differentiation operators

Let $f : E \to \mathbb{R}$ be a vector (or scalar) function

Directional derivative: if the following limit exists

$$\nabla_{\mathbf{d}} f(\mathbf{v}) := \lim_{h \to 0} \frac{f(\mathbf{v} + h\mathbf{d}) - f(\mathbf{v})}{h}$$

Fréchet derivative: if there exist $\nabla f(v) \in E$ such that

$$f(\mathbf{v} + \mathbf{u}) = f(\mathbf{v}) + \langle \nabla f(\mathbf{v}), \mathbf{u} \rangle + o(\|\mathbf{u}\|)$$

If the gradient of f exists, then:

$$\nabla_{\mathbf{d}} f(\mathbf{v}) = \langle \nabla f(\mathbf{v}), \mathbf{d} \rangle$$

Cost function differentiation

$$
\begin{aligned}
J(\mathbf{v}_0 + h\mathbf{u}) - J(\mathbf{v}_0) &= \frac{1}{2}\sum_{k=1}^{N} \|\mathbb{H} \circ \mathbf{v}^k(\mathbf{v}_0 + h\mathbf{u}) - \mathbf{v}_{ob}^k\|^2 \delta t + \frac{\alpha}{2}\|\mathbf{v}_0 + h\mathbf{u}\|^2 \\
&- \frac{1}{2}\sum_{k=1}^{N} \|\mathbb{H} \circ \mathbf{v}^k - \mathbf{v}_{ob}^k\|^2 \delta t - \frac{\alpha}{2}\|\mathbf{v}_0\|^2
\end{aligned}
$$

Rewritten the terms, we get:

$$
\begin{aligned}
&\|\mathbb{H} \circ \mathbf{v}^k(\mathbf{v}_0 + h\mathbf{u}) - \mathbf{v}_{ob}^k\|^2 - \|\mathbb{H} \circ \mathbf{v}^k - \mathbf{v}_{ob}^k\|^2 = \\
&+\langle \mathbb{H} \circ \mathbf{v}^k(\mathbf{v}_0 + h\mathbf{u}) + \mathbb{H} \circ \mathbf{v}^k - 2\mathbf{v}_{ob}^k, \mathbb{H} \circ \mathbf{v}^k(\mathbf{v}_0 + h\mathbf{u}) - \mathbb{H} \circ \mathbf{v}^k \rangle
\end{aligned}
$$

and

$$
\|\mathbf{v}_0 + h\mathbf{u}\|^2 - \|\mathbf{v}_0\|^2 = \langle 2\mathbf{v}_0 + h\mathbf{u}, h\mathbf{u} \rangle
$$

Thus:

$$
\nabla_{\mathbf{u}} J(\mathbf{v}_0) = \sum_{k=1}^{N} \langle \mathbb{H} \circ \mathbf{v}^k - \mathbf{v}_{ob}^k, \nabla \mathbb{H} \circ \mathbf{v}^k \cdot \nabla_{\mathbf{u}} \mathbf{v}^k \rangle \delta t + \alpha \langle \mathbf{v}_0, \mathbf{u} \rangle
$$

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Problem

Let us consider the one-dimensional ODE

$$\partial_t y = F(t).y, \qquad \text{with } F(t) \text{ a linear operator}$$

The continuous adjoint model is

$$-\partial_t \lambda = F(t)^T.\lambda$$

Discretizing with an explicit Euler scheme, we get:

$$y_{n+1} - y_n = \delta t F_n.y_n \Rightarrow y_{n+1} = (1 + \delta t F_n).y_n$$

For which we get

$$y_n^* = (1 + \delta t F_n)^T.y_{n+1}^*$$

Otherwise:

$$\lambda_n - \lambda_{n+1} = \delta t F_{n+1}^T.\lambda_{n+1} \Rightarrow \lambda_n = (1 + \delta t F_{n+1})^T.\lambda_{n+1}$$

$$(1 + \delta t F_{n+1})^T \neq (1 + \delta t F_n)^T$$

Linear tangent

One disturbs the initial condition: $\tilde{\mathbf{v}}_0 = \mathbf{v}_0 + h\mathbf{u}$. Then, we get:

$$L_{-1/2}\tilde{\mathbf{v}}^{n+1} - L_{1/2}\tilde{\mathbf{v}}^n + \frac{3}{2}\tilde{B}^n \circ \tilde{\mathbf{v}}^n - \frac{1}{2}\tilde{B}^{n-1} \circ \tilde{\mathbf{v}}^{n-1} = 0.$$

Taking the difference with the non disturbed equation, we have:

$$\begin{aligned} L_{-1/2}(\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^{n+1}) - L_{1/2}(\tilde{\mathbf{v}}^n - \mathbf{v}^n) &= -\frac{3}{2}(\tilde{B}^n \circ \tilde{\mathbf{v}}^n - B^n \circ \mathbf{v}^n) \\ &+ \frac{1}{2}(\tilde{B}^{n-1} \circ \tilde{\mathbf{v}}^{n-1} - B^{n-1} \circ \mathbf{v}^{n-1}) \end{aligned}$$

Multiplying with $1/h$ and taking the limit as $h \to 0$, we get:

$$\begin{aligned} L_{-1/2}\nabla_u \mathbf{v}^{n+1} - L_{1/2}\nabla_u \mathbf{v}^n &= -\frac{3}{2}\nabla B^n \circ \mathbf{v}^n \cdot \nabla_u \mathbf{v}^n \\ &+ \frac{1}{2}\nabla B^{n-1} \circ \mathbf{v}^{n-1} \cdot \nabla_u \mathbf{v}^{n-1} \end{aligned}$$

Adjoint variable

Taking the inner product of the linear tangent with λ^{n+1} yields

$$\langle \nabla_u \mathbf{v}^{n+1}, L_{-1/2}\lambda^{n+1}\rangle - \langle \nabla_u \mathbf{v}^{n}, L_{1/2}\lambda^{n+1}\rangle = -\langle \nabla_u \mathbf{v}^{n}, \frac{3}{2}\nabla^* B^n \cdot \lambda^{n+1}\rangle$$
$$+\langle \nabla_u \mathbf{v}^{n-1}, \frac{1}{2}\nabla^* B^{n-1} \cdot \lambda^{n+1}\rangle$$

Thus, making identification, the adjoint model is defined as:

$$L_{-1/2}\lambda^N = F^N,$$
$$L_{-1/2}\lambda^{N-1} - L_{1/2}\lambda^N + \frac{3}{2}\nabla^* B^{N-1} \cdot \lambda^N = F^{N-1}$$

with

$$F^n = \nabla^* \mathbb{H} \circ \mathbf{v}^n \cdot (\mathbb{H} \circ \mathbf{v}^n - \mathbf{v}^n_{ob}), \qquad 1 \leq n \leq N$$

For $1 \leq n \leq N-2$,

$$L_{-1/2}\lambda^n - L_{1/2}\lambda^{n+1} + \frac{3}{2}\nabla^* B^n \cdot \lambda^{n+1} - \frac{1}{2}\nabla^* B^n \cdot \lambda^{n+2} = F^n$$

$\longrightarrow \nabla J(\mathbf{v}_0) = \alpha \mathbf{v}_0 + L_{1/2}\lambda^1 - \frac{3}{2}\nabla^* B^0 \cdot \lambda^1 + \frac{1}{2}\nabla^* B^0 \cdot \lambda^2$

Model error

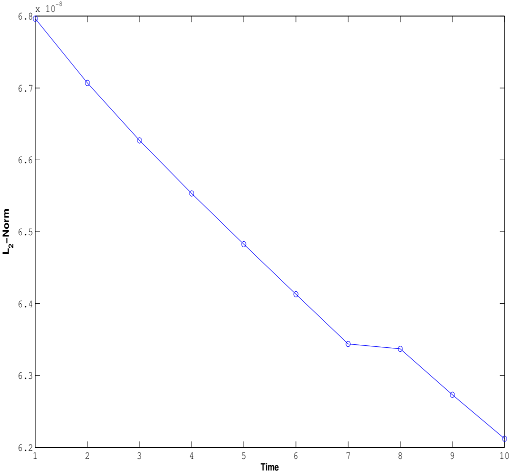


Figure: L^2 -norm error

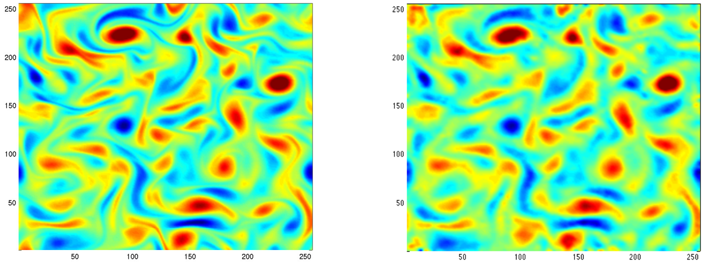
Error on real experience

Two types of observation

$$\mathbb{H} \circ \mathbf{v}^k - \mathbf{v}_{ob}^k := \mathbf{v}^k - I\mathbf{v}_{ob}^k, \qquad \mathbf{v}_{ob}^k \sim \text{ Optical-Flow}$$

$$\mathbb{H} \circ \mathbf{v}^k - \mathbf{v}_{ob}^k := I_1^k(x + \mathbf{v}^k) - I_0^k(x)$$

Pseudo observations error



(a) True vorticity

(b) Estimated vorticity

Figure: Optical-flow observation: RMSE=0.0544.

Pseudo observations error

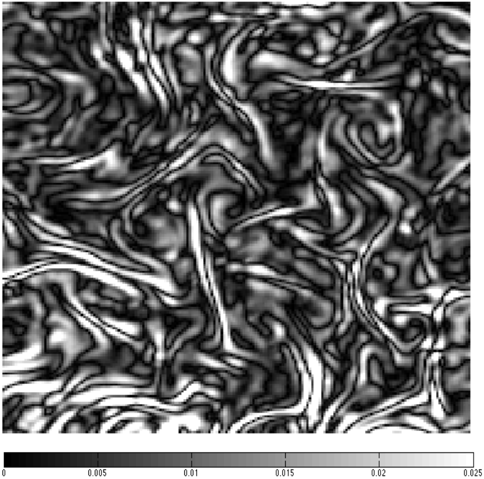
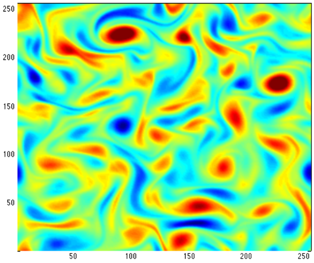
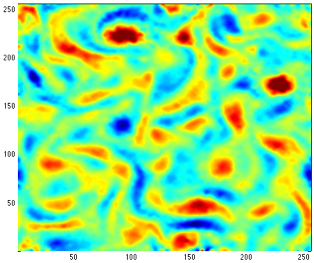


Figure: RSE on the vorticity = 0.0103

DFD observations error



(a) True vorticity



(b) Estimated vorticity

Figure: DFD observation: RMSE=0.0696, $j = 7$.

DFD observations error

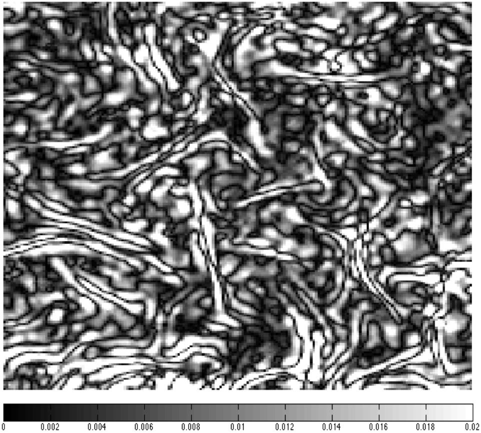
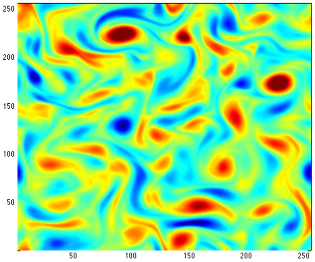
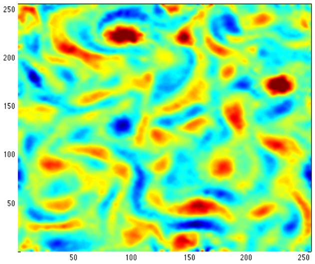


Figure: RSE on the vorticity

DFD observations and diffusion



(a) True vorticity



(b) Estimated vorticity

Figure: DFD observation: RMSE=0.0523,$\nu \simeq 2.73E^{-5}$, $Re^{-1} \simeq 1.33E^{-6}$.

Conclusion and Outlook

- **Navier-Stokes discretization**
- **Discrete adjoint models**

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- Navier-Stokes discretization
- Discrete adjoint models
- Models with low complexity
- Wavelet adaptativity in the simulation
- Use methods on a dynamic geophysical flow models