Divergence-free and Curl-free Wavelets on $[0, 1]^d$ with related applications

Souleymane Kadri-Harouna

FLUMINANCE-Centre de recherche INRIA Rennes, Bretagne Atlantique

Outline

1. Motivations

Outline

Outline

1. Motivations
2. Divergence-free and curl-free wavelets on $[0,1]^2$
   - Principle of the construction
   - Biorthogonal wavelet bases on $[0,1]$
   - Wavelet bases for $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{curl}(\Omega)$
3. Related applications
   - Divergence-free vector field analysis
   - Helmholtz-Hodge decomposition by wavelets
   - Navier-Stokes simulation

Outline

1. Motivations
2. Divergence-free and curl-free wavelets on $[0,1]^2$
   - Principle of the construction
   - Biorthogonal wavelet bases on $[0,1]$
   - Wavelet bases for $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{curl}(\Omega)$
3. Related applications
   - Divergence-free vector field analysis
   - Helmholtz-Hodge decomposition by wavelets
   - Navier-Stokes simulation
4. Conclusion and Outlook

Motivations

- **Electromagnetism** :

An electromagnetic field $\mathbf{E}$ that vanishes quickly at infinity can be decomposed as :

$$\mathbf{E} = \mathbf{B} + \nabla\Phi$$

Φ electric potential and $\mathbf{B}$ magnetic field which satisfies the Gauss law :

$$\oint \mathbf{B} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = 0$$

- **Incompressible fluid** :

Let $\mathbf{v}$ be the velocity of a fluid confined in $\Omega \subset \mathbb{R}^d$, we have :

$$\frac{d}{dt}V(t) = \int_{\partial\Omega(t)} \mathbf{v} \cdot \vec{\mathbf{n}} = \int_{\Omega(t)} \nabla \cdot \mathbf{v}$$

Volume variation = Integral of the velocity flux

The incompressibility condition gives :

$$\frac{d}{dt}V(t) = 0 \quad \Rightarrow \quad \mathbf{div}(\mathbf{v}) = \nabla \cdot \mathbf{v} = \sum_{i=1}^{d} \frac{\partial \mathbf{v}_i}{\partial x_i} = 0$$

More general : Helmholtz-Hodge Decomposition

[Girault-Raviart 86]

- **For u** $\in (L^2(\Omega))^d$, $\Omega \subset \mathbb{R}^d$ a *regular* open subset, we have :

$$\mathbf{u} = \nabla \wedge \chi + \nabla q + h \;\rightarrow \textbf{unique}$$

with

$$\nabla \cdot (\nabla \wedge \chi) = 0, \qquad \nabla \wedge (\nabla q) = 0, \qquad \nabla \cdot h = 0 \quad \text{and} \quad \nabla \wedge h = 0$$

- **In terms of spaces**, we obtain :

$$(L^2(\Omega))^d = \mathcal{H}_{div}(\Omega) \oplus \mathcal{H}_{curl}(\Omega) \oplus \mathcal{H}_{har}(\Omega) \;\rightarrow \textbf{orthogonal sum}$$

where

$$\mathcal{H}_{div}(\Omega) \;=\; \{\mathbf{u} \in (L^2(\Omega))^d \;;\; \nabla \cdot \mathbf{u} = 0 \; \underline{\text{and}} \; \mathbf{u} \cdot \vec{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$$

$\vec{\mathbf{n}}$ unit outward normal to $\partial\Omega$.

$\mathcal{H}_{curl}(\Omega) = \{\nabla q \;;\; q \in H_0^1(\Omega)\}$ and $\mathcal{H}_{har}(\Omega) = \{\nabla q \;;\; q \in H^1(\Omega) \; \underline{\text{and}} \; \Delta q = 0\}$

Outline

1. Motivations
2. Divergence-free and curl-free wavelets on $[0,1]^2$
   - Principle of the construction
   - Biorthogonal wavelet bases on $[0,1]$
   - Wavelet bases for $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{curl}(\Omega)$
3. Related applications
   - Divergence-free vector field analysis
   - Helmholtz-Hodge decomposition by wavelets
   - Navier-Stokes simulation
4. Conclusion and Outlook

Principle of the construction

(i) Starting with 1D regular biorthogonal MRA of $L^2(0,1)$: $(V_j^1, \tilde{V}_j^1)$.

(ii) Construct a 1D biorthogonal MRA $(V_j^0, \tilde{V}_j^0)$ linked to $(V_j^1, \tilde{V}_j^1)$ by :

$$\frac{d}{dx} V_j^1 = V_j^0 \quad \text{and} \quad \frac{d}{dx} \tilde{V}_j^0 \subset \tilde{V}_j^1, \quad \text{with} \quad \tilde{V}_j^0 \subset H_0^1(0,1)$$

(iii) Construct divergence-free and curl-free MRAs by :

$$\mathbf{V}_j^{div} = \mathbf{curl}[V_j^d \otimes V_j^d] \quad \text{and} \quad \mathbf{V}_j^{curl} = \mathbf{grad}[V_j^d \otimes V_j^d]$$

where : $V_j^d = V_j^1 \cap H_0^1(\Omega)$.

$$\longrightarrow \mathbf{div}[\mathbf{curl}(\mathbf{u})] = 0 \quad \text{and} \quad \mathbf{curl}[\mathbf{grad}(\mathbf{u})] = 0$$

Outline

(i) Biorthogonal wavelet bases on $[0,1]$

[Monasse-Perrier 98,Talocia-Tabacco 00]

Biorthogonal Multiresolution Analyses (BMRAs) of $L^2(0,1)$:

- $V_{j_0} \subset \cdots \subset V_j \subset V_{j+1} \cdots \subset L^2(0,1)$ and $\tilde{V}_{j_0} \subset \cdots \subset \tilde{V}_j \subset \tilde{V}_{j+1} \cdots \subset L^2(0,1)$
- $\overline{\cup V_j} = L^2(0,1)$ and $\overline{\cup \tilde{V}_j} = L^2(0,1)$
- $\forall\ j \geq j_0, \quad L^2(0,1) = V_j \oplus \tilde{V}_j^{\perp}$
- $V_j = \text{span} < \varphi^\flat_{j,\ell} = 2^{\frac{j}{2}} \varphi^\flat_\ell(2^j x), \varphi_{j,k} = 2^{\frac{j}{2}} \varphi(2^j x - k), \varphi^\sharp_{j,\ell} = 2^{\frac{j}{2}} \varphi^\sharp_\ell(2^j - 2^j x) >$

- $\varphi_{j,k}$ interior scaling functions whose support is included in $[0,1]$.
- $\varphi^\flat_{j,\ell}$ edge scaling functions at left and $\varphi^\sharp_{j,\ell}$ edge scaling functions at right.

$\longrightarrow$ the space $\tilde{V}_j$ has the same structure (edge and interior scaling functions).

(i) Biorthogonal wavelet bases on $[0,1]$

Biorthogonal wavelet bases :

- Bases for the detail spaces : $W_j = V_{j+1} \cap \tilde{V}_j^{\perp}$ and $\tilde{W}_j = \tilde{V}_{j+1} \cap V_j^{\perp}$.

- $W_j = \text{span} < \psi^{\flat}_{j,\ell} = 2^{\frac{j}{2}} \psi^{\flat}_{\ell}(2^j x), \psi_{j,k} = 2^{\frac{j}{2}} \psi(2^j x - k), \psi^{\sharp}_{j,\ell} = 2^{\frac{j}{2}} \psi^{\sharp}_{\ell}(2^j - 2^j x) >$

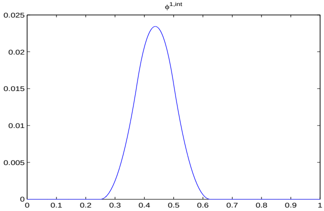
$\longrightarrow$ the space $\tilde{W}_j$ has the same structure (edge and interior wavelets).

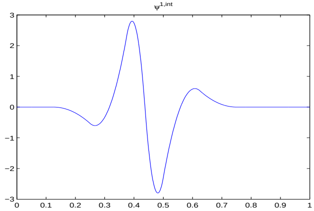
Finite dimensional space :

$$\# V_j = \# \tilde{V}_j = I_j < +\infty \quad \text{and} \quad \# W_j = \# \tilde{W}_j = 2^j$$

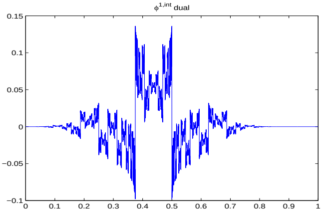
Biorthogonal B-Spline wavelets (3 vanishing moments)

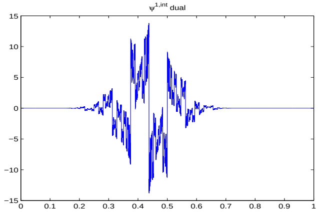
Primal internal scaling function (left) and wavelet (right) :





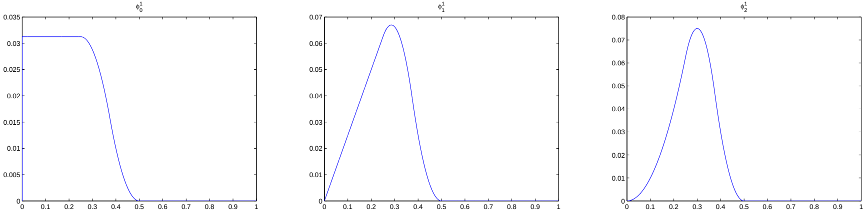
Dual internal scaling function (left) and wavelet (right) :



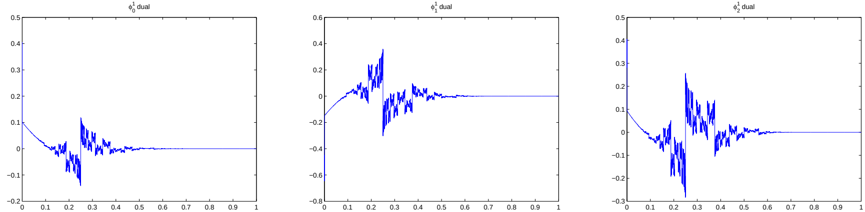


Biorthogonal Edge Scaling Functions (B-Spline 3.3)

Primal edge scaling functions of V_j :

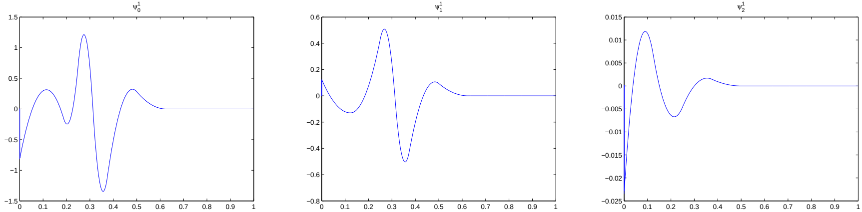


Dual edge scaling functions of $\tilde{V}_j$:

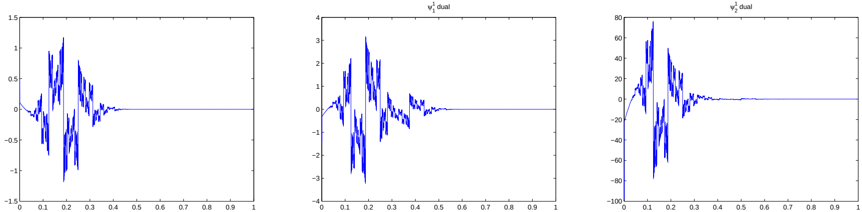


Biorthogonal Edge Wavelets (B-Spline 3.3)

Primal edge wavelets of W_j :



Dual edge wavelets of $\tilde{W}_j$:



(ii) BMRAs on $[0,1]$ linked by differentiation

Malgouyres-Lemarié's fundamental theorem : [Lemarié 92]

Let (φ^1, ψ^1) be 1D compactly supported scaling function and wavelet, with $\varphi^1 \in \mathcal{C}^{1+\epsilon}$. There exists (φ^0, ψ^0) compactly supported scaling function and wavelet such that :

$$(\varphi^1(x))' = \varphi^0(x) - \varphi^0(x-1) \quad \text{and} \quad (\psi^1(x))' = 4\psi^0$$

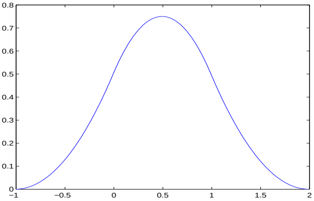
The associated spaces in whole $\mathbb{R}$ satisfy :

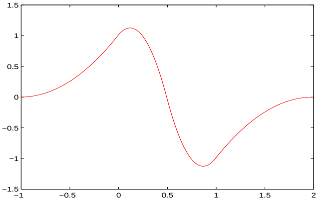
$$\frac{d}{dx} V_j^1(\mathbb{R}) = V_j^0(\mathbb{R}) \quad \text{and} \quad \frac{d}{dx} W_j^1(\mathbb{R}) = W_j^0(\mathbb{R})$$

For the duals : $(\tilde{\varphi}^0(x))' = \tilde{\varphi}^1(x+1) - \tilde{\varphi}^1(x)$ and $(\tilde{\psi}^0(x))' = -4\tilde{\psi}^1$.

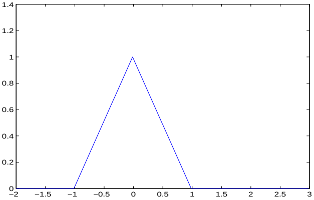
Quadratic and linear Spline linked by differentiation

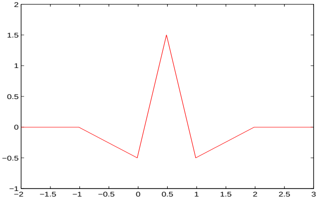
Quadratic Spline φ^1 (left) and ψ^1 (right) :





Linear Spline φ^0 (left) and ψ^0 (right) :





(ii) BMRAs on $[0, 1]$ linked by differentiation

Adaptation to the interval $[0, 1]$:

- Jouini-Lemarié (1993) provide a theoretical construction that conserves :

$$\frac{d}{dx} V_j^1 = V_j^0 \quad \text{and} \quad \frac{d}{dx} W_j^1 = W_j^0$$

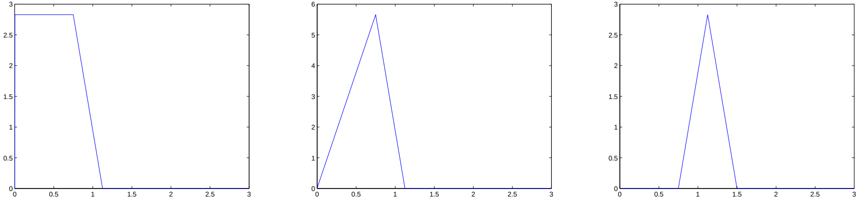
- Kadri-Perrier (2010) provide a practical construction that satisfies :

$$2^{-j}(\psi_{j,k}^1)' = \psi_{j,k}^0$$

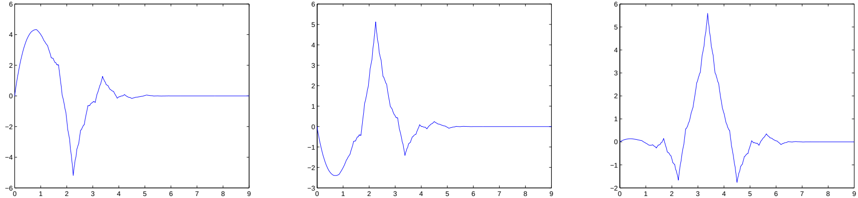
$\longrightarrow$ Even for the edge wavelets

Biorthogonal Scaling Functions from derivation/integration

- V_j^0 **primal edge scaling functions (derivative space) :**

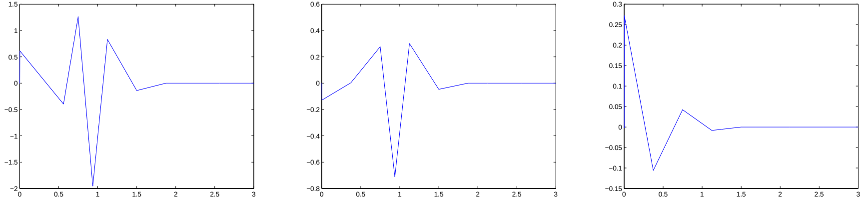


- $\tilde{V}_j^0$ **dual edge scaling functions (integrated space) :**

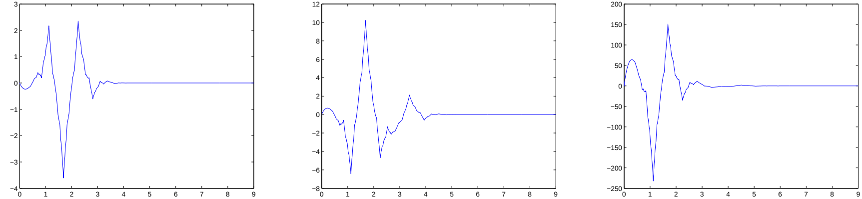


Biorthogonal Wavelets from derivation/integration

- W_j^0 **edge primal wavelet from derivation :**



- $\tilde{W}_j^0$ **dual edge wavelet from integration :**



Outline

(iii) Wavelet basis for $\mathcal{H}_{div}(\Omega)$

- **Divergence-free Multiresolution Analysis** $\mathbb{V}_j^{div}$:

$$\mathbb{V}_j^{div} = \text{span} < \Phi_{j,\mathbf{k}}^{div};\ \mathbf{k} \in I_j^2 >, \quad j \geq j_0$$

- **Divergence-free scaling functions on** $[0,1]^2$ **:**

$$\Phi_{j,\mathbf{k}}^{div} = \mathbf{curl}[\varphi_{j,k_1}^d \otimes \varphi_{j,k_2}^d] = \left| \begin{array}{l} \varphi_{j,k_1}^d \otimes (\varphi_{j,k_2}^d)' \\ -(\varphi_{j,k_1}^d)' \otimes \varphi_{j,k_2}^d \end{array} \right. , \quad \varphi_{j,k}^d \in V_j^d = V_j^1 \cap H_0^1$$

(As : $\frac{d}{dx} V_j^1 = V_j^0$)

$$\longrightarrow \ \mathbb{V}_j^{div} = (V_j^1 \otimes V_j^0) \times (V_j^0 \otimes V_j^1) \cap \mathcal{H}_{div}(\Omega) \ \longrightarrow \ \textbf{FWT !}$$

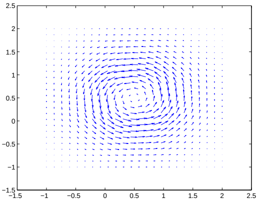
- **Anisotropic divergence-free wavelets on** $[0,1]^2$ **:**

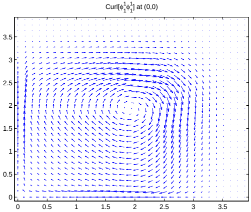
$$\Psi_{\mathbf{j},\mathbf{k}}^{div} = \mathbf{curl}[\psi_{j_1,k_1}^d \otimes \psi_{j_2,k_2}^d] = \left| \begin{array}{l} 2^{j_2} \psi_{j_1,k_1}^d \otimes \psi_{j_2,k_2}^0 \\ -2^{j_1} \psi_{j_1,k_1}^0 \otimes \psi_{j_2,k_2}^d \end{array} \right. \qquad (\psi_{j,k}^0 = 2^{-j} (\psi_{j,k}^1)')$$

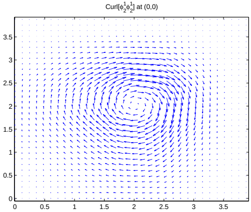
$\mathbf{j} = (j_1, j_2)$, with $j_1, j_2 \geq j_0$.

Divergence-free function basis (vector field)

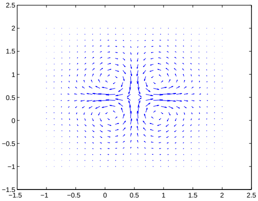
Scaling functions :

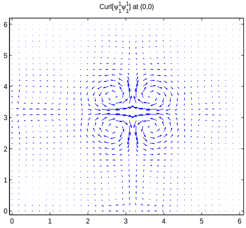


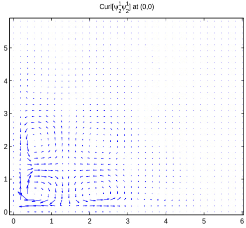




Wavelets :







(iii) Wavelet basis for $\mathcal{H}_{curl}(\Omega)$

- **Curl-free Multiresolution Analysis** $\mathbb{V}_j^{curl}$:

$$\mathbb{V}_j^{curl} = \text{span} < \Phi_{j,\mathbf{k}}^{curl};\ \mathbf{k} \in I_j^2 >, \quad j \geq j_0$$

- **Curl-free scaling functions on** $[0,1]^2$ **:**

$$\Phi_{j,\mathbf{k}}^{curl} \;=\; \nabla[\varphi_{j,k_1}^d \otimes \varphi_{j,k_2}^d] = \left| \begin{array}{l} (\varphi_{j,k_1}^d)' \otimes \varphi_{j,k_2}^d \\ \varphi_{j,k_1}^d \otimes (\varphi_{j,k_2}^d)' \end{array} \right. , \quad \varphi_{j,k}^d \in V_j^d = V_j^1 \cap H_0^1(\Omega)$$

(As : $\frac{d}{dx} V_j^1 = V_j^0$)

$$\longrightarrow\ \mathbb{V}_j^{curl} \subset (V_j^0 \otimes V_j^1) \times (V_j^1 \otimes V_j^0) \cap \mathcal{H}_{div}^{\perp}(\Omega) \ \longrightarrow\ \textbf{FWT !}$$

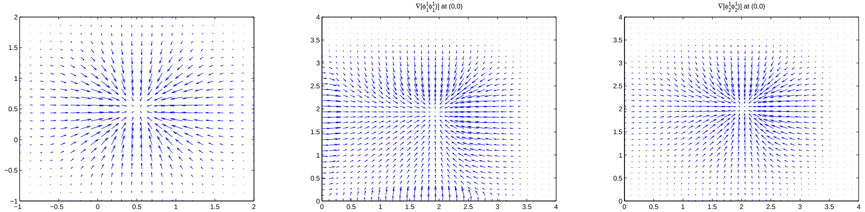
- **Anisotropic curl-free wavelets on** $[0,1]^2$ **:**

$$\Psi_{\mathbf{j},\mathbf{k}}^{curl} = \nabla[\psi_{j_1,k_1}^d \otimes \psi_{j_2,k_2}^d] = \left| \begin{array}{l} 2^{j_1} \psi_{j_1,k_1}^0 \otimes \psi_{j_2,k_2}^d \\ 2^{j_2} \psi_{j_1,k_1}^d \otimes \psi_{j_2,k_2}^0 \end{array} \right. \quad (\psi_{j,k}^0 = 2^{-j} \frac{d}{dx} \psi_{j,k}^1)$$

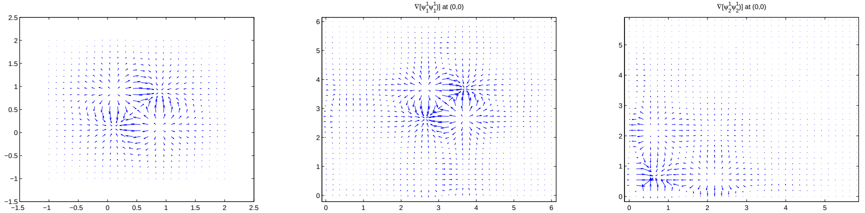
$\mathbf{j} = (j_1, j_2)$, with $j_1, j_2 \geq j_0$.

Curl-free function basis (vector field)

Scaling functions :



Wavelets :



Outline

1. Motivations
2. Divergence-free and curl-free wavelets on $[0,1]^2$
   - Principle of the construction
   - Biorthogonal wavelet bases on $[0,1]$
   - Wavelet bases for $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{curl}(\Omega)$
3. Related applications
   - Divergence-free vector field analysis
   - Helmholtz-Hodge decomposition by wavelets
   - Navier-Stokes simulation
4. Conclusion and Outlook

Divergence-free vector field analysis (ψ^1 B-Spline 3.3)

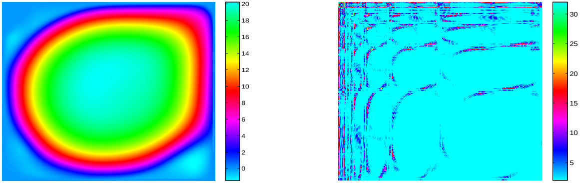


Figure: Scaling functions coeffs (left) and wavelets coeffs (right) : $j = 8$.

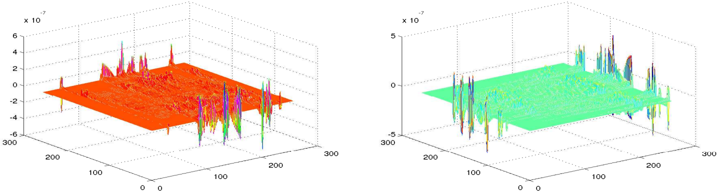


Figure: Residual error on $\mathbf{u}1$ and $\mathbf{u}2$ with 22% of the divergence-free coeffs : $j = 8$.

Divergence-free vector field analysis (ψ^1 B-Spline 3.3)

Non linear approximation :

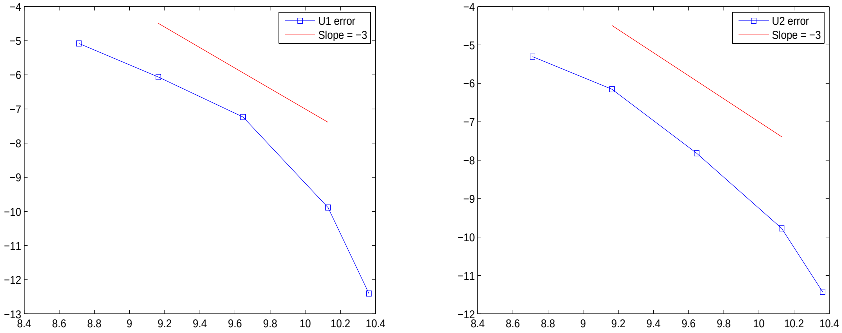


Figure: ℓ^2 error on $\mathbf{u}_1$ (left) and $\mathbf{u}_2$ (right) : logarithmic scale (X axis number of coefficients retained and Y axis error).

Outline

1. Motivations
2. Divergence-free and curl-free wavelets on $[0,1]^2$
   - Principle of the construction
   - Biorthogonal wavelet bases on $[0,1]$
   - Wavelet bases for $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{curl}(\Omega)$
3. Related applications
   - Divergence-free vector field analysis
   - Helmholtz-Hodge decomposition by wavelets
   - Navier-Stokes simulation
4. Conclusion and Outlook

Helmholtz-Hodge decomposition by wavelets

Practical computation :

$$\mathbf{u} = \mathbf{u}_{\rm div} + \mathbf{u}_{\rm curl} + \mathbf{u}_{\rm har}$$

Then :

$$\langle \mathbf{u} / \Psi^{\rm div}_{\mathbf{j},\mathbf{k}} \rangle = \langle \mathbf{u}_{\rm div} / \Psi^{\rm div}_{\mathbf{j},\mathbf{k}} \rangle \qquad \text{and} \qquad \langle \mathbf{u} / \Psi^{\rm curl}_{\mathbf{j},\mathbf{k}} \rangle = \langle \mathbf{u}_{\rm curl} / \Psi^{\rm curl}_{\mathbf{j},\mathbf{k}} \rangle$$

Searching $\mathbf{u}_{\rm div}$ and $\mathbf{u}_{\rm curl}$ in the divergence-free and curl-free form :

$$\mathbf{u}_{\rm div} = \sum_{\mathbf{j},\mathbf{k}} d^{\rm div}_{\mathbf{j},\mathbf{k}} \; \Psi^{\rm div}_{\mathbf{j},\mathbf{k}} \qquad \text{and} \qquad \mathbf{u}_{\rm rot} = \sum_{\mathbf{j},\mathbf{k}} d^{\rm curl}_{\mathbf{j},\mathbf{k}} \; \Psi^{\rm curl}_{\mathbf{j},\mathbf{k}}$$

leads to :

$$(d^{\rm div}_{\mathbf{j},\mathbf{k}}) = \mathbb{M}^{-1}_{\rm div}(\langle \mathbf{u} / \Psi^{\rm div}_{\mathbf{j},\mathbf{k}} \rangle) \qquad \text{and} \qquad (d^{\rm curl}_{\mathbf{j},\mathbf{k}}) = \mathbb{M}^{-1}_{\rm curl}(\langle \mathbf{u} / \Psi^{\rm curl}_{\mathbf{j},\mathbf{k}} \rangle)$$

$\mathbb{M}_{\rm div}$ and $\mathbb{M}_{\rm curl}$: Gram matrices of bases $(\Psi^{\rm div}_{\mathbf{j},\mathbf{k}})$ and $(\Psi^{\rm curl}_{\mathbf{j},\mathbf{k}})$ respectively.

$$\mathbf{u}_{\rm har} = \mathbf{u} - \mathbf{u}_{\rm div} - \mathbf{u}_{\rm curl}$$

$\longrightarrow$ In periodic, it is easier to use the Fourier basis !

Remarkable property

- **In the two dimensional space**, we have :

$$\forall\ \mathbf{u},\mathbf{v} \in H_0^1(\Omega) \quad \int_\Omega \mathbf{rot}(\mathbf{u})\cdot\mathbf{rot}(\mathbf{v}) = \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v} \quad \text{(same property in periodic)}$$

Then

$\longrightarrow$ $\mathbb{M}$ = Matrix of 2D scalar Laplacian $\longrightarrow$ **diagonal preconditioner !**

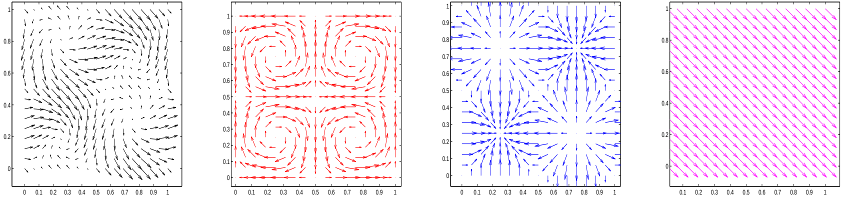
- **The tensor structure** of bases gives :

$$[\mathbb{M}(d^{\mathrm{div}}_{\mathbf{j},\mathbf{k}})] = \mathbf{M}[d^{div}_{\mathbf{j},\mathbf{k}}]\mathbf{R} + \mathbf{R}[d^{div}_{\mathbf{j},\mathbf{k}}]\mathbf{M}$$

with **M** and **R** the mass-matrix and stiffness-matrix of the 1D wavelet basis $(\psi^d_{j,k})$.

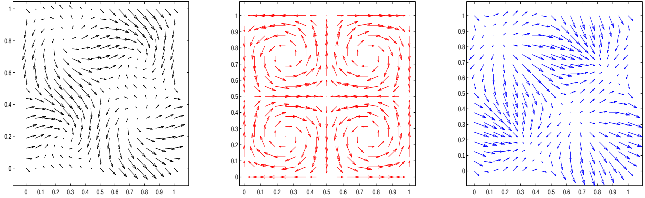
Example of Helmholtz-Hodge decomposition

- **Helmholtz-Hodge decomposition :**



$$\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \mathbf{u}_{\mathrm{curl}} + \mathbf{u}_{\mathrm{har}}$$

- **Helmholtz decomposition :** $\mathbf{u}_{\mathrm{div}} \cdot \vec{\mathbf{n}} = 0$



$$\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \nabla q$$

Example of Helmholtz-Hodge decomposition

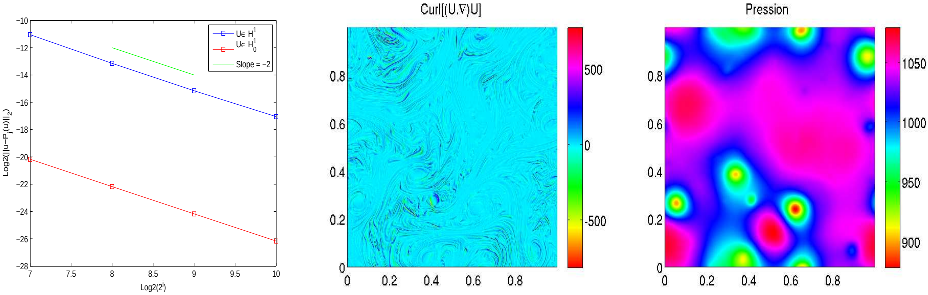


Figure: H^1 and H_0^1 error (left), vorticity of $(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \nabla\, \mathbf{p}$ (center) and the pressure $\mathbf{p}$ (right).

$$\mathbf{u} = \mathbf{curl}[\cos(4\pi x)x(1-x)\cos(4\pi y)y(1-y)] \in H^1(\Omega)$$

$$\mathbf{v} = \mathbf{curl}[\sin(4\pi x)x^3(1-x)^3\sin(4\pi y)y^3(1-y)^3] \in H_0^1(\Omega)$$

Outline

1. Motivations

2. Divergence-free and curl-free wavelets on $[0,1]^2$
   - Principle of the construction
   - Biorthogonal wavelet bases on $[0,1]$
   - Wavelet bases for $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{curl}(\Omega)$

3. Related applications
   - Divergence-free vector field analysis
   - Helmholtz-Hodge decomposition by wavelets
   - Navier-Stokes simulation

4. Conclusion and Outlook

Incompressible Navier-Stokes equations

Cauchy problem for **Navier-Stokes** :

$$(NS) \quad \begin{cases} \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathbf{p} = \mathbf{f}, & x\in\Omega,\ t\in[0,T] \\ \nabla\cdot\mathbf{v} = 0, & x\in\Omega,\ t\in[0,T] \\ \mathbf{v}(0,x) = \mathbf{v}_0(x), & x\in\Omega \\ \mathbf{v} = \mathbf{v}_b, & x\in\partial\Omega,\ t\in[0,T] \end{cases}$$

Unknowns : velocity $\mathbf{v}(t,x)$ and pressure $\mathbf{p}(t,x)$

Projecting (NS) onto $\mathcal{H}_{div}(\Omega)$ yields :

$$\partial_t\mathbf{v} = \mathbb{P}[\nu\Delta\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{f}] \qquad (NSP)$$

with $\mathbb{P}$ orthogonal projector from $(L^2(\Omega))^d$ to $\mathcal{H}_{div}(\Omega)$.

The pressure $\mathbf{p}$ is recovered through the Helmholtz-Hodge decomposition :

$$\nabla\mathbf{p} = \nu\Delta\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{f} - \mathbb{P}[\nu\Delta\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{f}]$$

New schemes for Navier-Stokes

Time discretization-classical projection method [Chorin 68, Temam 69] **:**

- Intermediate velocity computation :

$$\begin{cases} \frac{\mathbf{v}^* - \mathbf{v}_n}{\delta t} - \nu \Delta \mathbf{v}^* + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = 0, \text{ in } \Omega \\ \mathbf{v}^* = \mathbf{v}_b, \text{ on } \partial\Omega \end{cases}$$

- Pressure and velocity computation :

$$\begin{cases} \delta t \Delta \mathbf{p}_{n+1} - \nabla \cdot \mathbf{v}^* = 0, \text{ in } \Omega \\ \nabla \mathbf{p}_{n+1} = 0, \text{ on } \partial\Omega \end{cases} \quad \begin{cases} \mathbf{v}_{n+1} = \mathbf{v}^* - \delta t \nabla \mathbf{p}_{n+1}, \text{ in } \Omega \\ \mathbf{v}_{n+1} = \mathbf{v}_b, \text{ on } \partial\Omega \end{cases}$$

New schemes for Navier-Stokes

Time discretization-classical projection method [Chorin 68, Temam 69] **:**

- Intermediate velocity computation :

$$\begin{cases} \frac{\mathbf{v}^* - \mathbf{v}_n}{\delta t} - \nu \Delta \mathbf{v}^* + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = 0, \text{ in } \Omega \\ \mathbf{v}^* = \mathbf{v}_b, \text{ on } \partial\Omega \end{cases}$$

- Pressure and velocity computation :

$$\begin{cases} \delta t \Delta \mathbf{p}_{n+1} - \nabla \cdot \mathbf{v}^* = 0, \text{ in } \Omega \\ \nabla \mathbf{p}_{n+1} = 0, \text{ on } \partial\Omega \end{cases} \quad \begin{cases} \mathbf{v}_{n+1} = \mathbf{v}^* - \delta t \nabla \mathbf{p}_{n+1}, \text{ in } \Omega \\ \mathbf{v}_{n+1} = \mathbf{v}_b, \text{ on } \partial\Omega \end{cases}$$

Modified projection method

$$\begin{cases} \frac{\mathbf{v}^* - \mathbf{v}_n}{\delta t} - \nu \Delta \mathbf{v}^* + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = 0, \text{ in } \Omega \\ \mathbf{v}^* = \mathbf{v}_b, \text{ on } \partial\Omega \\ \mathbf{v}_{n+1} = \mathbb{P}(\mathbf{v}^*) \end{cases}$$

New schemes for Navier-Stokes

Time discretization-classical Gauge method : $\mathbf{a} = \mathbf{v} + \nabla\chi$

- Intermediate velocity computation :

$$\begin{cases} \frac{\mathbf{a}_{n+1}-\mathbf{a}_n}{\delta t} - \nu\Delta\mathbf{a}_{n+1} + (\mathbf{v}_n\cdot\nabla)\mathbf{v}_n = 0, \text{ in } \Omega \\ \mathbf{a}_{n+1}\cdot\vec{\mathbf{n}} = \mathbf{v}_b\cdot\vec{\mathbf{n}},\ \mathbf{a}_{n+1}\cdot\vec{\tau} = \mathbf{v}_b\cdot\vec{\tau} + 2\frac{\partial\chi_n}{\partial\vec{\tau}} - \frac{\partial\chi_{n-1}}{\partial\vec{\tau}}, \text{ on } \partial\Omega \end{cases}$$

- Velocity computation :

$$\begin{cases} \Delta\chi_{n+1} = \nabla\cdot\mathbf{a}_{n+1}, \text{ in } \Omega \\ \frac{\partial\chi_{n+1}}{\partial\vec{\mathbf{n}}} = 0, \text{ on } \partial\Omega \\ \mathbf{v}_{n+1} = \mathbf{a}_{n+1} - \nabla\chi_{n+1} \end{cases}$$

New schemes for Navier-Stokes

Time discretization-classical Gauge method : $\mathbf{a} = \mathbf{v} + \nabla\chi$

- Intermediate velocity computation :

$$\begin{cases} \frac{\mathbf{a}_{n+1}-\mathbf{a}_n}{\delta t} - \nu\Delta\mathbf{a}_{n+1} + (\mathbf{v}_n\cdot\nabla)\mathbf{v}_n = 0, \text{ in } \Omega \\ \mathbf{a}_{n+1}\cdot\vec{\mathbf{n}} = \mathbf{v}_b\cdot\vec{\mathbf{n}},\ \mathbf{a}_{n+1}\cdot\vec{\tau} = \mathbf{v}_b\cdot\vec{\tau} + 2\frac{\partial\chi_n}{\partial\vec{\tau}} - \frac{\partial\chi_{n-1}}{\partial\vec{\tau}}, \text{ on } \partial\Omega \end{cases}$$

- Velocity computation :

$$\begin{cases} \Delta\chi_{n+1} = \nabla\cdot\mathbf{a}_{n+1}, \text{ in } \Omega \\ \frac{\partial\chi_{n+1}}{\partial\vec{\mathbf{n}}} = 0, \text{ on } \partial\Omega \\ \mathbf{v}_{n+1} = \mathbf{a}_{n+1} - \nabla\chi_{n+1} \end{cases}$$

Modified Gauge method

$$\begin{cases} \frac{\mathbf{a}_{n+1}-\mathbf{a}_n}{\delta t} - \nu\Delta\mathbf{a}_{n+1} + (\mathbf{v}_n\cdot\nabla)\mathbf{v}_n = 0, \text{ in } \Omega \\ \mathbf{a}_{n+1}\cdot\vec{\mathbf{n}} = \mathbf{v}_b\cdot\vec{\mathbf{n}},\ \mathbf{a}_{n+1}\cdot\vec{\tau} = \mathbf{v}_b\cdot\vec{\tau} + 2\frac{\partial\chi_n}{\partial\vec{\tau}} - \frac{\partial\chi_{n-1}}{\partial\vec{\tau}}, \text{ on } \partial\Omega \\ \mathbf{v}_{n+1} = \mathbb{P}(\mathbf{a}_{n+1}) \end{cases}$$

Numerical resolution

- Scale separation :

$$\mathbf{v}(t,x) = \sum_{\mathbf{j},\mathbf{k}} d^{div}_{\mathbf{j},\mathbf{k}}(t)\ \Psi^{div}_{\mathbf{j},\mathbf{k}}(x)$$

$\longrightarrow$ ODE system on the coefficients $[d^{div}_{\mathbf{j},\mathbf{k}}(t)]$ in (NSP).

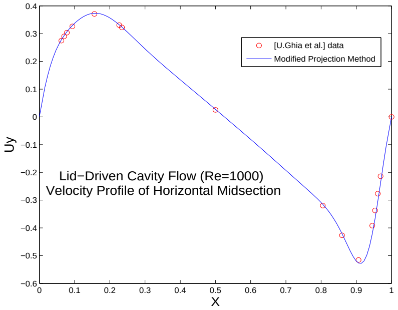
- Classical method for time and space discretization :

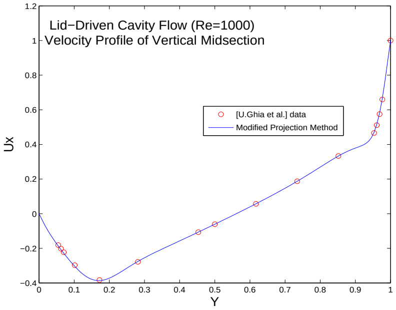
- Galerkin method in space with $\vec{\mathbf{V}}_j = (V_j^1 \otimes V_j^0) \times (V_j^0 \otimes V_j^1)$.

$\longrightarrow$ At each time step we need to compute the projector $\mathbb{P}$.

Lid Driven Cavity

Validation :





Velocity profile in mid horizontal (left) and vertical (right) section : $Re = 1000$ and $j = 7$ with $\mathbf{v}_b = 1$.

Reference : [U.Ghia et al. 82]

Lid Driven Cavity

Divergence-free scaling coefficients :

Div-free scaling function coefficients : $Re = 5000$ and mesh size 128×128.

Lid Driven Cavity

Complexity analysis :

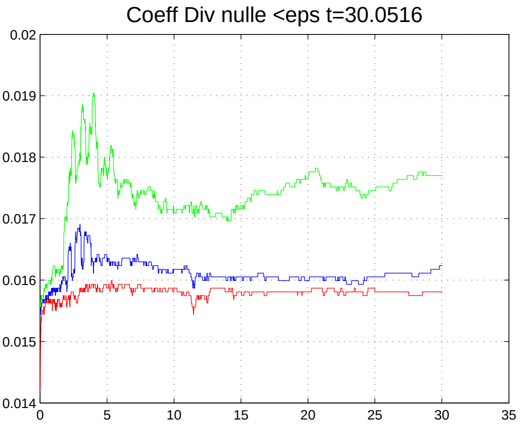


Figure: Percentage of divergence-free wavelet coefficients up to ϵ : $\epsilon = 8.10^{-4}, \epsilon = 16.10^{-4}, \epsilon = 32.10^{-4}$.

Lid Driven Cavity

Driven cavity with High Reynolds number :

Vorticity evolution : $Re = 50000$, $j = 9$ and $\mathbf{v}_b = 16x^2(1-x)^2$.

Dipole-vortex rebound from a wall

Vorticity evolution : $\nu = 1/4000$ and $j = 9$.

Conclusion and Outlook

- **Practical construction of divergence-free and curl-free wavelets**
- **Helmholtz-Hodge decomposition with boundary conditions**
- **Navier-Stokes simulation with physical boundary conditions**

Conclusion and Outlook

- **Practical construction of divergence-free and curl-free wavelets**
- **Helmholtz-Hodge decomposition with boundary conditions**
- **Navier-Stokes simulation with physical boundary conditions**
- **Use wavelet adaptativity in Navier-Stokes simulation**
- **Variational multiscale method**

Quelques Simulations

Reconnection de vortex :

Module de vorticité : $j = 6$

Vorticité dans le plan de reconnection : $j = 6$