# Random zonal eigenfunctions and Hölder version of the Paley-Zygmund theorem on compact manifolds

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**Abstract**

We study the convergence of Gaussian random series of radial/zonal eigenfunctions of the Laplace-Beltrami operator (on the Euclidean space and on the round sphere). More precisely, we obtain a simple necessary and sufficient condition of almost sure uniform convergence (we thus complete an analysis of Ayache and Tzvetkov). In dimension two, our strategy turns out to be linked with Hölder regularities. As a by-product, we also prove a Hölder version of the Paley-Zygmund theorem on a boundaryless Riemannian compact manifold.

## 1 Introduction

Let $\mathcal{M}$ be a Riemannian manifold (we shall precise assumptions in the sequel) and let us consider a sequence $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(\mathcal{M})$ made of eigenfunctions of a fixed operator (for instance a Laplace-Beltrami operator). The present paper gives a contribution in the research area interested in the following question : once a Banach space $B$ of functions on $\mathcal{M}$ is fixed, can we give a necessary and sufficient condition on a sequence of coefficients $(c_k)_{k\in\mathbb{N}}$ such that the Gaussian random series $\sum g_k(\omega)c_k\phi_k$ almost surely converges in $B$ (here $(g_k)_{k\in\mathbb{N}}$ is a standard i.i.d. Gaussian $\mathcal{N}(0,1)$ sequence of random variables defined on an abstract probability space $\Omega$). Actually such a question is too general and we shall give a satisfactory solution in two particular cases that we shall motivate in the present introduction.

We prefer to refer to the introduction of [MP81, Kah68] for more details about the history of such questions for $\mathcal{M}$ being the torus or a compact group or the introduction of [Ime22] for $\mathcal{M}$ being a manifold. We however shortly recall here the main contributions. One could separate results in two directions :

**Analysis on groups.** The story began with the paper of Paley and Zygmund [PZ] who gave a solution for random linear combinations[1] $\sum_{k\in\mathbb{N}} \pm c_k e^{ikx}$. Actually, for any fixed $p\in[2,+\infty)$, such a random series belongs, with probability 1, to $L^p(-\pi,\pi)$ if and only if $(c_k)$ belongs to $\ell^2(\mathbb{N})$. It is now known that considering random signs $\pm$ (namely i.i.d. Rademacher random variables) instead of Gaussian random variables leads to the same results[2]. The case $p=+\infty$ is much more delicate. One of the best "simple sufficient condition" is given[3] by a result of Salem and Zygmund [SZ54, page 291]

$$\sum_{k\geq 2}\frac{1}{k\sqrt{\ln(k)}}\sqrt{\sum_{n\geq k}|c_n|^2}<+\infty$$

that actually sharpens the Paley-Zygmund condition : there is $\gamma>1$ satisfying

$$\sum_{k\geq 2}|c_k|^2\ln^\gamma(k)<+\infty.$$

In order to obtain a condition which is sufficient and necessary on the sequence of $(c_k)$ in the case $p=+\infty$, one needs to combine two arguments : the theorem of Dudley [Dud67] that proves that an abstract condition, called the entropy condition, is sufficient and then the theorem of Fernique that proves that the entropy condition is necessary (by exploiting the stationarity of Gaussian processes, see [Fer74, Part 8]). The book [MP81] finally contains the

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[1] We prefer here to choose an index set equaling $\mathbb{N}$ instead of the natural choice $\mathbb{Z}$ by consistence with the sequel of the article.

[2] This equivalence has been proved by Marcus and Pisier.

[3] Actually, such a condition may become necessary under some assumptions (see [Mar75]).

complete solution for $\mathcal{M}$ being a compact group for adequate random series. We also refer to [LT91, Chapter 13] or [LQ18a, Chapters 3 and 6] for much more details on this aspect.

**Analysis for elliptic operators.** The story continues with the papers of Ayache-Tzvetkov [AT08] and Tzvetkov [Tzv09] by replacing the trigonometric functions $e^{ikx}$ with eigenfunctions of a Laplace operator. Let us explain with a few details the contribution of Ayache and Tzvetkov : let us denote by $(Z_n^{d,\mathrm{Dir}})_{n\geq 1}$ the sequence of the radial eigenfunctions of the Laplace operator on the closed unit ball $B_d(0,1)$ of $\mathbb{R}^d$ with Dirichlet conditions for $d\geq 2$ (see (1) for the exact definition). Ayache and Tzvetkov proved that, in contrast to the trigonometric case, for any sequence of coefficients $(c_n)_{n\in\mathbb{N}^\star}$, there is an exponent[4] $p_c\in[2,+\infty]$ (depending on $(c_n)$) such that

$$p<p_c \quad\Rightarrow\quad \sum_{n\geq 1} g_n(\omega)c_n Z_n^{d,\mathrm{Dir}} \quad \text{almost surely converges in } L^p(B_d(0,1)),$$
$$p>p_c \quad\Rightarrow\quad \sum_{n\geq 1} g_n(\omega)c_n Z_n^{d,\mathrm{Dir}} \quad \text{almost surely diverges in } L^p(B_d(0,1)).$$

In [AT08, Theorem 4], it is proved that $p_c=\frac{2d}{d-2}$ in the specific case $c_n\simeq\frac{1}{n}$. Then, Grivaux obtained in [Gri10] a simple formula giving $p_c$ for any arbitrary sequence of coefficients $(c_n)\in\ell^2(\mathbb{N}^\star)$. Actually, the issue behind such considerations is to decide whether or not the Gaussian random series $\sum g_n(\omega)c_nZ_n^{d,\mathrm{Dir}}$ almost surely converges in $L^p(B_d(0,1))$. For any finite exponent $p>\frac{2d}{d-1}$, a complete solution is given in [Ime18] for zonal eigenfunctions[5] on the sphere $\mathbb{S}^d$ for $d\geq 2$. The analysis of [Ime18] makes play a fundamental role to the finiteness of $p$ because of interpolation arguments. We note that a common ingredient in all the works [AT08, Gri10, Ime18] is the concentration of the radial (or zonal) eigenfunctions. We also refer to [IRT16] for the analysis of radial eigenfunctions of the harmonic oscillator $-\Delta+|x|^2$ on $L^2(\mathbb{R}^d)$.

**First contribution of the paper.** The first contribution of the present paper is to find a necessary and sufficient condition for $p=+\infty$ in the zonal/radial framework studied by Ayache and Tzvetkov in [AT08]. Let us briefly recall the two examples of functions we have in mind :

- the sequence of radial eigenfunctions of the Laplace operator $-\Delta_{\mathrm{Dir}}$ on the Euclidean closed ball $B_d(0,1)=\{x\in\mathbb{R}^d, |x|\leq 1\}$ (for $d\geq 2$) with Dirichlet conditions. Then the sequence of radial eigenfunctions $Z_n^{d,\mathrm{Dir}}$ is given by

$$Z_n^{d,\mathrm{Dir}}(x)=c_{d,n}\frac{J_{\frac{d}{2}-1}(\lambda_{d,n}|x|)}{|x|^{\frac{d}{2}-1}},\qquad \forall x\in\mathbb{R}^d,\qquad \forall n\in\mathbb{N}^\star \tag{1}$$

  in which $\lambda_{d,n}$ is the $n$-th zero the Bessel function $J_{\frac{d}{2}-1}$. We have $\lambda_{d,n}\simeq n$ and $c_{d,n}\simeq\sqrt{n}$ for $n\to+\infty$ (see [AT08, page 4431]). The functions $Z_n^{d,\mathrm{Dir}}$ concentrate around the origin (see (22)) and satisfy the eigenfunction equation

$$-\Delta Z_n^{d,\mathrm{Dir}}=\lambda_{d,n}^2 Z_n^{d,\mathrm{Dir}}.$$

- the second example is the sequence of zonal eigenfunctions of the round sphere $\mathbb{S}^d\subset\mathbb{R}^{d+1}$, for $d\geq 2$, with respect to the Laplace-Beltrami operator $-\boldsymbol{\Delta}$. It is usual to consider the zonal eigenfunctions around a point, for instance $P=(1,0,\dots,0)$. With such a formalism, the sequence of zonal eigenfunctions, denoted here by $Z_n^{\mathbb{S}^d}$ can be defined via the orthogonal Jacobi polynomials (or also Gegenbauer polynomials)

$$Z_n^{\mathbb{S}^d}(x)=c'_{d,n}P_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(x_1),\qquad \forall x=(x_1,\dots,x_{d+1})\in\mathbb{S}^d, \tag{2}$$

  in which $c'_{d,n}\simeq\sqrt{n}$ (for $n\to+\infty$) and $\|Z_n^{\mathbb{S}^d}\|_{L^2(\mathbb{S}^d)}=1$. We also have

$$-\boldsymbol{\Delta}Z_n^{\mathbb{S}^d}=n(n+d-1)Z_n^{\mathbb{S}^d}. \tag{3}$$

  Furthermore, $Z_n^{\mathbb{S}^d}$ concentrates around the point $P$ but also around the point $-P$ thanks to the formula $P_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(-x)=(-1)^nP_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(x)$.

[4] Actually, the inequality $p_c\geq\frac{2d}{d-1}$ holds because the eigenfunctions $Z_n^{d,\mathrm{Dir}}$ are uniformly bounded in $L^p(B_d(0,1))$ for $p<\frac{2d}{d-1}$ (we refer to [AT08] for explanations or the inequality in [Ime18, page 266, line (2)]).

[5] Actually it is known that the zonal eigenfunctions on $\mathbb{S}^d$ have in some sense a similar behavior than that of the radial eigenfunctions $Z_n^{d,\mathrm{Dir}}$ which play a sort of canonical model of eigenfunctions that concentrate around a point (see [AT08, page 4428, remark d)] and this point of view will also be clarified in Section 2.

It is known that these models are very similar (for instance via their $L^p$ bounds) and more precisely that the first example is more or less a sort of canonical model (see Section 2). Let us moreover fix a sequence $(g_n)_{n\geq 1}$ of i.i.d. Gaussian random variables $\mathcal{N}(0,1)$. We shall prove the following result whose main contribution is the proof of the implication i) $\Rightarrow$ ii).

**Theorem 1.** *We assume the dimension $d$ fulfills $d \geq 2$. For simplicity, let us write $Z_n^d : \mathcal{M} \to \mathbb{R}$ one of the previous two models ($Z_n^{d,\mathrm{Dir}}$ or $Z_n^{\mathbb{S}^d}$) for which $\mathcal{M}$ is understood as the corresponding manifold of dimension $d$ ($B_d(0,1)$ or $\mathbb{S}^d$). Let us fix a sequence $(c_n)_{n\geq 1}$ of coefficients, then the following three statements are equivalent*

*i) the series $\sum_{n\geq 1} |c_n|^2 n^{d-1}$ is convergent,*

*ii) with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d$ uniformly converges on $\mathcal{M}$,*

*iii) with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d$ weakly converges in the following sense to a function $f^{G,\omega}$ which belongs to $L^\infty(\mathcal{M})$ :*

$$\forall \psi \in \mathcal{C}^\infty(\mathcal{M}) \qquad \int_{\mathcal{M}} \Big( \sum_{n=1}^{N} g_n(\omega) c_n Z_n^d(x) \Big) \psi(x) \mathrm{d}x \xrightarrow[N\to+\infty]{} \int_{\mathcal{M}} f^{G,\omega}(x)\psi(x)\mathrm{d}x.$$

Let us make a few comments on the previous result :

- Theorem 1 remains true by replacing the sequence $(g_n)$ with a sequence $(\varepsilon_n)$ of i.i.d. Rademacher random variables. Actually if ii) holds true then the famous contraction principle[6] shows that the Rademacher random series $\sum_{n\geq 1} \varepsilon_n(\omega) c_n Z_n^d$ almost surely uniformly converges on $\mathcal{M}$ and thus almost surely weakly converges in a similar sense to the assertion iii). And it turns out that the proof of iii) $\Rightarrow$ i) developed in Section 16 would be totally similar in the Rademacher case.

- Let us explain why the condition in i) is the best one that we could expect. For simplicity, let us call $P$ a point of concentration of each $Z_n^d$ (even if $P$ is the origin for the model $Z_n^d = Z_n^{d,\mathrm{Dir}}$). Due to such a concentration, we may expect that the behavior of the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d$ is merely relevant at $P$, namely if the Gaussian numerical random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d(P)$ is convergent. But, it is a basic fact that, for any complex sequence $(a_n)_{n\geq 1}$, the Gaussian random series $\sum g_n(\omega) a_n$ is almost surely convergent if and only if $\sum_{n\geq 1} |a_n|^2 < +\infty$. As a consequence, the implication ii) $\Rightarrow$ i) is obvious since we have[7] $Z_n^d(P) \simeq n^{\frac{d-1}{2}}$. The implication iii) $\Rightarrow$ i) is not either difficult (see Section 16) from a probabilistic point of view (though the proof needs a few basic facts of semi-classical analysis and heat kernel theory).

- However, the previous geometric intuition turns out to be misleading in dimension $d = 3$ from a deterministic point of view. Actually, a result of Hardy and Littlewood (see Appendix A) allows to construct a sequence of coefficients $(c_n)_{n\geq 1}$ fulfilling the following statements for the zonal eigenfunctions on $\mathbb{S}^3$ :

  a) we have $\sum_{n\geq 1} |c_n|^2 n^2 < +\infty$ (more generally, for any $s < \frac{3}{2}$ the series $\sum_{n\geq 1} |c_n|^2 n^{2s}$ converges[8]),

  b) the sequence $\sum c_n Z_n^{\mathbb{S}^3}(P)$ is convergent (for the point $P = (1,0,0,0) \in \mathbb{S}^3$),

  c) but the zonal function $\sum_{n\geq 1} c_n Z_n^{\mathbb{S}^3}$ is not continuous at $P$ (and hence the series $\sum_{n\geq 1} c_n Z_n^{\mathbb{S}^3}$ does not uniformly converge on $\mathbb{S}^3$).

  In other words, by considering random coefficients, Theorem 1 ensures that the Hardy-Littlewood case is completely exceptional and the geometric intuition about concentration around $P$ takes over.

[6] See for instance [LQ18b, page 137, Th IV.4], [LT91, page 99, Lem 4.5] or [MP81, page 45, Th 4.9]. We also refer to a result of Hoffman-Jorgensen as stated in [IRT16, Theorem 5.2].

[7] See (21) below.

[8] Although we merely need to reach $s = 1$ for our purpose, we note that condition c) and the Sobolev embedding (4) imply that we cannot set $s$ strictly larger than $\frac{3}{2}$.

- Let us now give a quite unexpected interpretation of Theorem 1 via Sobolev embeddings. Remembering that the sequence of zonal eigenfunctions $Z_n^{\mathbb{S}^d}$ are orthogonal in $L^2(\mathbb{S}^d)$, the assertion i) of Theorem 1 means that the function $\sum_{n\geq 1} c_n Z_n^{\mathbb{S}^d}$ belongs to the Sobolev space $H^{\frac{d-1}{2}}(\mathbb{S}^d)$. For the compact manifold $\mathbb{S}^d$, Theorem 1 states the equivalence of the following two assertions[9]

  i') with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^{\mathbb{S}^d}$ converges in $H^{\frac{d-1}{2}}(\mathbb{S}^d)$,

  ii) with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^{\mathbb{S}^d}$ uniformly converges on $\mathbb{S}^d$.

  Clearly, such an equivalence is a probabilistic improvement of the sharp Sobolev embedding

$$H^s(\mathbb{S}^d) \subset \mathcal{C}^0(\mathbb{S}^d), \qquad \forall s > \frac{d}{2}. \tag{4}$$

  Not only the regularity exponent $s$ can be probabilistically decreased to $\frac{d-1}{2}$ but we obtain an equivalence.

At this point of the introduction, we shortly explain a few ideas of the proof of Theorem 1. After a reduction via a canonical model (see Section 2), we are led to study the almost sure uniform convergence for $t \in [0, +\infty)$ of Gaussian random series of the form

$$\sum_{n\geq 1} g_n(\omega) \lambda_n^{\frac{d-1}{2}} c_n W(\lambda_n t) \qquad \text{with} \qquad W(t) = \frac{J_{\frac{d}{2}-1}(t)}{t^{\frac{d}{2}-1}}. \tag{5}$$

and in which $\lambda_n$ is a positive and increasing sequence of numbers satisfying a gap condition like the following one for a suitable positive constant $C \geq 1$ :

$$\forall n \in \mathbb{N}^\star \qquad \frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C. \tag{6}$$

For $d \geq 4$, the conclusion will essentially come by making an Abel transformation on (5) and exploiting that the function $W$ is of bounded variations on $[0, +\infty)$.

For $d = 3$ and $d = 2$, the function $W$ in (5) is no more of bounded variations and we must make other reasonings.

For $d = 3$, the term $W(t)$ equals $\sqrt{\frac{2}{\pi}} \frac{\sin(t)}{t}$ and the problem is thus reduced to study the almost sure diffentiability at $t = 0$ of the Gaussian random series

$$\sum_{n\geq 1} g_n(\omega) c_n \sin(\lambda_n t).$$

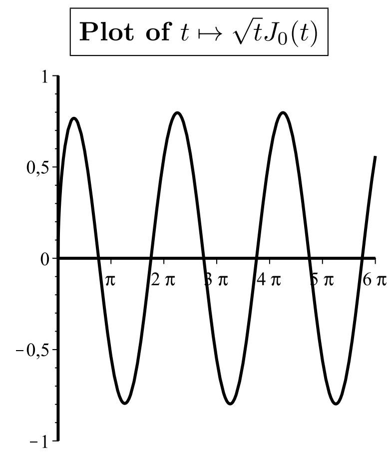
Among a few probabilistic arguments, we shall use an estimate of Paley and Wiener generalizing the Plancherel theorem (see (36)) and the fact that any function in the Sobolev space $H^1(\mathbb{R})$ is almost everywhere differentiable on $\mathbb{R}$.

For $d = 2$, the proof of Theorem 1 developed here is much more longer than those of the cases $d \geq 3$ and $d = 4$. Actually, even if one could simplify the strategy for $d = 2$, the machinery we develop is also useful to prove Theorem 2 below. Firstly, the Gaussian random series in (5) equals

$$\sum_{n\geq 1} g_n(\omega) \sqrt{\lambda_n} c_n J_0(\lambda_n t). \tag{7}$$

[9] One could also consider assertions with Rademacher random variables, see for instance [Kah68, page 30, Theorem 2].

The standard Bessel asymptotic $\sqrt{t}J_0(t) = \frac{\sqrt{2}}{\sqrt{\pi}}\cos\left(t-\frac{\pi}{4}\right)+\mathcal{O}\left(\frac{1}{t}\right)$ for $t \to +\infty$ suggests a behavior like a trigonometric function.




Then, a model of Gaussian series that should be understood before (7) is for instance

$$\sum_{n\geq 1} g_n(\omega)c_n\frac{\sin(\lambda_n t)}{\sqrt{t}} \qquad \text{with} \qquad \sum_{n\geq 1}|c_n|^2 n < +\infty. \tag{8}$$

In some sense, we will show that (7) is equal to a sum of a trigonometric model like (8) and a bounded variation perturbation like the case $d \geq 4$ (see more details at (44) and (48)). It remains to show that a Gaussian random series like (8) almost surely uniformly converges for $t \in [0, +\infty)$, or equivalently that the following Gaussian random series is pointwise $\frac{1}{2}$-Hölder at $t = 0$ :

$$\sum_{n\geq 1} g_n(\omega)c_n \sin(\lambda_n t). \tag{9}$$

To get such a pointwise Hölder regularity, we shall generalize in an optimal way a result of Kreit and Nicolay [KN18, Theorem 14] (see the precise statement Proposition 21 which is of self-interest). To use our Proposition 21, we will need two ingredients :

- an elementary probabilistic result (via the concentration of measure) giving a sufficient condition for the boundedness of a sequence of Gaussian processes (see Proposition 16).
- a proof of the almost sure continuity of (9). Here we stress that the sequence $(\lambda_n)$ is merely assumed to satisfy the gap estimate (6). In particular, despite we want to reach the uniform convergence of (8) on $[0, +\infty)$, it is hopeless to get the continuity of (9) via the uniform convergence on the whole half-closed interval $[0, +\infty)$ (we refer to (96) quoting the seminal work of Meyer [Mey73]). In our proof, we shall obtain almost sure bounds on $\left|\sum_{n\geq 1} g_n(\omega)c_n \sin(\lambda_n t)\right|$ on compact subsets of $[0, +\infty)$ via the Dudley theorem (see Proposition 27).

**Second contribution of the paper.** Actually, our probabilistic arguments may also be applied to another problem with a few additional arguments. In the sequel of the introduction, we shall denote by $\mathcal{M}$ a boundaryless compact Riemannian manifold of dimension $d \geq 2$. We denote by $(\phi_k)_{k\in\mathbb{N}}$ a Hilbert basis of $L^2(\mathcal{M})$ made of eigenfunctions of the Laplace-Beltrami operator $\boldsymbol{\Delta}$ :

$$\boldsymbol{\Delta}\phi_k = -\mu_k^2\phi_k, \quad 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \to +\infty. \tag{10}$$

Let us recall the notations of [Ime22, page 752]. For any $K > 0$ and any $n \in \mathbb{N}^\star$, we define the spectral subspace of $L^2(\mathcal{M})$ :

$$E_{(Kn-K,Kn]} = \mathrm{Span}\{\phi_k, \quad \mu_k \in (Kn-K, Kn]\}.$$

This spectral subspace is a good analogue of an eigenspace but on a general compact manifold[10]. We now define the multidimensional random series adapted to our framework. Let $U_n : \Omega \to \mathbb{R}^{\dim(E_{(Kn-K,Kn]})}$ be a uniform random vector on the usual Euclidean sphere of $\mathbb{R}^{\dim(E_{(Kn-K,Kn]})}$ and let $(g_{n,k})_{n,k}$ be a double sequence of i.i.d. Gaussian

[10] Such considerations are more or less unavoidable because it is known that, for generic compact Riemannian manifolds, the eigenspaces of the Laplace-Beltrami operator are of dimension 1 (see [Uhl76]).

random variables $\mathcal{N}(0,1)$, we define for any $f = \sum_{n\geq 1} f_n \in L^2(\mathcal{M})$ with $f_n \in E_{(Kn-K,Kn]}$ the following random series

$$f^\omega := \sum_{n\geq 1} f_n^\omega \qquad \text{with} \qquad f_n^\omega = \|f_n\|_{L^2(\mathcal{M})} \sum_{\mu_k \in (Kn-K,Kn]} U_{n,k}(\omega)\phi_k, \tag{11}$$

$$f^{G,\omega} := \sum_{n\geq 1} f_n^{G,\omega} \qquad \text{with} \qquad f_n^{G,\omega} = \frac{\|f_n\|_{L^2(\mathcal{M})}}{\sqrt{\dim(E_{(Kn-K,Kn]})}} \sum_{\mu_k \in (Kn-K,Kn]} g_{n,k}(\omega)\phi_k. \tag{12}$$

The random series $\sum f_n^\omega$ and $\sum f_n^{G,\omega}$ are multidimensional analogues of the Rademacher and Gaussian random series. Although not presented in the previous form, such random series implicitly appear in the papers [BL13, BL14] of Burq and Lebeau. In some sense, the previous random series translate the heuristic idea consisting in applying to the deterministic series $\sum f_n$ an infinite sequence of linear random perturbations that stabilize the sequence of spectral subspaces $(E_{(Kn-K,Kn]})_{n\geq 1}$. In particular, a consequence of the analysis of Burq and Lebeau is a manifold-version of the Paley-Zygmund theorem : with probability 1, the random functions $f^\omega$ and $f^{G,\omega}$ belong to $\bigcap_{p\in[2,+\infty)} L^p(\mathcal{M})$. In order to reach the case $p=+\infty$, we refer to [Ime19, Theorem 2.1] for a simple sufficient condition but the general necessary and sufficient condition on the sequence $(f_n)$ is given by [Ime22, Theorem 1]. In particular, for any $N\in\mathbb{N}^\star$, an equivalence[11] of Salem-Zygmund type holds true[12] :

$$\mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=1}^N f_n^\omega(x)\Big|\Big] \simeq \sum_{p=1}^N \frac{1}{p\sqrt{\ln(p+1)}}\Big(\sum_{n=p}^N \|f_n\|^2_{L^2(\mathcal{M})}\Big)^{\frac12}.$$

A simple consequence of that equivalence will be used at the end of our study : for any two integers $N_2 \geq N_1 \gg 1$, we get

$$\sqrt{\ln(N_1)}\Big(\sum_{n=N_1}^{N_2}\|f_n\|^2_{L^2(\mathcal{M})}\Big)^{\frac12} \lesssim \mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=N_1}^{N_2} f_n^\omega(x)\Big|\Big] \lesssim \sqrt{\ln(N_2)}\Big(\sum_{n=N_1}^{N_2}\|f_n\|^2_{L^2(\mathcal{M})}\Big)^{\frac12}. \tag{13}$$

Such an inequality is a sort of manifold version of a two-sided inequality known for the torus (see [LQ18a, page 259]) and the case $N_1 = N_2$ has been proved by Burq and Lebeau (see [BL13, page 930, Th 5]).

In order to state our second contribution, we need to set a last notation, for any $\alpha\in(0,1)$, we denote by $\mathcal{C}^{0,\alpha}(\mathcal{M})$ the Banach space of functions $f:\mathcal{M}\to\mathbb{C}$ that satisfy a Hölder condition of order $\alpha$ :

$$\sup_{x\neq y}\frac{|f(x)-f(y)|}{\delta_g(x,y)^\alpha} < +\infty, \tag{14}$$

in which $\delta_g$ stands for the Riemannian distance of $\mathcal{M}$.

**Theorem 2.** *There is $K_0>0$ depending only on the Riemannian structure of $\mathcal{M}$ such that for any $K\geq K_0$ and any sequence $(f_n)_{n\geq 1}$ satisfying $f_n\in E_{(Kn-K,Kn]}$, the following statements are equivalent :*

*i) there exists $C>0$ such that the following bounds hold for $j\gg 1$ :*

$$\sum_{n=2^j}^{2^{j+1}-1}\|f_n\|^2_{L^2(\mathcal{M})} \leq \frac{C}{j2^{2\alpha j}}, \tag{15}$$

*ii) the sum of the Gaussian random series $\sum_{n\geq1} f_n^{G,\omega}$ almost surely belongs to $\mathcal{C}^{0,\alpha}(\mathcal{M})$,*

*iii) the sum of the random series $\sum_{n\geq 1} f_n^\omega$ almost surely belongs to $\mathcal{C}^{0,\alpha}(\mathcal{M})$.*

Our proofs also allow to give a necessary and sufficient condition for the convergence in $\mathcal{C}^{0,\alpha}(\mathcal{M})$ of the partial sums of the random series $\sum_{n\geq1} f_n^{G,\omega}$ and $\sum_{n\geq1} f_n^\omega$ (see Remarks 30, 31 and 32).

The previous theorem should be compared to [Kah68, page 89, Theorem 3 and Proposition] that deals with sufficient or necessary conditions (not both) to get the Hölder regularity of random Fourier series on the torus. But the previous theorem has no satisfactory analogue for $d=1$. For instance, one may show the following two facts with lacunary series (see Appendix B) :

[11] with constants independent of $N\in\mathbb{N}^\star$.

[12] More precisely, the parameter $K>0$ is chosen large enough but fixed once and for all (see [Ime22, Theorem 1]).

- with probability 1, the Gaussian random series $\sum_{j\in\mathbb{N}} \frac{g_j(\omega)}{2^{\alpha j}} e^{i2^j x}$ does not belong to $\mathcal{C}^{0,\alpha}(\mathbb{T})$,

- with probability 1, the Rademacher random series $\sum_{j\in\mathbb{N}} \frac{\varepsilon_j(\omega)}{2^{\alpha j}} e^{i2^j x}$ belongs to $\mathcal{C}^{0,\alpha}(\mathbb{T})$.

Let us now comment on Theorem 2 about its interpretation for Sobolev embeddings (we refer to [BL13, BL14, Tzv09, Ime19] for more about probabilistic Sobolev embeddings). We recall the following Sobolev embedding for any $\alpha \in (0,1)$ :

$$H^{\alpha+\frac{d}{2}}(\mathcal{M}) \subset \mathcal{C}^{0,\alpha}(\mathcal{M}).$$

Such an embedding is well known for $\mathbb{R}^d$ instead of a boundaryless compact manifold $\mathcal{M}$ (see [AG07, page 82, Prop 1.4]) and can be proved for $\mathcal{M}$ by working on local charts. We now claim that Theorem 2 implies a probabilistic gain of almost $\frac{d}{2}$ derivatives in the following sense (see [Ime19, top of page 2734] for a similar gain for the spaces $L^\infty(\mathcal{M})$ and BMO$(\mathcal{M})$).

**Corollary 3.** *Let $K > 0$ and $(f_n)_{n\geq 1}$ be as in Theorem 2 and we consider $f_n^\omega$ in (11). For any $\alpha \in (0,1)$ and any $\eta > 0$, if $\sum_{n\geq 1} f_n$ belongs to the Sobolev space $H^{\alpha+\eta}(\mathcal{M})$, then the random function $\sum_{n\geq 1} f_n^\omega$ almost surely belongs to $\mathcal{C}^{0,\alpha}(\mathcal{M})$.*

Proof. Actually, we get a much stronger inequality than (15) :

$$\sum_{n=2^j}^{2^{j+1}-1} \|f_n\|_{L^2(\mathcal{M})}^2 \leq \frac{1}{2^{2j(\alpha+\eta)}} \sum_{n=2^j}^{2^{j+1}-1} n^{2(\alpha+\eta)} \|f_n\|_{L^2(\mathcal{M})}^2 \leq \frac{C}{2^{2j(\alpha+\eta)}} \Big\| \sum_{n\geq 1} f_n \Big\|_{H^{\alpha+\eta}(\mathcal{M})}^2 .$$

□

**Organization of the paper.**

In Section 2, we explain how the sense i) ⇒ ii) of Theorem 1 is a consequence of the analysis of a canonical model of zonal/radial eigenfuntions. For this canonical model, we state Theorem 4. The difficult part of Theorem 4 is also its implication i) ⇒ ii) and will be developed in the next sections.

In Section 3, we present a few elementary results about Abel summations for Gaussian random series (the main probabilistic ingredient is indeed the use of the Lévy's inequalities).

In Section 4, we explain how the case $d = 4$ is an immediate consequence of Section 3 and of well-known asymptotics of Bessel functions.

Section 5 is devoted to the proof of the case $d = 3$.

In Section 6, we state and prove a result in the spirit of the "closed graph theorem" in order to get quantitative bounds for Gaussian processes. This part is elementary upon using integrability properties of Gaussian processes. The idea is simply that we may get boundedness from continuity in some special cases.

Section 7 shows how the case $d = 2$ can be reduced to a probabilistic result about pointwise Hölder regularity of aperiodic trigonometric sums (namely Proposition 12).

Section 8 studies a consequence of the concentration measure phenomenon in order to obtain the almost sure boundedness of a family of Gaussian processes (such a result will be used for the proof of Proposition 12 and Theorem 2).

Section 9 contains the statements of two results about Hölder regularities in the language of the Littlewood-Paley theory :

- the first one is Proposition 21 and gives a necessary and sufficient condition to get the pointwise Hölder regularity on $\mathbb{R}^d$ (this is an improvement of a sufficient condition previously proved by Kreit and Nicolay). The proof is developed in Sections 10, 11 and 12;

- the second one is Proposition 22, which is more or less known, and gives a necessary and sufficient condition to get the global Hölder regularity on a boundaryless Riemannian compact manifold and moreover shows that the semi-classical multipliers of the Laplace-Beltrami operator behave well with respect to the global Hölder norm (these results are proved in Sections 13 and 14).

Section 15 contains, as announced in Section 7, the final argument of Theorem 4 (in particular, we prove Proposition 12). In other words, as explained in Section 2, we finally get the sense i) ⇒ ii) of Theorem 1.

Section 16 is devoted to show the sense iii) ⇒ i) of Theorem 1. Since the sense ii) ⇒ iii) of Theorem 1 is immediate, the proof of our first contribution will be completely finished.

In Sections 17, 18, 19 and 20, we prove our second contribution stated in Theorem 2, namely the Hölder version of the Paley-Zygmund theorem on compact manifolds.

Appendix A explains why a result of Hardy and Littlewood implies, for $d = 3$, that there is no deterministic analogue of Theorem 1.

Finally, Appendix B contains a few complements of classical random trigonometric sums with respect to the global Hölder norm on the unidimensional torus $\mathbb{T}$.

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# 2 The canonical model of zonal eigenfunctions

Let us consider the usual Laplace operator $\Delta$ on $\mathbb{R}^d$ for $d \geq 2$. For any $\xi \in \mathbb{R}^2$ whose Euclidean norm $|\xi|$ equals 1, we have $-\Delta e^{i\langle \xi, \cdot \rangle} = e^{i\langle \xi, \cdot \rangle}$. If one looks for an eigenfunction which is invariant under the action of all isometries around the origin, it is thus natural to consider the following function averaged over the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ :

$$x \in \mathbb{R}^d \mapsto \int_{\mathbb{S}^{d-1}} e^{i\langle \xi, x \rangle} \mathrm{d}\sigma_{d-1}(\xi) = (2\pi)^{\frac{d}{2}} \times \frac{J_{\frac{d}{2}-1}(|x|)}{|x|^{\frac{d}{2}-1}} \tag{16}$$

in which $\sigma_{d-1}$ is the surface measure of $\mathbb{S}^{d-1}$ and $J_{\frac{d}{2}-1}$ is the usual Bessel function.

Here is our canonical model : we fix a sequence of positive numbers $(\lambda_n)_{n\geq 1}$ satisfying for some constant $C \geq 1$ and any $n \geq 1$ the following estimates

$$\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C. \tag{17}$$

and we define for any $x \in \mathbb{R}^d$

$$Z_n^{d,\mathrm{can}}(x) = \sqrt{\lambda_n} \times \frac{J_{\frac{d}{2}-1}(\lambda_n |x|)}{|x|^{\frac{d}{2}-1}} = \lambda_n^{\frac{d-1}{2}} \frac{J_{\frac{d}{2}-1}(\lambda_n |x|)}{|\lambda_n x|^{\frac{d}{2}-1}}. \tag{18}$$

As announced in the introduction, the Dirichlet model $Z_n^{d,\mathrm{Dir}}$ is of that form[13] once we restrict the analysis on the unit ball of $\mathbb{R}^d$. Actually, the gap assumption (17) is a consequence of the McMahon's asymptotic expansions for large zeros of the Bessel functions (see [OLBC10, page 236]).

In order to motivate the factor $\lambda_n^{\frac{d-1}{2}}$ in (18), let us recall the following property[14] of the Bessel functions :

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T t J_{\frac{d}{2}-1}(t)^2 \mathrm{d}t = \frac{1}{\pi}$$

from which we immediately get, by a compactness argument, that there exists a constant $C_d > 1$ satisfying

$$T \geq 1 \qquad \Rightarrow \qquad \frac{\sqrt{T}}{C_d} \leq \sqrt{\int_0^T t J_{\frac{d}{2}-1}(t)^2 \mathrm{d}t} \leq C_d \sqrt{T}.$$

Let $(K_n)$ be a sequence of positive numbers and set $T = K_n \lambda_n$ in the foregoing facts. The factors $\sqrt{\lambda_n}$ and $\lambda_n^{\frac{d-1}{2}}$ are made to ensure the following normalization conditions (coming from a simple change of variables) :

$$\lim_{K_n \lambda_n \to +\infty} \frac{\|Z_n^{d,\mathrm{can}}\|_{L^2(B_d(0,K_n))}}{\sqrt{K_n}} = \frac{\sqrt{\mathrm{Vol}(\mathbb{S}^{d-1})}}{\sqrt{\pi}}, \tag{19}$$

$$K_n \lambda_n \geq 1 \quad \Rightarrow \quad \frac{\sqrt{K_n}}{C_d} \leq \|Z_n^{d,\mathrm{can}}\|_{L^2(B_d(0,K_n))} \leq C_d \sqrt{K_n}. \tag{20}$$

[13] Actually, if one sets $\|Z_n^{d,\mathrm{Dir}}\|_{L^2(B_d(0,1))} = 1$ then we have $Z_n^{d,\mathrm{Dir}} = \beta_{n,d} Z_n^{d,\mathrm{can}}$ with $\lim_{n \to +\infty} \beta_{n,d} = \frac{\sqrt{\pi}}{\sqrt{\mathrm{Vol}(\mathbb{S}^{d-1})}}$ thanks to (19) with the choice $K_n = 1$ and $\lambda_n \to +\infty$.

[14] Such a limit can be seen as a consequence of the asymptotics $J_\nu(t) = \frac{\sqrt{2}}{\sqrt{\pi t}} \Big( \cos\big(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\big) + O\big(\frac{1}{t}\big) \Big)$ for $t \to +\infty$.

Actually, the bounds (20) are essentially included in the proof of [AT08, Lemma 2.4]. By homogeneity and by using (16), we have

$$-\Delta Z_n^{d,\mathrm{can}} = \lambda_n^2 Z_n^{d,\mathrm{can}}$$

Note that we also have

$$Z_n^{d,\mathrm{can}}(0) = \frac{\lambda_n^{\frac{d-1}{2}}}{\Gamma(\frac{d}{2})2^{\frac{d}{2}-1}} \tag{21}$$

and that (16) implies

$$\|Z_n^{d,\mathrm{can}}\|_{L^\infty(\mathbb{R}^d)} \leq Z_n^{d,\mathrm{can}}(0).$$

The classical bound $J_{\frac{d}{2}-1}(t) = \mathcal{O}\Big(\frac{1}{\sqrt{t}}\Big)$ for $t \to +\infty$ shows that $Z_n^{d,\mathrm{can}}$ concentrates[15] on the ball $B\Big(0, \frac{1}{\lambda_n}\Big)$ in the following sense :

$$|x| \geq \frac{1}{\lambda_n} \quad \Rightarrow \quad |Z_n^{d,\mathrm{can}}(x)| \leq \frac{C}{|x|^{\frac{d-1}{2}}}. \tag{22}$$

In the sequel of the paper, the following result will be proved.

**Theorem 4.** *For any integer $d \geq 2$ and any sequence of coefficients $(c_n)_{n\geq 1}$, the following assertions are equivalent*

*i) the series $\sum_{n\geq 1} |c_n|^2 n^{d-1}$ is convergent,*

*ii) with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega)c_n Z_n^{d,\mathrm{can}}$ uniformly converges on $\mathbb{R}^d$,*

*iii) with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega)c_n Z_n^{d,\mathrm{can}}$ uniformly converges on a neighborhood of $0 \in \mathbb{R}^d$.*

Note that the implication ii) ⇒ iii) is obvious. And the implication iii) ⇒ i) is easy thanks to (21) and the asymptotic $\lambda_n \simeq n$. So it remains to give a proof of the implication i) ⇒ ii).

Also note that Theorem 4 implies the implication i) ⇒ ii) of Theorem 1 for the radial eigenfunctions $Z_n^{d,\mathrm{Dir}}$ on the unit ball of $\mathbb{R}^d$. We can also deduce the conclusion for the zonal eigenfunctions $Z_n^{\mathbb{S}^d}$ of the spheres, but we need to combine Theorem 4 with a result of Frenzen and Wong, namely the "Hilb's type asymptotic" (24), giving the asymptotic of Jacobi polynomials involving Bessel functions and thus comforting the universality of the canonical model (18).

**Corollary 5.** *For any integer $d \geq 2$ and any sequence of coefficients $(c_n)_{n\geq 1}$, the following two statements are equivalent*

*i) the series $\sum_{n\geq 1} |c_n|^2 n^{d-1}$ is convergent,*

*ii) with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega)c_n Z_n^{\mathbb{S}^d}$ uniformly converges on $\mathbb{S}^d$.*

Proof. The implication ii) ⇒ i) is obvious since $Z_n^{\mathbb{S}^d}(1,0,\ldots,0) \simeq n^{\frac{d-1}{2}}$. Let us show the converse implication i) ⇒ ii). Due to the formula (2), we have to prove that the following Gaussian random series uniformly converges for $t \in [0,\pi]$ :

$$\sum_{n\geq 1} g_n(\omega)c_n c'_{d,n} P_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(\cos(t)) \tag{23}$$

in which we recall the asymptotic $c'_{d,n} \simeq \sqrt{n}$.

**Step 1.** We claim that there is no loss to reduce the analysis to $[0, \frac{\pi}{2}]$. Indeed, for any $t \in [\frac{\pi}{2}, \pi]$ we may write

$$P_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(\cos(t)) = P_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(-\cos(\pi - t)) = (-1)^n P_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(\cos(\pi - t)).$$

Since $((-1)^n g_n)_{n\geq 1}$ is a sequence of i.i.d. Gaussian random variables $\mathcal{N}(0,1)$, the almost sure uniform convergence on $[0, \frac{\pi}{2}]$ implies the almost sure uniform convergence on $[\frac{\pi}{2}, \pi]$.

[15] See [Ime18, line (93)] for similar estimates for zonal eigenfunctions on the sphere $\mathbb{S}^d$.

**Step 2.** We choose $\lambda_n = n + \frac{d-1}{2}$ for the canonical model (hence (17) is true). Thanks to [FW85, pages 980 and 994 with $m = 1$ and $A_0 = 1$], the following asymptotic is uniform on $[0, \frac{\pi}{2}]$ for $n \to +\infty$ :

$$\begin{aligned} P_n^{(\frac{d}{2}-1,\frac{d}{2}-1)}(\cos(t)) &= \frac{\Gamma\Big(n+\frac{d}{2}\Big)}{n!}\frac{2^{\frac{d}{2}-1}}{\sin^{\frac{d}{2}-1}(t)}\sqrt{\frac{t}{\sin(t)}}\left[\frac{J_{\frac{d}{2}-1}(\lambda_n t)}{\lambda_n^{\frac{d}{2}-1}} + t^{\frac{d}{2}-1}\mathcal{O}\Big(\frac{1}{\lambda_n}\Big)\right] \\ &= 2^{\frac{d}{2}-1}\left(\frac{t}{\sin(t)}\right)^{\frac{d-1}{2}}\frac{\Gamma\Big(n+\frac{d}{2}\Big)}{n!}\frac{J_{\frac{d}{2}-1}(\lambda_n t)}{(\lambda_n t)^{\frac{d}{2}-1}} + \mathcal{O}\Big(n^{\frac{d}{2}-2}\Big). \end{aligned} \tag{24}$$

Let us plug this asymptotic in (23). Remembering $c'_{d,n} \simeq \sqrt{n}$, one sees that the contribution of the remainder is a Gaussian series of the form $\sum g_n(\omega)c_n n^{\frac{d-3}{2}}u_n(t)$ with $|u_n(t)| \leq C$. But the series $\sum c_n n^{\frac{d-3}{2}}$ is absolutely convergent thanks to i) and the Cauchy-Schwarz inequality. As a consequence, the Gaussian random series $\sum g_n(\omega)c_n n^{\frac{d-3}{2}}u_n(t)$ converges in $L^1(\Omega, \mathcal{C}^0([0,\pi/2]))$ and thus[16] almost surely uniformly converges on $[0, \frac{\pi}{2}]$.

**Step 3.** Let us now see the contribution of the principal part of (24). Theorem 4 and (18) ensure the almost sure uniform convergence of $\sum g_n(\omega)c_n\lambda_n^{\frac{d-1}{2}}\frac{J_{\frac{d}{2}-1}(\lambda_n t)}{(\lambda_n t)^{\frac{d}{2}-1}}$. The contraction principle on random series (see the proof of [LT91, page 98, Th 4.4]) and the equivalence

$$c'_{d,n}\frac{\Gamma\big(n+\frac{d}{2}\big)}{n!} \simeq \sqrt{n}n^{\frac{d}{2}-1} \simeq \lambda_n^{\frac{d-1}{2}}$$

ensure the almost sure uniform convergence of $\sum g_n(\omega)c_n c'_{d,n}\frac{\Gamma\big(n+\frac{d}{2}\big)}{n!}\frac{J_{\frac{d}{2}-1}(\lambda_n t)}{(\lambda_n t)^{\frac{d}{2}-1}}$. Since $t \in [0, \frac{\pi}{2}] \mapsto \left(\frac{t}{\sin(t)}\right)^{\frac{d-1}{2}}$ is a continuous and bounded function, we easily conclude that (24) implies the almost sure uniform convergence of (23). □

# 3 Gaussian random series and bounded variation assumptions

We recall that a function $W : [0, +\infty) \to \mathbb{R}$ is of bounded variation, and we write $W \in BV$, if the following condition holds true

$$\|W\|_{BV} := \sup_{N\in\mathbb{N}^\star}\sup_{0\leq x_0<\cdots<x_N}\sum_{n=1}^{N}|W(x_n) - W(x_{n-1})| < +\infty.$$

In particular, if $W : [0, +\infty) \to \mathbb{R}$ is continuously differentiable and satisfies $W' \in L^1(0, +\infty)$ then $W \in BV$ and

$$\|W\|_{BV} = \int_0^{+\infty}|W'(x)|\mathrm{d}x. \tag{25}$$

In order to motivate the elementary tools we shall use, let us write a very simple fact :

**Fact.** If a series of real coefficients $\sum a_n$ is convergent then, for any $W \in BV$, the sequence of functions $\sum a_n W(\lambda_n t)$ is uniformly convergent for $t \in [0, +\infty)$.

**Proof of the fact.** Note that $W$ is bounded on $[0, +\infty)$ by $\|W\|_{BV} + |W(0)|$. We then write the deterministic remainder $R_n = \sum\limits_{k\geq n} a_k$ and make an Abel summation for any positive integers $q > p$ :

$$\begin{aligned} \sum_{n=p}^{q} a_n W(\lambda_n t) &= \sum_{n=p}^{q}(R_n - R_{n+1})W(\lambda_n t) \\ &= R_p W(\lambda_p t) - R_{q+1}W(\lambda_q t) + \sum_{n=p+1}^{q} R_n(W(\lambda_n t) - W(\lambda_{n-1}t)) \\ \sup_{t\geq 0}\Big|\sum_{n=p}^{q} a_n W(\lambda_n t)\Big| &\leq \max_{n\geq p}|R_n| \times \big(2|W(0)| + 3\|W\|_{BV}\big). \end{aligned}$$

We achieve the proof of the fact thanks to the Cauchy convergence test.

The previous proof directly gives Point i) of the following probabilistic version.

[16] This implication will be extensively used in the present paper, see for instance the proof of Proposition 7.

**Lemma 6.** *Let us consider a function $W \in BV$, a sequence of real numbers $(a_n)_{n\geq 1} \in \ell^2(\mathbb{N}^\star)$. Then the following statements hold true :*

*i) with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) a_n W(\lambda_n t)$ uniformly converges for $t \in [0, +\infty)$,*

*ii) moreover, the following inequality holds*[17] *true :*

$$\mathbf{E}\Big[\sup_{t\geq 0}\Big|\sum_{n\geq 1} g_n(\omega) a_n W(\lambda_n t)\Big|\Big] \leq 3(\|W\|_{BV} + |W(0)|) \times \Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2}.$$

We see the previous lemma as an application of the following result for $u_n = a_n$ and $I = [0, +\infty)$. We state here a quite general result since Proposition 7 will also be used for the function $W = \mathbf{1}_{(1,+\infty)}$ of bounded variation (see the end of Section 7).

**Proposition 7.** *Let us consider a function $W : [0, +\infty) \to \mathbb{R}$ belonging to $BV$ and a sequence of bounded functions $(u_n)_{n\geq 1}$ on an interval $I \subset [0, +\infty)$. Then the following statements hold true*

*i) If the Gaussian random series $\sum_{n\geq 1} g_n(\omega) u_n(t)$ uniformly converges for $t \in I$ (with probability 1) then so does the Gaussian random series $\sum_{n\geq 1} g_n(\omega) u_n(t) W(\lambda_n t)$ (with probability 1).*

*ii) Moreover, the following inequality holds true :*

$$\mathbf{E}\Big[\sup_{t\in I}\Big|\sum_{n\geq 1} g_n(\omega) u_n(t) W(\lambda_n t)\Big|\Big] \leq 3(\|W\|_{BV} + |W(0)|) \times \mathbf{E}\Big[\sup_{t\in I}\Big|\sum_{n\geq 1} g_n(\omega) u_n(t)\Big|\Big]. \tag{26}$$

PROOF. **Step 1.** Let us explain why the result is a consequence of the following estimates for any $N \in \mathbb{N}^\star$ :

$$\mathbf{E}\Big[\sup_{t\in I}\Big|\sum_{n=1}^{N} g_n(\omega) u_n(t) W(\lambda_n t)\Big|\Big] \leq 3(\|W\|_{BV} + |W(0)|) \times \mathbf{E}\Big[\sup_{t\in I}\Big|\sum_{n=1}^{N} g_n(\omega) u_n(t)\Big|\Big]. \tag{27}$$

Let $B(I)$ be the Banach space of the bounded functions on the interval $I$. For point i), it is well-known that the almost sure convergence of a Gaussian random series in a Banach space, for instance $B(I)$, is equivalent to the convergence in $L^1(\Omega, B(I))$ (see [MP81, page 44, Th 4.7]). Combining this equivalence to (27), we see that $\sum_{n=1}^{N} g_n(\omega) u_n(t) W(\lambda_n t)$ is a Cauchy sequence in $L^1(\Omega, B(I))$ and thus almost surely converges in $B(I)$. Point i) is proved. We thus are allowed to make tend $N$ to $+\infty$ in (27) so that we get the inequality (26) of Point ii).

**Step 2.** Let us now turn to the proof of (27). For any $n \geq 1$, we define the random remainder $R_n = \sum_{k=n}^{N} g_k(\omega) u_k(t)$ (with the convention $R_{N+1} = 0$) and we use an Abel summation :

$$\begin{aligned}
\sum_{n=1}^{N} g_n(\omega) u_n(t) W(\lambda_n t) &= \sum_{n=1}^{N} (R_n - R_{n+1}) W(\lambda_n t) \\
&= R_1 W(\lambda_1 t) + \sum_{n=2}^{N} R_n (W(\lambda_n t) - W(\lambda_{n-1} t)) \\
\mathbf{E}\Big[\Big\|\sum_{n=1}^{N} g_n(\omega) u_n(t) W(\lambda_n t)\Big\|_{B(I)}\Big] &\leq \mathbf{E}[\|R_1\|_{B(I)}] \times \sup_{t\geq 0} |W(t)| + \mathbf{E}\Big[\max_{2\leq n\leq N} \|R_n\|_{B(I)}\Big] \times \|W\|_{BV}.
\end{aligned}$$

Thanks to the Lévy's inequality (see [LT91, page 42, Proposition 2.3]), we have

$$\mathbf{E}\Big[\max_{2\leq n\leq N} \|R_n\|_{B(I)}\Big] \leq \mathbf{E}\Big[\max_{1\leq n\leq N} \|R_n\|_{B(I)}\Big] \leq 2\mathbf{E}[\|R_1\|_{B(I)}].$$

The bound $\sup_{t\geq 0} |W(t)| \leq \|W\|_{BV} + |W(0)|$ allows us to get (27) as follows

$$\mathbf{E}\Big[\Big\|\sum_{n=1}^{N} g_n(\omega) u_n(t) W(\lambda_n t)\Big\|_{B(I)}\Big] \leq 3(\|W\|_{BV} + |W(0)|) \times \mathbf{E}[\|R_1\|_{B(I)}].$$

□

[17] We refer to [LT91, top of page 33] for the justification of the measurability of the supremum over the possibly uncountable set $I$.

We give here a variant that we will use in Section 15.

**Proposition 8.** *Let $B$ be a Banach space and consider a sequence $(u_n)_{n\geq 1}$ in $B$ such that the Gaussian random series $\sum_{n\geq 1} g_n(\omega)u_n$ almost surely converges in $B$. Let us moreover consider an arbitrary countable set $\mathcal{E}$ and a double sequence $(\alpha_{k,n})$ with $(k,n)\in\mathcal{E}\times\mathbb{N}^\star$ satisfying for a positive constant $M$ the following estimate*

$$\sup_{k\in\mathcal{E}}\Big(|\alpha_{k,1}|+\sum_{n\geq 1}|\alpha_{k,n+1}-\alpha_{k,n}|\Big)\leq M. \tag{28}$$

*Then, for any $k\in\mathcal{E}$, the Gaussian random series $\sum_{n\geq 1}\alpha_{k,n}g_n(\omega)u_n$ almost surely converges in $B$ and the following inequality holds true for any $T>0$*

$$\mathbf{P}\Big(\sup_{k\in\mathcal{E}}\Big\|\sum_{n\geq 1}\alpha_{k,n}g_n(\omega)u_n\Big\|_B>MT\Big)\leq 2\mathbf{P}\Big(\Big\|\sum_{n\geq 1}g_n(\omega)u_n\Big\|_B\geq T\Big). \tag{29}$$

*Proof.* The first statement about the Gaussian random series $\sum_{n\geq 1}\alpha_{k,n}g_n(\omega)u_n$ comes from the qualitative contraction principle (see for instance [LQ18b, page 133, Th IV.1]) and the boundedness of each sequence $(\alpha_{k,n})_{n\in\mathbb{N}^\star}$ :

$$\sup_{n\geq 1}|\alpha_{k,n}|\leq M.$$

In order to prove the inequality (29), we set $R_{n,N}=g_n(\omega)u_n+\cdots+g_N(\omega)u_N$ for any $N\in\mathbb{N}^\star$ and we repeat the Abel summation argument of Proposition 7 :

$$\begin{aligned}\Big\|\sum_{n=1}^{N}\alpha_{k,n}g_n(\omega)u_n\Big\|_B &= \Big\|\alpha_{k,1}R_{1,N}+\sum_{n=2}^{N}(\alpha_{k,n}-\alpha_{k,n-1})R_{n,N}\Big\|_B\\ &\leq M\max_{1\leq n\leq N}\|R_{n,N}\|_B.\end{aligned}$$

The end of the proof is completely standard via the Lévy's inequality but we give details for the sake of completeness. Making $N$ tend to $+\infty$ and then taking the upper bound over $k$, we get

$$\sup_{k\in\mathcal{E}}\Big\|\sum_{n=1}^{+\infty}\alpha_{k,n}g_n(\omega)u_n\Big\|_B\leq M\sup_{N\in\mathbb{N}^\star}\max_{1\leq n\leq N}\|R_{n,N}\|_B.$$

Then we obtain

$$\mathbf{P}\Big(\sup_{k\in\mathcal{E}}\Big\|\sum_{n\geq 1}\alpha_{k,n}g_n(\omega)u_n\Big\|_B>MT\Big)\leq\mathbf{P}\Big(\sup_{N\in\mathbb{N}^\star}\max_{1\leq n\leq N}\|R_{n,N}\|_B>T\Big).$$

We now rewrite the last upper bound as follows

$$\mathbf{P}\Big(\bigcup_{N\in\mathbb{N}^\star}\{\max_{1\leq n\leq N}\|R_{n,N}\|_B>T\}\Big)=\lim_{N\to+\infty}\mathbf{P}\Big(\max_{1\leq n\leq N}\|R_{n,N}\|_B>T\Big)$$

and the Lévy's inequality (see [LT91, page 42, Proposition 2.3]) shows that the last limit is less than or equal to

$$2\,\overline{\lim_{N\to+\infty}}\,\mathbf{P}\Big(\|g_1(\omega)u_1+\cdots+g_N(\omega)u_N\|_B>T\Big).$$

Fatou's lemma allows to bound the last term by

$$2\mathbf{P}\Big(\overline{\lim_{N\to+\infty}}\{\|g_1(\omega)u_1+\cdots+g_N(\omega)u_N\|_B>T\}\Big)\quad=\quad 2\mathbf{P}\Big(\bigcap_{n\in\mathbb{N}^\star}\bigcup_{N\geq n}\{\|g_1(\omega)u_1+\cdots+g_N(\omega)u_N\|_B>T\}\Big).$$

Since we have assumed that the Gaussian random series $\sum g_n(\omega)u_n$ almost surely converges in $B$, the last term is less than or equal to

$$2\mathbf{P}\Big(\Big\|\sum_{n\geq 1}g_n(\omega)u_n\Big\|_B\geq T\Big).$$

□

# 4 Proof of Theorem 4 for $d \geq 4$

Let $(\lambda_n)_{n\geq 1}$ be a positive sequence satisfying the gap estimates (17). For any $d \geq 4$, let $(Z_n^{d,\mathrm{can}})_{n\geq 1}$ be the canonical model of eigenfunctions of $\Delta$ on $\mathbb{R}^d$ introduced in Section 2. As already explained in Section 2, we merely have to prove the implication i) $\Rightarrow$ ii) of Theorem 4.

We set $a_n = c_n \lambda_n^{\frac{d-1}{2}}$ (which belongs to $\ell^2(\mathbb{N}^\star)$). Due to the definition (18) of $Z_n^{d,\mathrm{can}}$, we have to study the following Gaussian random series for $t \in [0, +\infty)$ :

$$\sum_{n\geq 1} g_n(\omega) a_n \frac{J_{\frac{d}{2}-1}(\lambda_n t)}{(\lambda_n t)^{\frac{d}{2}-1}}.$$

It is sufficient to apply Lemma 6 with $W(t) = t^{-\left(\frac{d}{2}-1\right)} J_{\frac{d}{2}-1}(t)$. In other words, we have to check that $W$ is of bounded variation on $[0, +\infty)$. We use (25) and the following two points :

• First of all, the origin is not a singularity of $W$. Indeed, $W$ is smooth on $[0, +\infty)$ since the following power series expansion holds :

$$W(t) = \sum_{p=0}^{+\infty} \frac{(-1)^p}{p!\Gamma(p+\frac{d}{2})} \left(\frac{t}{2}\right)^{2p} \frac{1}{2^{\frac{d}{2}-1}}.$$

• Let us now look at the behavior for $t \to +\infty$. We recall the known bounds $J_{\frac{d}{2}-1}(t) = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$ and $J'_{\frac{d}{2}-1}(t) = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$ (see [OLBC10, pages 228-229]). We then get the asymptotic $W'(t) = \mathcal{O}\left(\frac{1}{t^{\frac{d-1}{2}}}\right)$ for $t \to +\infty$ that shows that $W'$ is integrable for $d \geq 4$.

# 5 Proof of Theorem 4 for $d = 3$

We recall the identity $J_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}} \sin(t)$ and hence (18) gives

$$Z_n^{3,\mathrm{can}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lambda_n |x|)}{|x|}, \qquad \forall x \in \mathbb{R}^3.$$

As above, we merely have to prove the implication i) $\Rightarrow$ ii) of Theorem 4. We set $a_n = c_n \lambda_n$ for simplicity and hence $(a_n)_{n\geq 1} \in \ell^2(\mathbb{N}^\star)$. The core of the proof is the following proposition :

**Proposition 9.** *Let $(\lambda_n)_{n\geq 1}$ be a positive sequence satisfying the gap estimates* (17) *and let $(a_n)_{n\geq 1}$ be a sequence belonging to $\ell^2(\mathbb{N}^\star)$, then the following Gaussian Fourier series*

$$\begin{array}{rcl} f^\omega : \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \sum_{n\geq 1} g_n(\omega) \frac{a_n}{\lambda_n} \sin(\lambda_n t) \end{array} \tag{30}$$

*is almost surely continuous on $\mathbb{R}$ and differentiable at $t = 0$.*

Before proving the last result, let us explain its consequence for the proof of Theorem 4 (similarly to the case $d \geq 4$, we note that the uniform convergence holds on $[0, +\infty)$).

**Corollary 10.** *With the same assumptions as those of Proposition 9, with probability* 1*, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) a_n \frac{\sin(\lambda_n t)}{\lambda_n t}$ uniformly converges on $\mathbb{R}$.*

Proof. By parity, we merely have to understand the uniform convergence on $[0, +\infty)$. We note that the random function $t \mapsto \sum_{n\geq 1} g_n(\omega) a_n \frac{\sin(\lambda_n t)}{\lambda_n t}$ is almost surely continuous on $\mathbb{R}$ thanks to Proposition 9 (the continuity at $t = 0$ comes from the differentiability conclusion in Proposition 9). We then deduce the almost sure uniform convergence on $[0, 1]$ thanks to the Itô-Nisio theorem (see [LQ18a, page 238, Th II.2]). Let us now deal with $[1, +\infty)$ and let $\mathcal{BC}([1, +\infty))$ be the Banach space of the bounded and continuous functions on $[1, +\infty)$. We have $\left|\frac{\sin(\lambda_n t)}{\lambda_n t}\right| \leq \frac{1}{\lambda_n}$ for $t \geq 1$. Since the series $\sum \frac{a_n}{\lambda_n}$ is absolutely convergent, we deduce that the Gaussian random series $\sum_{n\geq 1} g_n(\omega) a_n \frac{\sin(\lambda_n t)}{\lambda_n t}$ converges in $L^1(\Omega, \mathcal{BC}([1, +\infty)))$. As recalled in the proof of Proposition 7, one deduces the almost sure uniform convergence on $[1, +\infty)$. $\square$

Let us now prove Proposition 9.

The series $\sum \frac{a_n}{\lambda_n}$ is absolutely convergent, thus the Gaussian series in (30) converges in $L^1(\Omega, \mathcal{BC}(\mathbb{R}))$ and so almost surely converges in $\mathcal{BC}(\mathbb{R})$. Hence, the function $f^\omega$, defined in (30), is continuous with probability 1. Let us now prove the differentiability at $t=0$. Let us denote $(\widetilde{g}_n)_{n\geq 1}$ a sequence of i.i.d. Gaussian variables $\mathcal{N}(0,1)$ which are moreover independent of the sequence $(g_n)_{n\geq 1}$ and set the following Gaussian function

$$\begin{array}{rcl} \widetilde{f}^\omega : \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \sum_{n\geq 1} \widetilde{g}_n(\omega) \frac{a_n}{\lambda_n} \cos(\lambda_n t). \end{array}$$

Since $f^\omega = \frac{1}{2}(f^\omega + \widetilde{f}^\omega) + \frac{1}{2}(f^\omega - \widetilde{f}^\omega)$, it is sufficient to study the differentiability of each $f^\omega \pm \widetilde{f}^\omega$. The main point is that the two Gaussian functions $f^\omega \pm \widetilde{f}^\omega$ are actually stationary Gaussian processes on $\mathbb{R}$ : the covariance function on $\mathbb{R}\times\mathbb{R}$ is clearly invariant by the translations $(t,t') \mapsto (t+\varpi, t'+\varpi)$, whatever is $\varpi \in \mathbb{R}$, as shown by the simple computation for each $\varrho = \pm 1$

$$\begin{aligned} \mathbf{E}[(f^\omega + \varrho\widetilde{f}^\omega)(t)(f^\omega + \varrho\widetilde{f}^\omega)(t')] &= \sum_{n\geq 1} \frac{a_n^2}{\lambda_n^2} \left(\sin(\lambda_n t)\sin(\lambda_n t') + \varrho^2 \cos(\lambda_n t)\cos(\lambda_n t')\right) \\ &= \sum_{n\geq 1} \frac{a_n^2}{\lambda_n^2} \cos\Big(\lambda_n (t-t')\Big). \end{aligned}$$

For simplicity, we now use the following notations

$$F^\omega(t) := f^\omega(t) + \widetilde{f}^\omega(t) = \sum_{n=1}^{+\infty} F_n^\omega(t) \qquad \text{with} \qquad F_n^\omega(t) := \frac{a_n}{\lambda_n}[g_n(\omega)\sin(\lambda_n t) + \widetilde{g}_n(\omega)\cos(\lambda_n t)].$$

One could invoke a general result by Cambanis (see [Cam73, Theorem 5]) but we shall give here a much more natural argument. We will prove that $F^\omega$ is differentiable at $t=0$ (the case of $f^\omega - \widetilde{f}^\omega$ is similar). Let us introduce the following subsets of pairs $(t,\omega) \in [0,\pi]\times\Omega$ :

$$\begin{aligned} \mathcal{A}_1 &:= \Big\{(t,\omega) \in [0,\pi]\times\Omega \quad \text{such that the numerical series } \sum_{n=1}^{+\infty} F_n^\omega(t+h) \text{ converges for every } h \in \mathbb{Q}\Big\}, \\ \mathcal{A}_2 &:= \bigcap_{\varepsilon\in\mathbb{Q}^{+\star}} \bigcup_{\delta\in\mathbb{Q}^{+\star}} \bigcap_{\substack{|h'|\in]0,\delta[\cap\mathbb{Q} \\ |h''|\in]0,\delta[\cap\mathbb{Q}}} \Big\{\Big| \overline{\lim_{N\to+\infty}} \sum_{n=1}^{N} \frac{F_n^\omega(t+h') - F_n^\omega(t)}{h'} - \frac{F_n^\omega(t+h'') - F_n^\omega(t)}{h''}\Big| \leq \varepsilon\Big\}. \end{aligned}$$

We note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are measurable subsets of $[0,\pi]\times\Omega$. We conclude via the following steps :

**Step 1.** The stationary properties of the Gaussian processes $(F_n^\omega)_{t\in\mathbb{R}}$ for each $n$ roughly mean that if a property almost surely holds for a specific $t$ then it will also almost surely hold for $t=0$. As a consequence, the following formula is intuitively clear (see below (32) for a detailed proof) :

$$\forall t \in [0,\pi] \qquad \int_\Omega \mathbf{1}_{\mathcal{A}_1\cap\mathcal{A}_2}(t,\omega)\mathrm{d}\mathbf{P}(\omega) = \int_\Omega \mathbf{1}_{\mathcal{A}_1\cap\mathcal{A}_2}(0,\omega)\mathrm{d}\mathbf{P}(\omega). \tag{31}$$

Before proving this formula, we note that (31) implies

$$\begin{aligned} \int_0^\pi \Big(\int_\Omega \mathbf{1}_{\mathcal{A}_1\cap\mathcal{A}_2}(t,\omega)\mathrm{d}\mathbf{P}(\omega)\Big)\mathrm{d}t &= \pi\int_\Omega \mathbf{1}_{\mathcal{A}_1\cap\mathcal{A}_2}(0,\omega)\mathrm{d}\mathbf{P}(\omega) \\ &= \pi\mathbf{P}((0,\omega)\in\mathcal{A}_1\cap\mathcal{A}_2). \end{aligned} \tag{32}$$

Let us now recall how to check (31) via the Sierpiński-Dynkin's $\pi$-$\lambda$ theorem. Let us endow the set $\mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}$ with its product measure space structure. In particular, for any $t\in[0,\pi]$, the map $\omega\in\Omega \mapsto (F_n^\omega(t+h))_{n,h} \in \mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}$ is measurable. We recall that the $\sigma$-algebra of $\mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}$ is generated by the set

$$\Big\{\mathcal{B} = \prod_{(n,h)} \mathcal{B}_{n,h}, \quad \text{such that} \quad \begin{array}{l} \mathcal{B}_{n,h} \text{ is a Borel subset of } \mathbb{R} \text{ for any } (n,h) \\ \mathcal{B}_{n,h} = \mathbb{R} \text{ for all but a finite number of } (n,h) \end{array}\Big\}. \tag{33}$$

We now remark that for any finite subset $\mathcal{J} \times \mathcal{H} \subset \mathbb{N} \times \mathbb{Q}$, by independence (with respect to $n$) and stationarity (with respect to $t$), we get

$$\begin{aligned}
\int_\Omega \Big( \prod_{(n,h)\in\mathcal{J}\times\mathcal{H}} \mathbf{1}_{\mathcal{B}_{n,h}}(F_n^\omega(t+h))\Big) \mathrm{d}\mathbf{P}(\omega) &= \prod_{n\in\mathcal{J}} \int_\Omega \Big( \prod_{h\in\mathcal{H}} \mathbf{1}_{\mathcal{B}_{n,h}}(F_n^\omega(t+h))\Big) \mathrm{d}\mathbf{P}(\omega) \\
&= \prod_{n\in\mathcal{J}} \int_\Omega \Big( \prod_{h\in\mathcal{H}} \mathbf{1}_{\mathcal{B}_{n,h}}(F_n^\omega(h))\Big) \mathrm{d}\mathbf{P}(\omega) \\
&= \int_\Omega \Big( \prod_{(n,h)\in\mathcal{J}\times\mathcal{H}} \mathbf{1}_{\mathcal{B}_{n,h}}(F_n^\omega(h))\Big) \mathrm{d}\mathbf{P}(\omega)
\end{aligned}$$

which means, for any elementary subset $\mathcal{B}$ as in (33), that the following formula holds

$$\int_\Omega \mathbf{1}_{\mathcal{B}}\Big( (F_n^\omega(t+h))_{(n,h)} \Big) \mathrm{d}\mathbf{P}(\omega) = \int_\Omega \mathbf{1}_{\mathcal{B}}\Big( (F_n^\omega(h))_{(n,h)} \Big) \mathrm{d}\mathbf{P}(\omega). \tag{34}$$

By seeing the previous two terms as two measures of $\mathcal{B}$ on $\mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}$, we may invoke[18] the Sierpiński-Dynkin's $\pi$-$\lambda$ theorem to ensure that (34) still holds true for any $\mathcal{B}$ in the $\sigma$-algebra of $\mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}$. Following the definitions of $\mathcal{A}_1$ and $\mathcal{A}_2$, we leave the reader check that the following two subsets belong to the $\sigma$-algebra of the product space $\mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}$ :

$$\begin{aligned}
\mathcal{B}_1 &= \bigcap_{h'\in\mathbb{Q}} \Big\{ (x_{n,h}) \in \mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}, \quad \sum_{n=1}^{+\infty} x_{n,h'} \text{ converges} \Big\}, \\
\mathcal{B}_2 &= \bigcap_{\varepsilon\in\mathbb{Q}^{+\star}} \bigcup_{\delta\in\mathbb{Q}^{+\star}} \bigcap_{\substack{|h'|\in]0,\delta[\cap\mathbb{Q} \\ |h''|\in]0,\delta[\cap\mathbb{Q}}} \Big\{ (x_{n,h}) \in \mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}} \quad \Big| \overline{\lim_{N\to+\infty}} \sum_{n=1}^{N} \frac{x_{n,h'} - x_{n,0}}{h'} - \frac{x_{n,h''} - x_{n,0}}{h''} \Big| \le \varepsilon \Big\}.
\end{aligned}$$

For the particular choice $\mathcal{B} := \mathcal{B}_1 \cap \mathcal{B}_2$, the formula (34) is exactly (31).

**Step 2.** The convergence of the series $\sum \frac{a_n}{\lambda_n}$ ensures that, with probability 1, the series $\sum F_n^\omega$ uniformly converges on $\mathbb{R}$. As a consequence, the function $F^\omega = \sum_{n=1}^{+\infty} F_n^\omega$ satisfies the following equality in the sense of the theory of distributions on $\mathbb{R}$ :

$$(F^\omega)'(t) = \sum_{n\ge1} a_n \left[ g_n(\omega)\cos(\lambda_n t) - \widetilde{g}_n(\omega)\sin(\lambda_n t) \right] \tag{35}$$

with moreover the following condition (coming from the condition $(a_n) \in \ell^2(\mathbb{N}^\star)$) :

$$\sum_{n\ge1} |g_n(\omega)a_n|^2 + |\widetilde{g}_n(\omega)a_n|^2 < +\infty.$$

One may check via an adequate use of the Plancherel theorem the following result about almost periodic functions (see the argument in [PW34, page 117, line (31.08)] by Paley and Wiener or [Ing36, page 371] by Ingham) : under the gap assumption (17), for any real number $T \gg 1$ and any integer $N \ge 1$, the following holds

$$\int_{-T}^{T} \left| \sum_{n=1}^{N} a_n \left[ g_n(\omega)\cos(\lambda_n t) - \widetilde{g}_n(\omega)\sin(\lambda_n t) \right] \right|^2 \mathrm{d}t \le C(T) \sum_{n=1}^{N} |g_n(\omega)a_n|^2 + |\widetilde{g}_n(\omega)a_n|^2, \tag{36}$$

where $C(T)$ is independent of $N$. We infer that the series (35) converges in $L^2(-T,T)$ and hence $(F^\omega)'$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R})$ and so also to $L^1_{\mathrm{loc}}(\mathbb{R})$. In other words, with probability 1, the function $t \mapsto F^\omega(t)$ is indeed locally absolutely continuous and thus is differentiable for almost every $t$.

Looking at the beginning of this step and at the definition of $\mathcal{A}_1$, we see that for almost every $\omega \in \Omega$, we have the inclusion $[0,\pi] \times \{\omega\} \subset \mathcal{A}_1$. As a consequence of the almost everywhere differentiability of $F^\omega$, for almost every $\omega \in \Omega$ there is a subset $\mathcal{E}_\omega \subset [0,\pi]$ of full measure satisfying the inclusion $\mathcal{E}_\omega \times \{\omega\} \subset \mathcal{A}_1 \cap \mathcal{A}_2$. Hence, we have

$$\int_\Omega \Big( \int_0^\pi \mathbf{1}_{\mathcal{A}_1\cap\mathcal{A}_2}(t,\omega)\mathrm{d}t \Big) \mathrm{d}\mathbf{P}(\omega) = \int_\Omega \Big( \int_0^\pi 1 \mathrm{d}t \Big) \mathrm{d}\mathbf{P}(\omega) = \pi.$$

[18] The $\pi$-system is the set in (33) and the $\lambda$-system is the set of $\mathcal{B}$ in the $\sigma$-algebra of $\mathbb{R}^{\mathbb{N}^\star\times\mathbb{Q}}$ satisfying (34) and such that the functions inside (34) are measurable on $\Omega$.

**Step 3.** The equality (32) then becomes $\mathbf{P}((0,\omega)\in\mathcal{A}_1\cap\mathcal{A}_2)=1$. For almost every $\omega$, we know that $t\mapsto F^\omega(t)$ is continuous. Then the density of $]0,\delta[\cap\mathbb{Q}$ in $]0,\delta[$ shows that, with probability 1 the pair $(0,\omega)$ belongs to

$$\bigcap_{\varepsilon\in\mathbb{Q}^{+\star}}\bigcup_{\delta\in\mathbb{Q}^{+\star}}\bigcap_{\substack{0<|h'|<\delta\\0<|h''|<\delta}}\Big\{\Big|\frac{F^\omega(t+h')-F^\omega(t)}{h'}-\frac{F^\omega(t+h'')-F^\omega(t)}{h''}\Big|\leq\varepsilon\Big\}.$$

It is easy to see that the previous considerations lead to the differentiability of $F^\omega$ at $t=0$ with probability 1.

# 6 From boundedness to continuity of Gaussian processes

In the proof of Corollary 10, we used the Itô-Nisio theorem in order to get the unifom convergence of some Gaussian random series and the absolute convergence of $\sum\frac{a_n}{\lambda_n}$ for any $(a_n)\in\ell^2(\mathbb{N}^\star)$ (since we recall the asymptotic $\lambda_n\simeq n$). For the case $d=2$, the same argument would rather lead to consider series like $\sum\frac{a_n}{\sqrt{\lambda_n}}$ which clearly may be divergent for some $(a_n)\in\ell^2(\mathbb{N}^\star)$. We explain here another abstract argument that can be considered as a sort of "closed graph theorem" for Gaussian processes and will be used in the next section.

**Proposition 11.** *Let $(u_n)_{n\geq1}$ be a sequence of complex-valued continuous functions on $[0,+\infty)$ satisfying the following two hypothesis :*
**(H1)** *The sequence of functions $(u_n)_{n\geq1}$ is pointwise bounded :*

$$\forall t\in[0,+\infty)\qquad\sup_{n\geq1}|u_n(t)|<+\infty.$$

*In particular, for any $(a_n)\in\ell^2(\mathbb{N}^\star)$, the following Gaussian process $(X_t)$ is well-defined :*

$$X_t(\omega)=\sum_{n\geq1}g_n(\omega)a_nu_n(t).\tag{37}$$

**(H2)** *For any $(a_n)\in\ell^2(\mathbb{N}^\star)$, the Gaussian process $(X_t)$ is bounded in the following sense*[19] *: for any countable subset $\mathcal{E}\subset[0,+\infty)$, the random variable $\sup_{t\in\mathcal{E}}X_t$ is almost surely finite.*

*Then the following assertions are true*

*i) with probability 1, the function $t\mapsto X_t$ is almost surely continuous on $[0,+\infty)$,*

*ii) with probability 1, the Gaussian random series $\sum_{n\geq1}g_n(\omega)a_nu_n$ uniformly converges on $[0,+\infty)$,*

*iii) there exists $C>0$ such that the bound $\mathbf{E}\Big[\sup_{t\in[0,+\infty)}|X_t|\Big]\leq C\Big(\sum_{n\geq1}|a_n|^2\Big)^{1/2}$ holds true,*

*iv) the sequence of functions $(u_n)_{n\geq1}$ is globally bounded : $\sup_{t\in[0,+\infty)}\sup_{n\geq1}|u_n(t)|<+\infty$.*

The core of the proof is iii).

**Step 1.** We note that the Gaussian random variable $X_t$ is well defined thanks to the uniform boundedness of $(u_n)_{n\geq1}$. The well-known integrability properties of Gaussian processes (see for instance [Led01, page 134, Th 7.1]) implies the finiteness of the expectation $\mathbf{E}\Big[\sup_{t\in\mathcal{E}}X_t\Big]$ for any countable subset $\mathcal{E}\subset[0,+\infty)$. Without loss of generality, we may assume $0\in\mathcal{E}$. In particular, we also have by symmetry

$$\mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_t|\Big]\leq\mathbf{E}[|X_0|]+\mathbf{E}\Big[\sup_{(s,t)\in\mathcal{E}\times\mathcal{E}}|X_s-X_t|\Big]=\mathbf{E}[|X_0|]+2\mathbf{E}\Big[\sup_{t\in\mathcal{E}}X_t\Big]<\infty.\tag{38}$$

We now define the following semi-norm on $\ell^2(\mathbb{N}^\star)$ in which $X_t$ is defined in (37) :

$$\|a\|:=\sup_{\substack{\mathcal{E}\subset[0,+\infty)\\\mathcal{E}\text{ countable}}}\mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_t|\Big].\tag{39}$$

[19] This statement is indeed one possible definition of a bounded Gaussian process.

We claim that $\|\!|a|\!\|$ is finite for the following reason : if $(\mathcal{E}_k)_{k\geq 1}$ is a sequence of countable subsets of $[0,+\infty)$ such that $\lim_{k\to+\infty} \mathbf{E}\Big[\sup_{t\in\mathcal{E}_k}|X_t|\Big] = \|\!|a|\!\|$ then, by setting $\mathcal{E} = \bigcup_{k\geq 1}\mathcal{E}_k$, we get thanks to (38)

$$\|\!|a|\!\| \leq \mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_t|\Big] < +\infty.$$

**Step 2.** We claim that there is $C>0$ such that

$$\forall a \in \ell^2(\mathbb{N}^\star) \qquad \|\!|a|\!\| \leq C\|a\|_{\ell^2(\mathbb{N}^\star)}. \tag{40}$$

We now use an argument which can be interpreted as a probabilistic version of the closed graph theorem (see [MP81, p 49]). Assume that there does not exist $C>0$ satisfying (40). Then, for any $M\in\mathbb{N}^\star$, there is sequence $(a_n^M)_{n\in\mathbb{N}^\star}\in\ell^2(\mathbb{N}^\star)$ such that

$$\|a^M\|_{\ell^2(\mathbb{N}^\star)} \leq 2^{-M} \qquad \text{and} \qquad 2^M \leq \|\!|a^M|\!\|. \tag{41}$$

We now consider a double sequence $(g_{n,M})$ of i.i.d. Gaussian random variables and we define the Gaussian process

$$X_t(\omega) := \sum_{n\geq 1}\Big(\sum_{M\geq 1} a_n^M g_{n,M}(\omega)\Big)u_n(t).$$

We note that $\Big(\sum_{M\geq 1} a_n^M g_{n,M}(\omega)\Big)_{n\geq 1}$ is a sequence of independent Gaussian centered random variables with variance $\sigma_n^2 := \sum_{M\geq 1}|a_n^M|^2$ which is finite since we may write

$$\sum_{n\geq 1}\sigma_n^2 = \sum_{M\geq 1}\|a^M\|^2_{\ell^2(\mathbb{N}^\star)} \leq \sum_{M\geq 1} 2^{-2M} < +\infty.$$

As a consequence, the Gaussian process $(X_t)$ is well-defined. Moreover, the assumption **(H2)** applied to $(\sigma_n)_{n\geq 1}\in\ell^2(\mathbb{N}^\star)$ shows that $\sup_{t\in\mathcal{E}} X_t$ is finite for any countable subset $\mathcal{E}\subset[0,+\infty)$. The analysis of Step 1 then proves

$$\sup_{\substack{\mathcal{E}\subset[0,+\infty)\\ \mathcal{E}\text{ countable}}} \mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_t|\Big] < +\infty. \tag{42}$$

For any $M_0\in\mathbb{N}^\star$ we define $X_{t,M_0}(\omega) := \sum_{n\geq 1} a_n^{M_0} g_{n,M_0}(\omega)u_n(t)$ and we decompose

$$X_t = X_{t,M_0} + (X_t - X_{t,M_0}).$$

Since $X_{t,M_0}$ and $(X_t - X_{t,M_0})$ are independent and centered, it is well-known that the following inequality holds true for any countable subset $\mathcal{E}\subset[0,+\infty)$ :

$$\mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_{t,M_0}|\Big] \quad\leq\quad \mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_t|\Big].$$

For the convenience of the reader, let us give a few ideas of the proof of the last inequality (see also Remark 28). By independence, the upper bound can be written (see [Ime19, Appendix F] for more details) :

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_t|\Big] &= \mathbf{E}\Big[\sup_{t\in\mathcal{E}}\Big|X_{t,M_0} + (X_t - X_{t,M_0})\Big|\Big]\\ &= \mathbf{E}_{\omega_1}\mathbf{E}_{\omega_2}\Big[\sup_{t\in\mathcal{E}}\Big|X_{t,M_0}(\omega_1) + (X_t(\omega_2) - X_{t,M_0}(\omega_2))\Big|\Big].\end{aligned}$$

The triangular inequality with respect to $\mathbf{E}_{\omega_2}$ gives us

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_t|\Big] &\geq \mathbf{E}_{\omega_1}\Big[\sup_{t\in\mathcal{E}}\Big|\mathbf{E}_{\omega_2}\Big[X_{t,M_0}(\omega_1) + (X_t(\omega_2) - X_{t,M_0}(\omega_2))\Big]\Big|\Big]\\ &= \mathbf{E}_{\omega_1}\Big[\sup_{t\in\mathcal{E}}|X_{t,M_0}(\omega_1) + 0 - 0|\Big] = \mathbf{E}\Big[\sup_{t\in\mathcal{E}}|X_{t,M_0}|\Big].\end{aligned}$$

The expected inequality is thus proved. As a consequence, we obtain $\|a^{M_0}\| \leq \|(\sigma_n)_{n\geq 1}\| < +\infty$ thanks to (39) and (42). This is a contradiction with (41) by choosing $M_0$ large enough.

**Step 3.** We now apply (40) for sequences $(a_n)_{n\geq 1}$ with finite support. Since each $u_n$ is continuous, one may choose one dense subset $\mathcal{E}$ of $[0,+\infty)$ to get

$$\mathbf{E}\Big[\sup_{t\in[0,+\infty)}\Big|\sum_{n\geq 1} g_n(\omega)a_n u_n(t)\Big|\Big] \leq C\Big(\sum_{n\geq 1}|a_n|^2\Big)^{\frac{1}{2}}. \tag{43}$$

Let us recall that $\mathcal{BC}([0,+\infty))$ denotes the Banach space of the bounded and continuous functions on $[0,+\infty)$ (see the proof of Corollary 10). For the general case, we do not assume $(a_n)$ to be with finite support anymore. One infers that the sequence of the partial sums of the Gaussian random series $\sum_{n\geq 1} g_n(\omega)a_n u_n(t)$ converges in $L^1(\Omega, \mathcal{BC}([0,+\infty)))$ and hence (43) still holds true. In other words, iii) is proved.

**Step 4.** We then easily get ii) and i) by considering Cauchy sequences[20] and iv) by considering for $(a_n)$ the elements of the canonical basis of $\ell^2(\mathbb{N}^\star)$.

# 7 Proof of Theorem 4 for $d = 2$

The goal of this part is to explain how the following analogue of Proposition 9 and Corollary 10 and some extra-arguments allow for a proof of Theorem 4 in the last case $d = 2$.

**Proposition 12.** *Let $(a_n)_{n\geq 1}$ be a sequence in $\ell^2(\mathbb{N}^\star)$, then, with probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega)a_n\frac{\sin(\lambda_n t)}{\sqrt{\lambda_n}}$ and $\sum_{n\geq 1} g_n(\omega)a_n\frac{\cos(\lambda_n t)-1}{\sqrt{\lambda_n}}$ uniformly converge on any compact subset of $[0,+\infty)$ and their sums are pointwise $\frac{1}{2}$-Hölder at $t = 0$ (see Definition 20 below) : there almost surely exists $C > 0$ satisfying*

$$\forall t \geq 0 \qquad \Big|\sum_{n\geq 1} g_n(\omega)a_n\frac{\sin(\lambda_n t)}{\sqrt{\lambda_n}}\Big| + \Big|\sum_{n\geq 1} g_n(\omega)a_n\frac{\cos(\lambda_n t)-1}{\sqrt{\lambda_n}}\Big| \leq C\sqrt{t}.$$

Proof. See Section 15. □

**Corollary 13.** *Let $(a_n)_{n\geq 1}$ be a sequence in $\ell^2(\mathbb{N}^\star)$, then, with probability 1, the Gaussian random series*

$$\sum_{n\geq 1} g_n(\omega)a_n\frac{\sin(\lambda_n t)}{\sqrt{\lambda_n t}} \qquad \text{and} \qquad \sum_{n\geq 1} g_n(\omega)a_n\frac{\cos(\lambda_n t)-1}{\sqrt{\lambda_n t}}$$

*uniformly converge on $[0,+\infty)$ (in particular, their sums are almost surely continuous on $[0,+\infty)$).*

Proof. We apply Proposition 11 with the following two choices of $u_n$ (in which we note the additional factor $\sqrt{t}$) :

$$u_n^{(1)}(t) = \frac{\sin(\lambda_n t)}{\sqrt{\lambda_n t}} \qquad \text{and} \qquad u_n^{(2)}(t) = \frac{\cos(\lambda_n t)-1}{\sqrt{\lambda_n t}}$$

which are clearly continuous and uniformly bounded for t belonging to $[0,+\infty)$. Hence the hypothesis **(H1)** of Proposition 11 is fulfilled. The following two associated Gaussian processes are thus well defined

$$\begin{aligned} X_t^{(1)} &:= \sum_{n\geq 1} g_n(\omega)a_n u_n^{(1)}(t) & \text{and} \qquad X_t^{(2)} &:= \sum_{n\geq 1} g_n(\omega)a_n u_n^{(2)}(t) \\ &= \sum_{n\geq 1} g_n(\omega)a_n\frac{\sin(\lambda_n t)}{\sqrt{\lambda_n t}} & &= \sum_{n\geq 1} g_n(\omega)a_n\frac{\cos(\lambda_n t)-1}{\sqrt{\lambda_n t}}. \end{aligned}$$

In order to check the hypothesis **(H2)** of Proposition 11 we merely have to see the conclusion of Proposition 12 which now directly reads :

$$\forall t \geq 0 \qquad |X_t^{(1)}| + |X_t^{(2)}| \leq C.$$

In particular, for any countable subset $\mathcal{E}$ of $[0,+\infty)$, we have

$$\forall t \in \mathcal{E} \qquad |X_t^{(1)}| + |X_t^{(2)}| \leq C.$$

Point ii) of the conclusion of Proposition 11 achieves the proof. □

[20] See step 1 of Proposition 7.

As above, we merely focus on the implication i) $\Rightarrow$ ii) of Theorem 4. In other words, by setting $a_n = c_n\sqrt{\lambda_n}$, we have to prove the following result.

**Theorem 14.** *For any sequence* $(a_n) \in \ell^2(\mathbb{N}^\star)$*, with probability 1, the Gaussian random series* $\sum_{n\geq 1} g_n(\omega)a_nJ_0(\lambda_n\cdot)$ *uniformly converges on* $[0,+\infty)$*.*

We cannot directly apply Lemma 6 with $W = J_0$ since $J_0$ is not of bounded variation on $[0,+\infty)$. The idea is to consider separately the following two random series for $t \in [0,+\infty)$

$$\sum_{n\geq 1} g_n(\omega)a_nJ_0(\lambda_n t)\mathbf{1}_{[0,1]}(\lambda_n t) \qquad \text{and} \qquad \sum_{n\geq 1} g_n(\omega)a_nJ_0(\lambda_n t)\mathbf{1}_{(1,+\infty)}(\lambda_n t). \tag{44}$$

The almost sure uniform convergence of the first random series on $[0,+\infty)$ is a direct consequence of Lemma 6 once we note that the function $t \in [0,+\infty) \mapsto J_0(t)\mathbf{1}_{[0,1]}(t)$ is of bounded variation.

We now focus on the second random series in (44). We need the simple but crucial following property of the Bessel function $J_0$.

**Proposition 15.** *There exists a differentiable function* $\theta : [0,+\infty[\to \mathbb{R}$ *satisfying* $\theta' \in L^1(0,+\infty)$ *and*

$$\forall t \geq 1 \qquad J_0(t) = \frac{\sin(t)+\cos(t)-1}{\sqrt{\pi t}} + \theta(t). \tag{45}$$

Proof. It is sufficient to show the integrability on $[1,+\infty)$ of the derivative of the following function

$$\forall t \geq 1 \qquad \varphi(t) := J_0(t) - \frac{\sin(t)+\cos(t)}{\sqrt{\pi t}}. \tag{46}$$

Then one may choose a function $\phi \in \mathcal{C}^\infty([0,+\infty),\mathbb{R})$ satisfying $\phi \equiv 0$ on $[0,\frac{1}{2}]$ and $\phi \equiv 1$ on $[1,+\infty)$ so that $\theta(t) = \phi(t)(\varphi(t) + \frac{1}{\sqrt{\pi t}})$ will be convenient for (45). The proof of the integrability of $\varphi'$ will need the following integral representation for the Hankel function $H_0^1$ (see [Wat44, page 168, line (3)]) for any $t > 0$ :

$$J_0(t) = \mathrm{Re}(H_0^1(t)) = \frac{1}{\sqrt{t}}\,\mathrm{Re}\left(e^{i\left(t-\frac{\pi}{4}\right)}S(t)\right) \quad \text{with} \quad S(t) = \frac{\sqrt{2}}{\pi}\int_0^{+\infty} \frac{e^{-u}}{\sqrt{1+\frac{iu}{2t}}}\frac{\mathrm{d}u}{\sqrt{u}} \tag{47}$$

in which $\frac{1}{\sqrt{1+\frac{iu}{2t}}}$ is understood as its principal value (with $\frac{u}{t} > 0$) :

$$\begin{aligned}
\frac{1}{\sqrt{1+\frac{iu}{2t}}} &= \frac{1}{\left(1+\frac{u^2}{4t^2}\right)^{\frac{1}{4}}}\exp\left(-\frac{i}{2}\arctan\left(\frac{u}{2t}\right)\right) \\
&= \frac{1}{\sqrt{2}(1+\frac{u^2}{4t^2})^{\frac{1}{4}}}\left(\left(1+\frac{1}{\sqrt{1+\frac{u^2}{4t^2}}}\right)^{\frac{1}{2}} - i\left(1-\frac{1}{\sqrt{1+\frac{u^2}{4t^2}}}\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Note that the integral in (47) is absolutely convergent on $(0,+\infty)$, by bounding by $\frac{e^{-u}}{\sqrt{u}}$, which leads to $|S(t)| \leq \frac{\sqrt{2}}{\sqrt{\pi}}$. In the sequel, we need the more accurate asymptotic both coming from the exact formula (47) :

$$\begin{aligned}
S(t) &= \frac{\sqrt{2}}{\pi}\int_0^{+\infty}\frac{e^{-u}}{\sqrt{u}}\Big(1-\overbrace{\frac{i}{2}\int_0^{\frac{u}{2t}}\frac{\mathrm{d}\alpha}{(1+i\alpha)^{3/2}}}^{=\mathcal{O}(\frac{u}{2t})}\Big)\mathrm{d}u = \frac{\sqrt{2}}{\sqrt{\pi}} + \mathcal{O}\Big(\frac{1}{t}\Big), \\
S'(t) &= \frac{i\sqrt{2}}{4\pi}\times\frac{1}{t^2}\int_0^{+\infty}\frac{e^{-u}\sqrt{u}}{(1+\frac{iu}{2t})^{3/2}}\mathrm{d}u = \mathcal{O}\Big(\frac{1}{t^2}\Big).
\end{aligned}$$

We then are able to come back to (46) :

$$\begin{aligned}
J_0(t) &= \frac{\cos\Big(t-\frac{\pi}{4}\Big)\mathrm{Re}(S(t)) - \sin\Big(t-\frac{\pi}{4}\Big)\mathrm{Im}(S(t))}{\sqrt{t}} \\
J_0'(t) &= \frac{-\sin\Big(t-\frac{\pi}{4}\Big)\mathrm{Re}(S(t)) - \cos\Big(t-\frac{\pi}{4}\Big)\mathrm{Im}(S(t))}{\sqrt{t}} + \mathcal{O}\Big(\frac{1}{t^{3/2}}\Big) \\
J_0'(t) &= \frac{-\sqrt{2}\sin\Big(t-\frac{\pi}{4}\Big)}{\sqrt{\pi t}} + \mathcal{O}\Big(\frac{1}{t^{3/2}}\Big) \\
\varphi'(t) &= \mathcal{O}\Big(\frac{1}{t^{3/2}}\Big).
\end{aligned}$$

□

By looking at Proposition 15, we now to apply Proposition 7 for $W = \mathbf{1}_{(1,+\infty)} \in BV$ and

$$u_n(t) = \frac{\sin(\lambda_n t) + \cos(\lambda_n t) - 1}{\sqrt{\pi\lambda_n t}} + \theta(\lambda_n t) \qquad \text{on } I = [0,+\infty).$$

So the almost sure uniform convergence on $[0,+\infty)$ of the second Gaussian random series in (44) will be a consequence of the almost sure uniform convergence on $[0,+\infty)$ of the following Gaussian random series

$$\sum_{n\geq 1} g_n(\omega) a_n \left(\frac{\sin(\lambda_n t) + \cos(\lambda_n t) - 1}{\sqrt{\pi\lambda_n t}} + \theta(\lambda_n t)\right). \tag{48}$$

Thanks to Lemma 6 and the conditions $\theta' \in L^1(0,+\infty)$ and $(a_n) \in \ell^2(\mathbb{N}^\star)$, the Gaussian random series $\sum g_n(\omega) a_n \theta(\lambda_n t)$ almost surely uniformly converges on $[0,+\infty)$. Finally, the trigonometric part is dealt thanks to Corollary 13 (relying on the admitted Proposition 12). We get the conclusion of Theorem 14 and hence of Theorem 4 for all $d \geq 2$.

# 8 Boundedness of countable families of Gaussian processes

For each $j \in \mathbb{N}$, let us denote by $(F_j^\omega(x))_{x\in\mathcal{M}}$ a centered Gaussian process on a manifold $\mathcal{M}$ (for instance $\mathcal{M} = \mathbb{R}^d$ or $\mathcal{M}$ being a boundaryless compact manifold). We moreover denote by $\mathcal{M}_j$ a compact subset of $\mathcal{M}$. We will always make the following assumption :

**(H-C)** with probability 1, for any $j \in \mathbb{N}$, the function $x \in \mathcal{M} \mapsto F_j^\omega(x) \in \mathbb{R}$ is continuous.

In particular, **(H-C)** implies that the expectation $\mathbf{E}\Big[\sup_{x\in\mathcal{M}_j} |F_j^\omega(x)|\Big]$ does not pose any measurability issue (by reducing $\mathcal{M}_j$ to a countable subset).

We are interested in giving simple sufficient conditions for ensuring uniform estimates like the following ones :

$$\text{with probability } 1 \qquad \sup_{j\geq 1} \sup_{x\in\mathcal{M}_j} |F_j^\omega(x)| < +\infty.$$

As well-known in the theory of Gaussian processes (see [LT91, Part 3.1]), the following so-called weak variance is of interest

$$\sigma_j = \sup_{x\in\mathcal{M}_j} \sqrt{\mathbf{E}[|F_j^\omega(x)|^2]}. \tag{49}$$

Additionally to the above notations, let $(\rho_j)_{j\geq 1}$ be a sequence of positive numbers. The tool we shall use in the sequel is the following one (see converse results in Proposition 18 and Lemma 19).

**Proposition 16.** *Assume* **(H-C)** *and the following two assertions :*

*i) there exists* $\beta > 0$ *such that the series* $\sum \exp\Big(-\beta^2 \frac{\rho_j^2}{2\sigma_j^2}\Big)$ *is convergent*[21],

[21] We make the convention $e^{-\frac{1}{0}} = 0$.

*ii) there exists $C>0$ such that $\mathbf{E}\Big[\sup_{x\in\mathcal{M}_j}|F_j^\omega(x)|\Big]\leq C\rho_j$ holds true for any $j\in\mathbb{N}$,*

*then we have*

$$\exists T>0 \qquad \sum_{j\in\mathbb{N}}\mathbf{P}\Big(\sup_{x\in\mathcal{M}_j}|F_j^\omega(x)|\geq T\rho_j\Big)<+\infty. \tag{50}$$

*Moreover, with probability 1, we have*

$$\overline{\lim_{j\to+\infty}}\Big(\frac{1}{\rho_j}\sup_{x\in\mathcal{M}_j}|F_j^\omega(x)|\Big)\leq\beta+\overline{\lim_{j\to+\infty}}\frac{1}{\rho_j}\mathbf{E}\Big[\sup_{x\in\mathcal{M}_j}|F_j^\omega(x)|\Big]. \tag{51}$$

*And finally, there almost surely exists $C_\omega>0$ such that $|F_j^\omega(x)|\leq C_\omega\rho_j$ for any $j\in\mathbb{N}$ and $x\in\mathcal{M}_j$ .*

**Remark 17.** *In some situations, the assumption i) may hold for any small enough $\beta>0$, for instance if $\sum_{j=0}^{\infty}\big|\frac{\sigma_j}{\rho_j}\big|^p<+\infty$ for some $p\in[1,+\infty)$. If in addition, we have $\lim_{j\to+\infty}\frac{1}{\rho_j}\mathbf{E}\Big[\sup_{x\in\mathcal{M}_j}|F_j^\omega(x)|\Big]=0$, then the conclusion of Proposition 16 ensures that almost surely $\lim_{j\to+\infty}\frac{1}{\rho_j}\sup_{x\in\mathcal{M}_j}|F_j^\omega(x)|=0$.*

**Proof of Proposition 16.** We may assume that $\rho_j=1$ for any $j$ since there is no loss in the proof provided that we replace $F_j^\omega$ with $\frac{1}{\rho_j}F_j^\omega$. We shall use the concentration of measure phenomenon for the Gaussian random vector $F_j^\omega$ seen as a function from the probability space $\Omega$ to the Banach space $\mathcal{C}^0(\mathcal{M}_j)$ of the continuous real-valued functions on the compact subset $\mathcal{M}_j$ (we refer for instance to [LT91, page 53, Lemma 3.1]). Let $M_j\in\mathbb{R}^+$ be a median of $\omega\mapsto\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}$, namely a number satisfying

$$\mathbf{P}\Big(\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}\leq M_j\Big)\geq\frac{1}{2}\quad\text{and}\quad\mathbf{P}\Big(\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}\geq M_j\Big)\geq\frac{1}{2}.$$

Then the following concentration inequality holds true

$$\mathbf{P}\Big(\Big|\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}-M_j\Big|>t\Big)\leq e^{-t^2/2\sigma_j^2},\qquad\forall t>0. \tag{52}$$

By integration, we get

$$\Big|\mathbf{E}\Big[\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}\Big]-M_j\Big|\leq\int_0^{+\infty}e^{-t^2/2\sigma_j^2}\mathrm{d}t=\frac{\sqrt{\pi}}{\sqrt{2}}\sigma_j.$$

Since the assumption i) means the convergence of the series $\sum e^{-\beta^2/2\sigma_j^2}$ for some constant $\beta>0$, it is clear that $(\sigma_j)$ tends to $0^+$ and we immediately have

$$\overline{\lim_{j\to+\infty}}M_j=\overline{\lim_{j\to+\infty}}\mathbf{E}\Big[\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}\Big]. \tag{53}$$

In particular, the sequence of medians $(M_j)_{j\geq1}$ is bounded. Let us fix $J\in\mathbb{N}$ and let us define

$$T_J=\beta+\sup_{j\geq J}M_j. \tag{54}$$

Again, (52) implies the following bound

$$\begin{aligned}\sum_{j\geq J}\mathbf{P}\Big(\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}>T_J\Big)\quad&\leq\quad\sum_{j\geq J}\mathbf{P}\Big(\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}-M_j>T_J-\sup_{j\geq J}M_j\Big)\\&\leq\quad\sum_{j\geq J}\exp\Big(-\frac{1}{2\sigma_j^2}(T_J-\sup_{j\geq J}M_j)^2\Big)\end{aligned}$$

and the last sum is finite thanks to the assumption i) and to the choice of $T_J$ in (54). We have thus proved (50) (for instance for $J=0$ and $T=2T_0$). The inequality (51) is a quite direct consequence of the Borel-Cantelli lemma since almost surely, for any $J\in\mathbb{N}$ and any $j\gg1$ (depending on $J$) we have $\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}\leq\beta+\sup_{j\geq J}M_j$ and hence

$$\overline{\lim_{j\to+\infty}}\|F_j^\omega\|_{\mathcal{C}^0(\mathcal{M}_j)}\leq\beta+\inf_{J\in\mathbb{N}}\big(\sup_{j\geq J}M_j\big)=\beta+\overline{\lim_{j\to+\infty}}M_j.$$

We conclude with the equality (53). The proof of Proposition 16 is finished. □

Let us explain a short argument showing that Proposition 16 is essentially sharp.

**Proposition 18.** *Assume* **(H-C)** *and the following two assertions :*

- *there almost surely exists* $C_\omega > 0$ *such that* $|F_j^\omega(x)| \le C_\omega \rho_j$ *for any* $j \in \mathbb{N}$ *and* $x \in \mathcal{M}_j$*,*
- *the Gaussian processes* $(F_j^\omega)_{j\in\mathbb{N}}$ *are independent.*

*Then*

*i) there exists* $\beta > 0$ *such that the series* $\sum \exp\Big(-\beta^2 \frac{\rho_j^2}{2\sigma_j^2}\Big)$ *is convergent,*

*ii) there exists* $C > 0$ *such that* $\mathbf{E}\Big[\sup_{x\in\mathcal{M}_j} |F_j^\omega(x)|\Big] \le C\rho_j$ *holds true for any* $j \in \mathbb{N}$.

We need the following lemma.

**Lemma 19.** *Let* $(X_j)_{j\in\mathbb{N}}$ *be a sequence of random variables belonging to* $L^2(\Omega)$ *satisfying, for a suitable constant* $C \ge 1$*, the following estimates for any* $j \in \mathbb{N}$ *:*

$$\mathbf{E}[|X_j|^2] \le C\mathbf{E}[|X_j|]^2. \tag{55}$$

*Then the following assertions are true*

*i) if the sequence* $(X_j(\omega))$ *is almost surely bounded then the expectations* $\mathbf{E}[|X_j|]$ *are bounded,*

*ii) if the sequence* $(X_j(\omega))$ *almost surely converges to* $0$ *then* $\lim_{j\to 0} \mathbf{E}[|X_j|] = 0$.

Proof. In the proof of [Ime22, Proposition 23, ii) ⇒ iii)], we will replace the dominated convergence theorem with the Fatou's lemma. The Paley-Zygmund inequality and (55) imply for any $j \in \mathbb{N}$

$$\frac{1}{4C} \le \mathbf{P}\Big(|X_j| \ge \frac{1}{2}\mathbf{E}[|X_j|]\Big).$$

Now choose $\mathcal{J}$ an infinite subset of $\mathbb{N}$ such that $\lim_{\substack{j\to+\infty\\ j\in\mathcal{J}}} \mathbf{E}[|X_j|] = \overline{\lim_{j\to+\infty}} \mathbf{E}[|X_j|]$. The Fatou's lemma then gives

$$\begin{aligned}\frac{1}{4C} \le \overline{\lim_{\substack{j\to+\infty\\ j\in\mathcal{J}}}} \mathbf{P}\Big(|X_j| \ge \frac{1}{2}\mathbf{E}[|X_j|]\Big) &\le \mathbf{P}\Big(\overline{\lim_{\substack{j\to+\infty\\ j\in\mathcal{J}}}} \Big\{|X_j| \ge \frac{1}{2}\mathbf{E}[|X_j|]\Big\}\Big)\\ &\le \mathbf{P}\Big(\overline{\lim_{j\to+\infty}} |X_j| \ge \frac{1}{2}\overline{\lim_{j\to+\infty}} \mathbf{E}[|X_j|]\Big).\end{aligned}$$

Then i) and ii) are a direct consequence of the last inequality. □

**Proof of Proposition 18.** Without loss of generality, we can set $\rho_j = 1$ for any $j \in \mathbb{N}$ as in the proof of Proposition 16. The previous lemma can be used with $X_j = \sup_{x\in\mathcal{M}_j} |F_j^\omega(x)|$ since (55) is satisfied thanks to the Gaussian version of the Kahane-Khintchine inequalities (see [LT91, page 56, Cor 3.2] or [LQ18b, p 256, Cor V.27]). So Lemma 19 shows the boundedness of the expectations of $X_j$.

Let us now prove the existence of $\beta > 0$ such that $\sum_{j\ge 0} \exp\Big(-\frac{\beta^2}{2\sigma_j^2}\Big) < +\infty$. Due to the definition (49), for each $j \in \mathbb{N}$ there exists $x_j \in \mathcal{M}_j$ satisfying

$$\sigma_j' \ge \frac{\sigma_j}{2} \qquad \text{with} \quad \sigma_j' := \sqrt{\mathbf{E}[|F_j^\omega(x_j)|]}. \tag{56}$$

The assumptions of Proposition 18 ensure that the centered Gaussian random variables $F_j^\omega(x_j)$ are independent and may be bounded independently of $j$ (with probability 1). In other words, if $(g_j)$ is a sequence of i.i.d. $\mathcal{N}(0,1)$-Gaussian random variable, then the sequence $(\sigma_j' g_j)_{j\ge 1}$ is almost surely bounded. The second Borel-Cantelli lemma shows that there is $T \in \mathbb{N}^\star$ such that the series $\sum_{j\in\mathbb{N}} \mathbf{P}(|\sigma_j' g_j| > T)$ converges (otherwise, with probability 1, the sequence $(|\sigma_j' g_j(\omega)|)_{j\in\mathbb{N}}$ goes beyond any $T \in \mathbb{N}^\star$). Then (56) implies the convergence of $\sum_{j\in\mathbb{N}} \mathbf{P}(|\sigma_j g_j| > 2T)$ and we conclude with the bound from below $\mathbf{P}(|g_j| > t) \ge C\exp(-ct^2)$ for adequate constants[22] $C > 0$ and $c > \frac{1}{2}$. □

[22] See the argument in the proof of [LQ18a, page 2, Proposition II.1].

# 9 Pointwise and global Hölder regularity via Littlewood-Paley decompositions

We denote by $\mathcal{M}$ a Riemannian manifold ($\mathcal{M}$ being compact or being $\mathbb{R}^d$) and by $\delta_g$ its Riemannian distance. We begin by recalling the notion of pointwise Hölder regularity (see more about this notion in [ACJM16]).

**Definition 20.** *Let us consider $\alpha \in (0,1)$ and $x_0 \in \mathcal{M}$. A function $f : \mathcal{M} \to \mathbb{C}$ belongs to $\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})$, the space of the pointwise Hölder functions $f : \mathcal{M} \to \mathbb{C}$ at $x_0$ of order $\alpha$, if there exists $C > 0$ such that the following holds for any $x \in \mathcal{M}$ :*

$$|f(x) - f(x_0)| \leq C\delta_g(x, x_0)^\alpha.$$

Let us set the following notations :

$$\|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})} := |f(x_0)| + \sup_{x \in \mathcal{M}\setminus\{x_0\}} \frac{|f(x) - f(x_0)|}{\delta_g(x, x_0)^\alpha},$$

$$\text{and} \qquad |\!|\!| f |\!|\!|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})} := \sup_{\substack{x \in \mathcal{M}\setminus\{x_0\} \\ \delta_g(x,x_0)\leq 1}} \frac{|f(x) - f(x_0)|}{\delta_g(x, x_0)^\alpha}.$$

Note that the global $\alpha$-Hölder norm can also be defined as

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})} := \sup_{x_0 \in \mathcal{M}} \|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})} \simeq \|f\|_{L^\infty(\mathcal{M})} + \sup_{\substack{(x,x_0)\in\mathcal{M}^2 \\ x \neq x_0}} \frac{|f(x) - f(x_0)|}{\delta_g(x, x_0)^\alpha}.$$

The following equivalence is easy to check :

$$\frac{1}{4}\|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})} \leq |\!|\!| f |\!|\!|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})} + \sup_{x\in\mathcal{M}} \frac{|f(x)|}{(1 + \delta_g(x, x_0))^\alpha} \leq 2\|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})}. \tag{57}$$

Moreover, the term $|\!|\!| f |\!|\!|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})}$ can be understood via the following equivalence[23]

$$\frac{1}{2}|\!|\!| f |\!|\!|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})} \leq \sup_{j\in\mathbb{N}} \Big(2^{\alpha j} \sup_{\substack{x\in\mathcal{M} \\ \delta_g(x,x_0)\leq 2^{-j}}} |f(x) - f(x_0)|\Big) \leq |\!|\!| f |\!|\!|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})}$$

but it will be more useful to consider the following equivalence[24]

$$\frac{1}{2}|\!|\!| f |\!|\!|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})} \leq \sup_{j\in\mathbb{N}} \Big(2^{\alpha j} \sup_{\substack{(x,x')\in\mathcal{M}^2 \\ \delta_g(x,x_0)\leq 2^{-j} \\ \delta_g(x',x_0)\leq 2^{-j}}} |f(x) - f(x')|\Big) \leq 2|\!|\!| f |\!|\!|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathcal{M})}. \tag{58}$$

Let us now study another equivalence via the Littlewood-Paley decomposition. Let $\Theta \in \mathcal{C}^\infty_c([0,+\infty),\mathbb{R})$ be a function such that $\Theta \equiv 1$ near $0^+$. We also consider the function $\theta \in \mathcal{C}^\infty_c(0,+\infty)$ defined as follows

$$\theta(\lambda) = \Theta(\lambda) - \Theta(4\lambda). \tag{59}$$

The pair $(\Theta, \theta)$ satisfies the Littlewood-Paley relation

$$\forall \lambda \geq 0 \qquad 1 = \Theta(\lambda) + \sum_{j\geq 1} \theta(2^{-2j}\lambda).$$

We refer to (65) and (66) for reminders about the Fourier multipliers. Then for any $f \in \mathcal{C}^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and any $\alpha \in (0,1)$, the equivalence of the following three statements is well-known (see the statements and the proofs in [AG07, page 81, Proposition 1.3] and [Zwo12, Th 7.16]) :

i) the function $f$ is globally $\alpha$-Hölder,

[23] For the lower bound, we may consider the rings $2^{-(j+1)} \leq \delta_g(x, x_0) \leq 2^{-j}$ and use the bound $2^{-\alpha j} = 2^\alpha 2^{-\alpha(j+1)} \leq 2\delta_g(x, x_0)^\alpha$.
[24] We just have to bound $|f(x) - f(x')| \leq |f(x) - f(x_0)| + |f(x') - f(x_0)|$ and use the previous inequality.

ii) there exists $C > 0$ such that $\|f - \Theta(-h^2\Delta)f\|_{L^\infty(\mathbb{R}^d)} \leq Ch^\alpha$ holds for any $h \in (0, 1]$,

iii) there exists $C > 0$ such that $\|\theta(-2^{-2j}\Delta)f\|_{L^\infty(\mathbb{R}^d)} \leq C2^{-j\alpha}$ holds for any $j \in \mathbb{N}^\star$.

We now need a pointwise analogue of the last equivalence in which we moreover weaken the condition $f \in L^\infty(\mathbb{R}^d)$ by a condition of polynomial growth like $|f(x)| \lesssim (1 + |x|)^\alpha$. The following result is inspired from a result of Kreit and Nicolay (see [KN18, Theorem 14]) but states an equivalence (actually, in contrast to the result of Kreit and Nicolay, we do not need to assume $f$ to belong in $\mathcal{C}^{0,\alpha'}(\mathbb{R}^d)$ for some $\alpha' \in (0, \alpha)$ and we do not need the operator $\Theta(-2^{-2k}\Delta)$ to be a convolution by a compactly supported function).

**Proposition 21.** *Let us fix $\alpha \in (0, 1)$ and $x_0 \in \mathbb{R}^d$. For any continuous function $f : \mathbb{R}^d \to \mathbb{C}$, the following equivalence holds true :*

$$\|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)} \simeq \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1 + |x - x_0|)^\alpha} + \sup_{j\in\mathbb{N}} 2^{\alpha j} \Big( \sup_{k\geq j} \|f - \Theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-j}))} \Big) \tag{60}$$

*with constants of equivalence independent of $f$ and $x_0$.*

In particular, we note that the equivalence (60) implies that our results are independent of the choice of the function $\Theta$ (actually such an independence should be completely expected for any result involving Littlewood-Paley decompositions).

In order to prove Theorem 2, we need the following global analogue result on compact manifolds.

**Proposition 22.** *We assume that $\mathcal{M}$ is a boundaryless compact Riemannian manifold and let $\boldsymbol{\Delta}$ be its Laplace-Beltrami operator. For any $\alpha \in (0, 1)$ and any $f \in L^2(\mathcal{M})$, the following qualitative equivalence is true*

$$f \in \mathcal{C}^{0,\alpha}(\mathcal{M}) \quad \Leftrightarrow \quad \sup_{j\geq 1} 2^{j\alpha} \|\theta(-2^{-2j}\boldsymbol{\Delta})f\|_{L^\infty(\mathcal{M})} < +\infty. \tag{61}$$

*Moreover, for any sequence $(h_j)_{j\in\mathbb{N}}$ of $(0, 1]$ which tends to $0^+$, the following quantitative equivalence holds :*

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})} \simeq \sup_{j\in\mathbb{N}} \|\Theta(-h_j^2\boldsymbol{\Delta})f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}. \tag{62}$$

# 10 Proof of Proposition 21, preliminary lemmas

The proof of Proposition 21 will be developed in the current section and in Sections 11 and 12. Lemmas 23 and 24 are written for the clarity of the exposition but Lemma 25 is the main purpose of the current section.

For any $\vartheta \geq 0$, we define the space $L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$ of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ satisfying

$$\|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1 + |x - x_0|)^\vartheta} < +\infty. \tag{63}$$

The vector space $L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$ is of course independent of $x_0$.

We now recall a few formulas of Fourier analysis. For any $\Psi$ in the Schwartz class $\mathcal{S}(\mathbb{R})$, we use the following convention and inversion formula :

$$\widehat{\Psi}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle}\Psi(x)\mathrm{d}x \qquad \text{and} \qquad \Psi(x) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\widehat{\Psi}(\xi)\frac{\mathrm{d}\xi}{(2\pi)^d}. \tag{64}$$

The semi-classical Fourier multiplier $\Psi(hD)$ for any $h \in (0, 1]$ is defined as follows : for any $\Psi \in \mathcal{S}(\mathbb{R}^d)$, any $f \in L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$, any $h \in (0, 1]$ and $x \in \mathbb{R}^d$, we have the convolution identity

$$(\Psi(hD)f)(x) = \int_{\mathbb{R}^d} f(x - hy)\widehat{\Psi}(-y)\frac{\mathrm{d}y}{(2\pi)^d}. \tag{65}$$

Such a formula[25] shows that, for $f \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$, we indeed have the usual formula

$$\widehat{\Psi(hD)f}(\xi) = \Psi(h\xi)\widehat{f}(\xi). \tag{66}$$

[25] The convergence of the integral is contained in the proof of Lemma 23.

For any $\Phi \in \mathcal{S}(\mathbb{R}^d)$, we deduce the composition formula

$$\Phi(hD) \circ \Psi(hD) = (\Phi\Psi)(hD). \tag{67}$$

We moreover easily check the following formula for any $\ell \in \{1, \ldots, d\}$ :

$$\Big(\frac{\partial}{\partial x_\ell}\{\Psi(hD)f\}\Big)(x) = \frac{1}{h}(\Psi_\ell(hD)f)(x) \qquad \text{with} \quad \Psi_\ell(\xi) = i\xi_\ell\Psi(\xi). \tag{68}$$

We will need the following lemmas.

**Lemma 23.** *Let us consider $\vartheta \in [0, +\infty)$ and $\Psi \in \mathcal{S}(\mathbb{R}^d)$, then for any $h \in (0, 1]$ and $x_0 \in \mathbb{R}^d$, the Fourier multipliers $\Psi(hD)$ are uniformly bounded on $L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$ with respect to $(x_0, h)$ :*

$$\sup_{x_0\in\mathbb{R}^d} \sup_{0<h\le 1} \|\Psi(hD)\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)\to L^\infty_{x_0,\vartheta}(\mathbb{R}^d)} < +\infty.$$

Proof. For any $f \in L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the convolution identity allows to bound

$$\begin{aligned}
|(\Psi(hD)f)(x)| &\le \|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)} \int_{\mathbb{R}^d} (1 + |x - x_0 - hy|)^\vartheta |\widehat{\Psi}(-y)| \frac{\mathrm{d}y}{(2\pi)^d} \\
&\le \|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)} \int_{\mathbb{R}^d} (1 + |x - x_0| + |y|)^\vartheta |\widehat{\Psi}(-y)| \frac{\mathrm{d}y}{(2\pi)^d} \\
&\le \|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)} (1 + |x - x_0|)^\vartheta \int_{\mathbb{R}^d} (1 + |y|)^\vartheta |\widehat{\Psi}(-y)| \frac{\mathrm{d}y}{(2\pi)^d}
\end{aligned}$$

where the last integral is finite because $\widehat{\Psi}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. ☐

**Lemma 24.** *Let $\Psi$ and $g$ be two functions in $\mathcal{S}(\mathbb{R}^d)$ and let us fix a function $f \in L^\infty_{\vartheta,x_0}(\mathbb{R}^d)$ for some $\vartheta \ge 0$. If $\Psi$ is real-valued then the operator $\Psi(D)$ is self-adjoint in the following sense*

$$\int_{\mathbb{R}^d} (\Psi(D)f)(x) \times \overline{g(x)} \mathrm{d}x = \int_{\mathbb{R}^d} f(x) \times \overline{(\Psi(D)g)(x)} \mathrm{d}x. \tag{69}$$

Proof. Formally, the result is expected via the relation (66). Here is indeed a direct argument. Note that both functions in the two sides of (69) are integrable because Lemma 23 ensures that $\Psi(D)f$ belongs to $L^\infty_{\vartheta,x_0}(\mathbb{R}^d)$ and because the standard Fourier analysis (via (65) or (66)) shows that $\Psi(D)g$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. We also remark the formula $\overline{\widehat{\Psi}(-y)} = \widehat{\Psi}(y)$ holding true because $\Psi$ is real-valued. We then conclude with the Fubini theorem, a change of variables and by reformulating (69) as

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x - y)\widehat{\Psi}(-y)\overline{g(x)} \frac{\mathrm{d}x\mathrm{d}y}{(2\pi)^d} = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\overline{g(x - y)}\overline{\widehat{\Psi}(-y)} \frac{\mathrm{d}x\mathrm{d}y}{(2\pi)^d}.$$

We note that, in both sides, the considered functions are integrable on $\mathbb{R}^d \times \mathbb{R}^d$. ☐

**Lemma 25.** *Let us consider $\alpha \in (0, 1), \vartheta \in [0, +\infty)$ and $x_0 \in \mathbb{R}^d$. Let $\Psi$ and $\Phi$ be two functions in the Schwartz class on $\mathbb{R}^d$ and let us consider $f \in L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$ such that there is a constant $M(f) \ge 0$ satisfying for any $k \in \mathbb{N}^\star$*

$$\|\Psi(2^{-k}D)f\|_{L^\infty(B(x_0,2^{-k}))} \le M(f)2^{-k\alpha}. \tag{70}$$

*Then there is a positive constant $C$ (independent of $k$, $x_0$ and $f$) such that for any $k \in \mathbb{N}$ the following inequality holds true*

$$\|\Phi(2^{-k}D) \circ \Psi(2^{-k}D)f\|_{L^\infty(B(x_0,2^{-k}))} \le C\Big(\|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)} + M(f)\Big)2^{-k\alpha}. \tag{71}$$

Proof. For any $g \in L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$ and any $R \ge 0$, the following inequality is obvious (see (63)) :

$$\|g\|_{L^\infty(B(x_0,R))} \le (1 + R)^\vartheta \|g\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)}. \tag{72}$$

Moreover, Lemma 23 shows that the Fourier mulipliers $\Phi(2^{-k}D)$ and $\Psi(2^{-k}D)$ are uniformly bounded (with respect to $k$ and $x_0$) on $L^\infty_{x_0,\vartheta}(\mathbb{R}^d)$. Combining these two simple remarks lead to

$$\|\Phi(2^{-k}D)\circ\Psi(2^{-k}D)f\|_{L^\infty(B(x_0,2^{-k}))} \le C\|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)}.$$

As a consequence, it is sufficient to restrict the proof of (71) for $k\ge 2$. Let us note $f_k = \Psi(2^{-k}D)f$ and let us use the convolution formula for any $x\in\mathbb{R}^d$ :

$$\Big(\Phi(2^{-k}D)\circ\Psi(2^{-k}D)f\Big)(x) = \Big(\Phi(2^{-k}D)f_k\Big)(x) = \int_{\mathbb{R}^d} f_k\Big(x-\frac{y}{2^k}\Big)\widehat{\Phi}(-y)\frac{\mathrm{d}y}{(2\pi)^d}$$

so that we can bound (by using $2^{-k}+2^{K-k}\le 2^{1+K-k}$ for any $K\in\mathbb{N}$) :

$$\begin{aligned}\Big\|\Phi(2^{-k}D)\circ\Psi(2^{-k}D)f\Big\|_{L^\infty(B(x_0,2^{-k}))} \quad\le\quad & \|f_k\|_{L^\infty(B(x_0,2^{1-k}))}\int_{|y|\le 1}|\widehat{\Phi}(-y)|\frac{\mathrm{d}y}{(2\pi)^d}\\ &+\sum_{K=1}^{+\infty}\|f_k\|_{L^\infty(B(x_0,2^{1+K-k}))}\int_{2^{K-1}\le|y|\le 2^K}|\widehat{\Phi}(-y)|\frac{\mathrm{d}y}{(2\pi)^d}.\end{aligned}$$

Since $\widehat{\Phi}$ belongs to the Schwartz space, we can bound $|\widehat{\Phi}(y)|\le\dfrac{C}{(1+|y|)^{1+d+\vartheta+\alpha}}$, for a suitable constant $C$ independent of $(k,K,x_0)$, and so we get (upon changing $C$)

$$\Big\|\Phi(2^{-k}D)\circ\Psi(2^{-k}D)f\Big\|_{L^\infty(B(x_0,2^{-k}))}\le C\sum_{K=0}^{+\infty}\frac{\|f_k\|_{L^\infty(B(x_0,2^{1+K-k}))}}{2^{(1+\vartheta+\alpha)K}}.$$

We shall cut the last sum by separating the cases $K\ge k-1$ and $0\le K\le k-2$.

**Case $K\ge k-1$.** Thanks to (72) and Lemma 23, we get for any $R\ge 0$ :

$$\|f_k\|_{L^\infty(B(x_0,R))}\le C(1+R)^\vartheta\|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)}.$$

By bounding $1+K-k\le 1+K$, we thus obtain

$$\sum_{K=k-1}^{+\infty}\frac{\|f_k\|_{L^\infty(B(x_0,2^{1+K-k}))}}{2^{(1+\vartheta+\alpha)K}}\le C\|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)}\sum_{K=k-1}^{+\infty}\frac{(1+2^{1+K})^\vartheta}{2^{(1+\vartheta+\alpha)K}}\le C\frac{\|f\|_{L^\infty_{x_0,\vartheta}(\mathbb{R}^d)}}{2^{k(1+\alpha)}},$$

which is much better than the bound $\mathcal{O}(2^{-k\alpha})$ expected in (71).

**Case $0\le K\le k-2$.** We consider the remaining terms thanks to the assumption (70) and we shall see that the rate $2^{-k\alpha}$ is the good one :

$$\begin{aligned}\sum_{K=0}^{k-2}\frac{\|f_k\|_{L^\infty(B(x_0,2^{1+K-k}))}}{2^{(1+\vartheta+\alpha)K}} \quad&\le\quad M(f)\sum_{K=0}^{k-2}\frac{2^{(1+K-k)\alpha}}{2^{(1+\vartheta+\alpha)K}}\\ &\le\quad\frac{M(f)}{2^{k\alpha}}\sum_{K=0}^{+\infty}\frac{2^\alpha}{2^{(1+\vartheta)K}}\\ &\le\quad\frac{M(f)}{2^{k\alpha}}\sum_{K=0}^{+\infty}\frac{2}{2^K}=\frac{4M(f)}{2^{k\alpha}}.\end{aligned}$$

□

# 11 Proof of Proposition 21, part A

We will control the right-hand side of (60) by $\|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)}$. Due to (57), we merely have to show for any $j\in\mathbb{N}$

$$\sup_{k\ge j}\|f-\Theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-j}))}\lesssim 2^{-\alpha j}\|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)}. \tag{73}$$

We write $\Theta(-2^{-2k}\Delta)f = \widetilde{\Theta}(2^{-k}D)f$ with $\widetilde{\Theta}(\xi) = \Theta(|\xi|^2)$. Hence (64) gives us

$$\int_{\mathbb{R}^d} \widehat{\widetilde{\Theta}}(y) \frac{\mathrm{d}y}{(2\pi)^d} = \widetilde{\Theta}(0) = \Theta(0) = 1.$$

Using the convolution expression (65) of $\widetilde{\Theta}(2^{-k}D)f$, we get for any $x \in B(x_0, 2^{-j})$ the following bound

$$\begin{aligned} f(x) - \Big(\Theta(-2^{-2k}\Delta)f\Big)(x) &= f(x) - f(x_0) + \int_{\mathbb{R}^d} \Big(f(x_0) - f\Big(x - \frac{y}{2^k}\Big)\Big)\widehat{\widetilde{\Theta}}(-y)\frac{\mathrm{d}y}{(2\pi)^d} \\ \Big|f(x) - \Big(\Theta(-2^{-2k}\Delta)f\Big)(x)\Big| &\leq \|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)}\Big(|x-x_0|^\alpha + \int_{\mathbb{R}^d}\Big|x - x_0 - \frac{y}{2^k}\Big|^\alpha |\widehat{\widetilde{\Theta}}(-y)|\frac{\mathrm{d}y}{(2\pi)^d}\Big) \\ &\leq \|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)}\Big(|x-x_0|^\alpha + \int_{\mathbb{R}^d}(|x-x_0|^\alpha + 2^{-k\alpha}|y|^\alpha)|\widehat{\widetilde{\Theta}}(-y)|\frac{\mathrm{d}y}{(2\pi)^d}\Big). \end{aligned}$$

Since the Fourier transform of $\widetilde{\Theta}$ belongs to the Schwartz class, one may bound $|y|^\alpha \leq 1 + |y|$ to see that there is a constant $C_\Theta > 0$ merely depending on $\Theta$ such that the following holds

$$\|f - \Theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-j}))} \leq C_\Theta \|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)}(2^{-\alpha j} + 2^{-k\alpha}).$$

Such an estimate implies (73).

## 12 Proof of Proposition 21, part B

We will control $\|f\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)}$ by the right-hand side of (60). Thanks to (57), we have to prove

$$\|\!|f|\!\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)} \lesssim \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1+|x-x_0|)^\alpha} + \sup_{j\in\mathbb{N}} 2^{\alpha j}\Big(\sup_{k\geq j}\|f - \Theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-j}))}\Big). \tag{74}$$

**Step 1.** Let us briefly explain the strategy : let us imagine that we are able to prove that there is a constant $C_1(f) \geq 0$ such that for any $j \in \mathbb{N}$, for any functions $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ with support in the ball $B(x_0, 2^{-j})$ we have

$$\begin{aligned} &\left|\int_{B(x_0,2^{-j})\times B(x_0,2^{-j})} \Big(f(x) - f(x')\Big)\overline{\varphi(x)\psi(x')}\mathrm{d}x\mathrm{d}x'\right| \\ &\qquad \leq C_1(f)2^{-\alpha j}\int_{B(x_0,2^{-j})}|\varphi(x)|\mathrm{d}x\int_{B(x_0,2^{-j})}|\psi(x')|\mathrm{d}x'. \end{aligned}\tag{75}$$

Let us now see a consequence of such an assertion. For any $(a, a')$ inside the product of balls $B(x_0, 2^{-j}) \times B(x_0, 2^{-j})$, one can consider two approximations of identity $\varphi_n$ and $\psi_n$ on $\mathbb{R}^d$ which respectively tend to the Dirac measures $\delta_a$ and $\delta_{a'}$. In particular, the sequence $(\varphi_n \otimes \psi_n)_n$ is an approximation of identity on $\mathbb{R}^d \times \mathbb{R}^d$ which tends to $\delta_{(a,a')}$. Then, since $f$ is assumed to be continuous in Proposition 21, (75) would imply

$$|f(a) - f(a')| \leq C_1(f)2^{-\alpha j}.$$

Hence, (58) would show

$$\|\!|f|\!\|_{\mathcal{C}^{0,\alpha}_{x_0}(\mathbb{R}^d)} \leq 2C_1(f). \tag{76}$$

In other words, our strategy is to show that we may choose in (75) a constant $C_1(f)$ which is bounded by the right-hand side of (74).

**Step 2.** We now want to begin the proof of (75) for a suitable constant $C_1(f) \geq 0$. Since the right-hand side of (74) may be assumed to be finite, the function $f$ belongs to the space $L^\infty_{x_0,\alpha}(\mathbb{R}^d)$ introduced in (63). Let us cut the Littlewood-Paley decomposition of $f$ as follows. For any $p \in \mathbb{N}$ we may write thanks to (59) :

$$f = S_p f + R_p f \qquad \text{with} \qquad S_p f = \Theta(-\Delta)f + \sum_{k=1}^{p}\theta(-2^{-2k}\Delta)f = \Theta(-2^{-2p}\Delta)f. \tag{77}$$

We now explain why (75) will come from the following inequalities for any $p \geq j$ and any $(x, x') \in B(x_0, 2^{-j}) \times B(x_0, 2^{-j})$ (for suitable constants $C_2(f)$ and $C_3(f)$) :

$$\begin{aligned} |(S_p f - S_j f)(x) - (S_p f - S_j f)(x')| &\leq C_2(f) 2^{-\alpha j}, &(78)\\ |S_j f(x) - S_j f(x')| &\leq C_3(f) 2^{-\alpha j}. &(79) \end{aligned}$$

Actually, by using the Fubini theorem and the self-adjointness of the Fourier multiplier $S_p$ (see Lemma 24 and the identity (77)), we obtain

$$\begin{aligned} &\int_{B(x_0,2^{-j})\times B(x_0,2^{-j})} \Big(S_p f(x) - S_p f(x')\Big)\overline{\varphi(x)\psi(x')}\mathrm{d}x\mathrm{d}x' \qquad (80)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(S_p f(x) - S_p f(x')\Big)\overline{\varphi(x)\psi(x')}\mathrm{d}x\mathrm{d}x'\\ &= \int_{\mathbb{R}^d} \Big(S_p f(x)\Big)\overline{\varphi(x)}\mathrm{d}x \int_{\mathbb{R}^d} \overline{\psi(x')}\mathrm{d}x' - \int_{\mathbb{R}^d} \overline{\varphi(x)} \int_{\mathbb{R}^d} \Big(S_p f(x')\Big)\overline{\psi(x')}\mathrm{d}x'\\ &= \int_{\mathbb{R}^d} f(x)\overline{S_p\varphi(x)}\mathrm{d}x \int_{\mathbb{R}^d} \overline{\psi(x')}\mathrm{d}x' - \int_{\mathbb{R}^d} \overline{\varphi(x)} \int_{\mathbb{R}^d} f(x')\overline{S_p\psi(x')}\mathrm{d}x'. \qquad (81) \end{aligned}$$

Taking account (79) and (78), we see that the integral (80) is bounded by

$$(C_2(f) + C_3(f))2^{-\alpha j}\|\varphi\|_{L^1(\mathbb{R}^d)}\|\psi\|_{L^1(\mathbb{R}^d)}.$$

Since $\varphi$ and $\psi$ belong to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, the following formulas (see (77) and (66))

$$\widehat{S_p\varphi}(\xi) = \Theta(2^{-2p}|\xi|^2)\widehat{\varphi}(\xi) \qquad \text{and} \qquad \widehat{S_p\psi}(\xi) = \Theta(2^{-2p}|\xi|^2)\widehat{\psi}(\xi)$$

easily show the convergences $S_p\varphi \to \varphi$ and $S_p\psi \to \psi$ in $\mathcal{S}(\mathbb{R}^d)$ as $p \to +\infty$. Remembering now that the bound $|f(x)| \lesssim (1 + |x|)^\alpha$ holds true, we get the convergences $f\overline{S_p\varphi} \to f\overline{\varphi}$ and $f\overline{S_p\psi} \to f\overline{\psi}$ in $L^1(\mathbb{R}^d)$. By making $p$ tend to $+\infty$ in the integrals (81), we obtain (75) for

$$C_1(f) = C_2(f) + C_3(f). \qquad (82)$$

It finally remains to get suitable constants in (79) and (78).

**Step 3, proof of** (78)**.** We write $S_p f - S_j f = (S_p f - f) - (S_j f - f)$ and note the equality $f - S_p(f) = f - \Theta(-2^{-2p}\Delta)f$ thanks to (77). In other words, we have for any $j \in \mathbb{N}$ and $p \geq j$ :

$$|(S_p f - S_j f)(x) - (S_p f - S_j f)(x')| \leq 4 \sup_{k\geq j} \|f - \Theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-j}))}.$$

Looking at (74), we get (78) for the choice

$$C_2(f) = 4 \sup_{j\in\mathbb{N}} 2^{\alpha j}\Big(\sup_{k\geq j} \|f - \Theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-j}))}\Big). \qquad (83)$$

**Step 4, proof of** (79)**.** Thanks to (77) and to the inequality $|x - x'| \leq |x - x_0| + |x' - x_0| \leq \frac{2}{2^j}$, we can bound $|S_j f(x) - S_j f(x')|$ by

$$\begin{aligned} &|x - x'| \sup_{|y-x_0|<\frac{1}{2^j}} \Big(\|\nabla\{\Theta(-\Delta)f\}(y)\| + \sum_{k=1}^{j} \|\nabla\{\theta(-2^{-2k}\Delta)f\}(y)\|\Big)\\ &= |x - x'| \sup_{|y-x_0|<\frac{1}{2^j}} \left(\sqrt{\sum_{\ell=1}^{d}\left|\Big(\frac{\partial\{\Theta(-\Delta)f\}}{\partial y_\ell}\Big)(y)\right|^2} + \sum_{k=1}^{j}\sqrt{\sum_{\ell=1}^{d}\left|\Big(\frac{\partial\{\theta(-2^{-2k}\Delta)f\}}{\partial y_\ell}\Big)(y)\right|^2}\right)\\ &\leq \frac{2}{2^j}\sum_{\ell=1}^{d}\left(\sup_{|y-x_0|<1}\left|\Big(\frac{\partial\{\Theta(-\Delta)f\}}{\partial y_\ell}\Big)(y)\right| + \sum_{k=1}^{j}\sup_{|y-x_0|<\frac{1}{2^k}}\left|\Big(\frac{\partial\{\theta(-2^{-2k}\Delta)f\}}{\partial y_\ell}\Big)(y)\right|\right). \qquad (84) \end{aligned}$$

We then use (68) to write

$$\frac{\partial\{\Theta(-\Delta)f\}}{\partial y_\ell}(y) = \Big(\Theta_\ell(D)f\Big)(y) \qquad \text{with} \qquad \Theta_\ell(\xi) = i\xi_\ell\Theta(|\xi|^2).$$

Since $\Theta_\ell$ belongs to $\mathcal{C}_c^\infty(\mathbb{R}^d)$, we can apply Lemma 23 to get

$$\left\| \frac{\partial}{\partial y_\ell}\{\Theta(-\Delta)f\}\right\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} \le C\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)}.$$

Then the bound (72) allows to deal with the first term in the upper bound (84) :

$$\left\| \frac{\partial}{\partial y_\ell}\{\Theta(-\Delta)f\}\right\|_{L^\infty(B(x_0,1))} \le 2^\alpha \left\| \frac{\partial}{\partial y_\ell}\{\Theta(-\Delta)f\}\right\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} \le 2C\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)}. \tag{85}$$

Let us explain how other terms in (84) can be similarly understood. Actually, we need a sort of analogue of [KN12, Lemma 4.2] allowing to bound $\left\|\frac{\partial}{\partial y_\ell}\{\theta(-2^{-2k}\Delta)f\}\right\|_{L^\infty(B(x_0,2^{-k}))}$ in (84). We may factorize $i\xi_\ell\theta(|\xi|^2) = \Phi(\xi)\Psi(\xi)$ with

- $\Psi(\xi) = \theta(|\xi|^2)$,
- $\Phi : \mathbb{R}^d \to \mathbb{C}$ a smooth compactly supported function that coincides with $\xi \mapsto i\xi_\ell$ on the support of $\Psi(\xi) = \theta(|\xi|^2)$.

As a consequence of (68) for $h = 2^{-k}$ and of (67), we have

$$\frac{\partial\{\theta(-2^{-2k}\Delta)f\}}{\partial y_\ell}(y) = \frac{\partial\{\Psi(hD)f\}}{\partial y_\ell}(y) = \Big(2^k\Phi(2^{-k}D)\circ\Psi(2^{-k}D)f\Big)(y). \tag{86}$$

We now aim to apply Lemma 25 whose assumption consists of bounding $\theta(-2^{-2k}\Delta)f$ on the ball $B(x_0, 2^{-k})$ for any $k \in \mathbb{N}^\star$. We write

$$\begin{aligned} \theta(-2^{-2k}\Delta)f &= S_kf - S_{k-1}f = (S_kf - f) - (S_{k-1}f - f) \\ \|\theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-k}))} &\le \|(S_kf - f)\|_{L^\infty(B(x_0,2^{-k}))} + \|(S_{k-1}f - f)\|_{L^\infty(B(x_0,2^{-(k-1)}))}. \end{aligned}$$

In particular, we get

$$\begin{aligned} &2^{k\alpha}\|\theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-k}))} \\ &\quad\le 2^{k\alpha}\|(S_kf - f)\|_{L^\infty(B(x_0,2^{-k}))} + 2^\alpha 2^{(k-1)\alpha}\|(S_{k-1}f - f)\|_{L^\infty(B(x_0,2^{-(k-1)}))} \\ &\quad\le 3\sup_{j\in\mathbb{N}} 2^{j\alpha}\|S_jf - f\|_{L^\infty(B(x_0,2^{-j}))} \\ &\quad\le 3\sup_{j\in\mathbb{N}} 2^{j\alpha}\Big(\sup_{p\ge j}\|S_pf - f\|_{L^\infty(B(x_0,2^{-j}))}\Big). \end{aligned}$$

In other words, $\Psi$ satisfies (70) with

$$M(f) = 3\sup_{j\in\mathbb{N}} 2^{\alpha j}\Big(\sup_{k\ge j}\|f - \Theta(-2^{-2k}\Delta)f\|_{L^\infty(B(x_0,2^{-j}))}\Big). \tag{87}$$

Then (86) and Lemma 25 show

$$\left\|\frac{\partial}{\partial y_\ell}\{\theta(-2^{2k}\Delta)f\}\right\|_{L^\infty(B(x_0,2^{-k}))} \le C\Big(\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} + M(f)\Big)2^{k-k\alpha}.$$

Combining the previous bound with (85) and (84), , we see that $|S_jf(x) - S_jf(x')|$ is less than or equal to (upon changing $C$)

$$\begin{aligned} &\frac{2}{2^j}\left(C\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} + C\Big(\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} + M(f)\Big)\sum_{k=1}^j 2^{k-k\alpha}\right) \\ &\quad\le \frac{C}{2^j}\Big(\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} + M(f)\Big)\Big(1 + \sum_{k=1}^j 2^{k-k\alpha}\Big) \\ &\quad\le \frac{C}{2^j}\Big(\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} + M(f)\Big)2^{j(1-\alpha)} = C\Big(\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} + M(f)\Big)2^{-j\alpha}. \end{aligned}$$

This is exactly (79) with $C_3(f) = C\Big(\|f\|_{L^\infty_{x_0,\alpha}(\mathbb{R}^d)} + M(f)\Big)$.

**Conclusion.** Thanks to (82), (83) and (87), we may conclude by plugging $C_1(f) = C_2(f) + C_3(f)$ in (76) (the strategy explained in Step 1 is finally realized).

## 13 Proof of Proposition 22, part A

The core of the proof of Proposition 22 will be the following quantitative equivalence :

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})} \simeq \|\Theta(-\mathbf{\Delta})f\|_{L^\infty(\mathcal{M})} + \sup_{j\geq 1} 2^{j\alpha}\|\theta(-2^{-2j}\mathbf{\Delta})f\|_{L^\infty(\mathcal{M})}. \tag{88}$$

The goal of the current section is to prove (61) which turns out to be a direct consequence of (88) since $\Theta(-\mathbf{\Delta})f$ automatically belongs to $L^\infty(\mathcal{M})$ (note that $\Theta(-\mathbf{\Delta})f$ is a finite linear combination of eigenfunctions of $\mathbf{\Delta}$).

The sense $\lesssim$ of (88) can be proved *mutatis mutandis* by following the proof in [AG07, page 81, Proposition 1.3, ii)]. The mere additional ingredients are the following ones :

- in order to work in a local chart, we have to keep in mind that the $\alpha$-Hölder regularity is indeed a local property, one may easily check that for any finite open cover $\mathcal{M} = \bigcup_{i=1}^{N}$ we have the equivalence

$$f \in \mathcal{C}^{0,\alpha}(\mathcal{M}) \qquad \Leftrightarrow \qquad \sup_{1\leq i\leq N} \|f_{|V_i}\|_{\mathcal{C}^{0,\alpha}(V_i)} < +\infty.$$

- a semi-classical Bernstein inequality holds true on $\mathcal{M}$ : for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^+,\mathbb{R})$, any $f \in L^\infty(\mathcal{M})$ and any $h \in (0,1)$ we have

$$\|\nabla\Psi(-h^2\mathbf{\Delta})f\|_{L^\infty(\mathcal{M})} \leq \frac{C}{h}\|f\|_{L^\infty(\mathcal{M})}.$$

  Such a semi-classical Bernstein inequality is known (we refer to [IO22, Theorem 2.2 and Section 8.4] and [FM10, Theorem 2.1] but also [Ime19, Theorem D.1] for similar bounds for other operators).

Let us now prove the sense $\gtrsim$ of (88). We shall use a strategy of the paper [BGT04] consisting in transferring a Littlewood-Paley theory on $\mathbb{R}^d$ to a compact manifold (see also [Ime19, Section 9] for a use of that strategy for $BMO$). Such a technique needs the semi-classical functional calculus of the Laplace-Beltrami operator (we refer to [AG07, Chapter I] or [Hör85, Chapter XVIII] for the theory of pseudo-differential operators and the notations of the form $\psi_j(x, hD)$ in the next statement and proofs). Actually [BGT04, Proposition 2.1] implies the following result for the semi-classical operators $\psi(-h^2\mathbf{\Delta})$.

**Proposition 26.** *Let us consider $\psi \in \mathcal{C}_c^\infty((0,+\infty),\mathbb{R})$ being constant near 0, let $\rho : U \subset \mathbb{R}^d \to V \subset \mathcal{M}$ be a local chart of $\mathcal{M}$, let us fix $\chi_1$ and $\chi_2$ two functions in $\mathcal{C}_c^\infty(V)$ satisfying $\chi_2 = 1$ in a neighborhood of the support of $\chi_1$. Then there exists a sequence of functions $(\psi_k)_{k\geq 0}$ in $\mathcal{C}_c^\infty(U\times\mathbb{R}^d)$ such that for any integer $N \in \mathbb{N}^\star$, for any $h \in (0,1]$, any $\sigma \in [0,N]$ and any $f \in \mathcal{C}^\infty(\mathcal{M})$, we have the bound*

$$\Big\|\Big(\chi_1\psi(-h^2\mathbf{\Delta})f\Big)\circ\rho - \sum_{k=0}^{N-1} h^k\psi_k(x,hD)((\chi_2 f)\circ\rho)\Big\|_{H^\sigma(\mathbb{R}^d)} \leq C_N h^{N-\sigma}\|f\|_{L^2(\mathcal{M})},$$

*where $H^\sigma(\mathbb{R}^d)$ stands for the usual Sobolev spaces. Moreover, the support of $\psi_k(x,\xi)$ is supported in the ring $\frac{1}{C} \leq |\xi| \leq C$ for a suitable constant $C \geq 1$ independent of $k$.*

To get the sense $\gtrsim$ of (88), we begin by fixing a function $f \in \mathcal{C}^{0,\alpha}(\mathcal{M})$.

The term $\Theta(-\mathbf{\Delta})f$ in (88) can be directly handled by using the continuity of the operator $\Theta(-\mathbf{\Delta}) : L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M})$ (see [BGT04, Corollary 2.2]) :

$$\|\Theta(-\mathbf{\Delta})f\|_{L^\infty(\mathcal{M})} \leq C\|f\|_{L^\infty(\mathcal{M})} \leq C'\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

Let us now focus on the terms $\theta(-2^{-2j}\mathbf{\Delta})f$ in (88). By introducing a finite open cover of $\mathcal{M}$, one sees that it is sufficient to prove that for any local chart $\rho : U \subset \mathbb{R}^d \to V \subset \mathcal{M}$ and any $\chi_1 \in \mathcal{C}_c^\infty(V)$ the following inequality holds true

$$\Big\|\Big(\chi_1\theta(-2^{-2j}\mathbf{\Delta})f\Big)\circ\rho\Big\|_{L^\infty(U)} \leq C2^{-j\alpha}\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

Let $\chi_2 \in \mathcal{C}_c^\infty(V)$ be a function satisfying $\chi_2 = 1$ on a neighborhood of the support of $\chi_1$. Let us moreover remember that $\theta$ has compact support in $(0,+\infty)$. By using the notations of the semi-classical functional calculus given by Proposition 26 for $h = 2^{-j}$, the following equality holds true for any fixed integer $N \in \mathbb{N}^\star$ :

$$\Big(\chi_1\theta(-2^{-2j}\mathbf{\Delta})f\Big)\circ\rho = \sum_{k=0}^{N-1} 2^{-kj}\psi_k(x,2^{-j}D)((\chi_2 f)\circ\rho) + R_j$$

in which the remainder $R_j$ fulfills the bound $\|R_j\|_{H^{N-1}(\mathbb{R}^d)} \lesssim 2^{-j}\|f\|_{L^2(\mathcal{M})}$. If one chooses $N > 1 + \frac{d}{2}$, the usual Sobolev embedding $H^{N-1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and the simple embedding $\mathcal{C}^{0,\alpha}(\mathcal{M}) \subset L^2(\mathcal{M})$ show

$$\|R_j\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-j}\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})} \lesssim 2^{-j\alpha}\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

The proof will be finished if one succeeds to show the bounds (for a fixed $k \in \{0, \dots, N-1\}$) :

$$\|\psi_k(x, 2^{-j}D)((\chi_2 f) \circ \rho)\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-j\alpha}\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}. \tag{89}$$

We note that the function $(\chi_2 f) \circ \rho$ belongs to $\mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ and thus we have for any $\ell \in \mathbb{N}^\star$ :

$$\Big\|\theta(-2^{-2\ell}\Delta)\Big((\chi_2 f) \circ \rho\Big)\Big\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-\ell\alpha}\|(\chi_2 f) \circ \rho\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^d)} \lesssim 2^{-\ell\alpha}\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})} \tag{90}$$

(this is a quantitative estimate coming from the proof of [AG07, page 81, Proposition 1.3], see below (59)). And hence the uniform convergence on $\mathbb{R}^d$ of its Littlewood-Paley series holds :

$$(\chi_2 f) \circ \rho = \Theta(-\Delta)\Big((\chi_2 f) \circ \rho)\Big) + \sum_{\ell=1}^{+\infty} \theta(-2^{-2\ell}\Delta)\Big((\chi_2 f) \circ \rho\Big). \tag{91}$$

Since it is known that the pseudo-differential operator $\psi_k(x, 2^{-j}D)$ is bounded on $L^\infty(\mathbb{R}^d)$ (see [Ime19, line (9.6)]), we may write for any $j \geq 1$

$$\begin{aligned}\psi_k(x, 2^{-j}D)\Big((\chi_2 f) \circ \rho)\Big) &= \psi_k(x, 2^{-j}D)\Theta(-\Delta)\Big((\chi_2 f) \circ \rho)\Big) \\ &\quad + \sum_{\ell \geq 1} \psi_k(x, 2^{-j}D) \circ \theta(-2^{-2\ell}\Delta)\Big((\chi_2 f) \circ \rho\Big).\end{aligned}$$

Thanks to the last statement of Proposition 26, the support of $\psi_k(x, 2^{-j}\xi)$ is inside a ring of the form $\frac{1}{C}2^j \leq |\xi| \leq C2^j$. As a consequence, the symbol $\psi_k(x, 2^{-j}\xi)$ is controlled by $2^{-j}$ for any semi-norm in the pseudo-differential Hörmander class $S^1_{1,0}$ : for any $(a, b) \in \mathbb{N}^d \times \mathbb{N}^d$ we have

$$\sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} (1 + |\xi|)^{-1+|b|}|\partial_x^a\partial_\xi^b\{\psi_k(x, 2^{-j}\xi)\}| \lesssim 2^{-j}.$$

Note also that the symbol of $\theta(-2^{-2\ell}\Delta)$, namely $\theta(2^{-2\ell}|\xi|^2)$, has support in a ring of the form $\frac{1}{C}2^\ell \leq |\xi| \leq C2^\ell$ (because $\theta$ vanishes near 0). For the same reason, we have

$$\sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} (1 + |\xi|)^{-1+|b|}|\partial_x^a\partial_\xi^b\{\theta(2^{-2\ell}|\xi|^2)\}| \lesssim 2^{-\ell}.$$

The previous support conditions imply that one may find a fixed constant $\nu$ large enough such that the support of the symbols $\psi_k(x, 2^{-j}\xi)$ and $\theta(2^{-2\ell}|\xi|^2)$ are disjoint provided that $|j - \ell| > \nu$ is assumed. Following the same argument as that of [Ime19, pages 2760-2761] based on symbolic calculus, we may bound

$$\|\psi_k(x, 2^{-j}D) \circ \theta(-2^{-2\ell}\Delta)\|_{L^2(\mathbb{R}^d)\to H^{N-1}(\mathbb{R}^d)} \lesssim 2^{-j-\ell}.$$

As a consequence, we get the following bounds (uniformly in $j$) :

$$\begin{aligned}\Big\|\sum_{\substack{\ell\geq 1\\ |j-\ell|\geq\nu}} \psi_k(x, 2^{-j}D) \circ \theta(-2^{-2\ell}\Delta)\Big((\chi_2 f) \circ \rho\Big)\Big\|_{H^{N-1}(\mathbb{R}^d)} &\lesssim \sum_{\ell=1}^{+\infty} 2^{-j-\ell}\|(\chi_2 f) \circ \rho\|_{L^2(\mathbb{R}^d)} \\ &\lesssim 2^{-j}\|f\|_{L^2(\mathcal{M})} \\ &\lesssim 2^{-j}\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.\end{aligned}$$

The same support argument gives the same following upper bound for the first term of the Littlewood-Paley decomposition (91) :

$$\Big\|\psi_k(x, 2^{-j}D)\Theta(-\Delta)\Big((\chi_2 f) \circ \rho\Big)\Big\|_{H^{N-1}(\mathbb{R}^d)} \lesssim 2^{-j}\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

It remains to deal with the part $|j-\ell|\leq\nu$ which turns out to contain a finite number of terms. Since $k$ runs over a finite set and since, we may use again the uniform boundedness on $L^\infty(\mathbb{R}^d)$ of each pseudo-differential operator $\psi_k(x,2^{-j}D)$ to get

$$\Big\|\sum_{\substack{\ell\geq1\\|j-\ell|\leq\nu}}\psi_k(x,2^{-j}D)\circ\theta(-2^{-2\ell}\Delta)\Big((\chi_2 f)\circ\rho\Big)\Big\|_{L^\infty(\mathbb{R}^d)}\lesssim\sum_{\substack{\ell\geq1\\|j-\ell|\leq\nu}}\Big\|\theta(-2^{-2\ell}\Delta)\Big((\chi_2 f)\circ\rho\Big)\Big\|_{L^\infty(\mathbb{R}^d)}.$$

Finally, (90) achieves the proof of the expected bounds (89).

# 14 Proof of Proposition 22, part B

It remains to prove the equivalence (62). We will still use the equivalence (88) proved in the previous section.

Since $\Theta\equiv1$ near $0^+$ and since $\theta$ has compact support, we directly see that, for an $j\in\mathbb{N}^\star$, one may find $\ell\in\mathbb{N}$ such that $2^jh_\ell$ is small enough so that the following holds true

$$\forall\lambda\geq0\qquad\theta(2^{-2j}\lambda)\Theta(h_\ell^2\lambda)=\theta(2^{-2j}\lambda).$$

As a consequence of the previous equality and of (88), we get

$$\begin{aligned}\sup_{j\geq1}2^{j\alpha}\|\theta(-2^{-2j}\boldsymbol{\Delta})f\|_{L^\infty(\mathcal{M})}&\leq\sup_{j\geq1}\sup_{\ell\in\mathbb{N}}2^{j\alpha}\|\theta(-2^{-2j}\boldsymbol{\Delta})\Theta(-h_\ell^2\boldsymbol{\Delta})f\|_{L^\infty(\mathcal{M})}\\&\leq\sup_{\ell\in\mathbb{N}}\sup_{j\geq1}2^{j\alpha}\|\theta(-2^{-2j}\boldsymbol{\Delta})\Theta(-h_\ell^2\boldsymbol{\Delta})f\|_{L^\infty(\mathcal{M})}\\&\leq C\sup_{\ell\in\mathbb{N}}\|\Theta(-h_\ell^2\boldsymbol{\Delta})f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.\end{aligned}$$

Similarly, (88) and a factorization of the form $\Theta(-\boldsymbol{\Delta})=\Theta(-\boldsymbol{\Delta})\Theta(-h_\ell^2\boldsymbol{\Delta})$ imply

$$\|\Theta(-\boldsymbol{\Delta})f\|_{L^\infty(\mathcal{M})}\leq C\sup_{\ell\in\mathbb{N}}\|\Theta(-h_\ell^2\boldsymbol{\Delta})f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

So (88) gives

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}\leq C\sup_{\ell\in\mathbb{N}}\|\Theta(-h_\ell^2\boldsymbol{\Delta})f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

Let us explain how to reverse this inequality. We use (88) to bound $\sup_{0<h\leq1}\|\Theta(-h^2\boldsymbol{\Delta})f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}$ by

$$\sup_{0<h\leq1}\|\Theta(-\boldsymbol{\Delta})\Theta(-h^2\boldsymbol{\Delta})f\|_{L^\infty(\mathcal{M})}+\sup_{0<h\leq1}\sup_{j\geq1}2^{j\alpha}\|\theta(-2^{-2j}\boldsymbol{\Delta})\Theta(-h^2\boldsymbol{\Delta})f\|_{L^\infty(\mathcal{M})}.$$

We now remember the equalities

$$\Theta(-\boldsymbol{\Delta})\Theta(-h^2\boldsymbol{\Delta})=\Theta(-h^2\boldsymbol{\Delta})\Theta(-\boldsymbol{\Delta})\quad\text{and}\quad\theta(-2^{-2j}\boldsymbol{\Delta})\Theta(-h^2\boldsymbol{\Delta})=\Theta(-h^2\boldsymbol{\Delta})\theta(-2^{-2j}\boldsymbol{\Delta})$$

and we invoke the uniform boundedness, with respect to $h\in(0,1]$, of the operators $\Theta(-h^2\boldsymbol{\Delta})$ from the space $L^\infty(\mathcal{M})$ to itself (see [BGT04, Corollary 2.2]). We again use a last time (88) to get the inequality

$$\sup_{0<h\leq1}\|\Theta(-h^2\boldsymbol{\Delta})f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}\leq C\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

# 15 Proof of Proposition 12 and final step of Theorem 4

At this stage of the article, we recall that Proposition 12 had been admitted for proving Theorem 4. The goal of this part is to complete this missing point, namely to show, under the assumption $(a_n)_{n\geq1}\in\ell^2(\mathbb{N}^\star)$, that the function

$$\begin{aligned}f^\omega:\mathbb{R}&\to\mathbb{C}\\t&\mapsto\sum_{n\geq1}g_n(\omega)\tfrac{a_n}{\sqrt{\lambda_n}}e^{i\lambda_nt}\end{aligned}$$

is almost surely given by a uniformly convergent series on compact subsets of $[0,+\infty)$ (see Proposition 27 below) and is almost surely pointwise $\frac{1}{2}$-Hölder at $t=0$. For this last point, thanks to Proposition 21, we have to prove that

- the random function $f^\omega$ is almost surely continuous on $\mathbb{R}$ (this is of course a consequence of the uniform convergence on compact subsets, see Proposition 27 below),

- the random function $f^\omega$ has a growth like $|f^\omega(t)| \leq C_\omega \sqrt{1+|t|}$ (see Corollary 29),

- with probability 1, $f^\omega$ satisfies the following bounds

$$\sup_{j\in\mathbb{N}} \sqrt{2^j}\Big(\sup_{k\geq j} \|f^\omega - \Theta(-2^{-2k}\Delta)f^\omega\|_{L^\infty(-2^{-j},2^{-j})}\Big) < +\infty. \tag{92}$$

The bounds (92) will be proved at the end of the current section. For all these points, we need the next result.

**Proposition 27.** *Let $(a_n)_{n\geq 1}$ be a sequence in $\ell^2(\mathbb{N}^\star)$ and let us fix an exponent $\alpha \in \big(0,\frac{1}{2}\big]$ and an increasing positive sequence $(\lambda_n)$ satisfying $\lambda_n \simeq n$. With probability 1, the Gaussian random series $\sum g_n(\omega)\frac{a_n}{\lambda_n^\alpha}e^{i\lambda_n t}$ uniformly converges on any compact subset of $\mathbb{R}$. More precisely, for any $K>0$ we have the general bound*

$$\mathbf{E}\Big[\sup_{|t|\leq K}\Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha}e^{i\lambda_n t}\Big|\Big] \leq C\Big(\sum_{n\geq 1}\frac{|a_n|^2}{\lambda_n^{2\alpha}}\Big)^{\frac{1}{2}} + CK^\alpha\Big(\sum_{n\geq 1}|a_n|^2\Big)^{\frac{1}{2}}. \tag{93}$$

*For any $K \gg 1$ we have*

$$\mathbf{E}\Big[\sup_{|t|\leq K}\Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha}e^{i\lambda_n t}\Big|\Big] \leq C\sqrt{\ln(K)}\Big(\sum_{n\geq 1}|a_n|^2\Big)^{\frac{1}{2}}. \tag{94}$$

*The same conclusion obviously holds true for $\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha}e^{-i\lambda_n t}$.*

**Remark 28.** *We have not pursued the best bounds but the last ones are sufficient for our purpose. Actually, here are two simple but opposed examples that show that improving the last bounds, at least for $K\to+\infty$, would need more informations :*

- *if the series $\sum \frac{a_n}{\lambda_n^\alpha}$ is absolutely convergent then the expectations in Proposition 27 are directly bounded independently of $K$;*

- *another interesting situation is the case $\lambda_n = \sqrt{n(n+1)}$ for $\alpha = \frac{1}{2}$. Let us fix a real sequence $(a_n) \in \ell^2(\mathbb{N}^\star)$ such that $\sum_{n\geq 1}\frac{|a_n|}{\sqrt{\lambda_n}} = +\infty$ (for instance $a_n = \frac{1}{\sqrt{n}\ln(n+1)}$). Then we claim that the left-hand side of (93) cannot be bounded independently of $K$ (in contrast with the proof of Corollary 10). By considering the limit as $K\to+\infty$ and the imaginary part, it is sufficient to explain the equality*

$$\mathbf{E}\Big[\sup_{t\in\mathbb{Q}}\Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\sqrt{\lambda_n}}\sin(\lambda_n t)\Big|\Big] = +\infty. \tag{95}$$

*We write $S_N(t,\omega) = \sum_{n=1}^{N} g_n(\omega)\frac{a_n}{\sqrt{\lambda_n}}\sin(\lambda_n t)$ and $R_N(t,\omega) = \sum_{n=N+1}^{+\infty} g_n(\omega)\frac{a_n}{\sqrt{\lambda_n}}\sin(\lambda_n t)$. Note the equality $\mathbf{E}[R_N(t,\omega)] = 0$ and the independence of $R_N$ and $S_N$ so that we can write*[26]

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\in\mathbb{Q}}|S_N(t,\omega)+R_N(t,\omega)|\Big] &= \mathbf{E}_{\omega_1}\mathbf{E}_{\omega_2}\Big[\sup_{t\in\mathbb{Q}}|S_N(t,\omega_1)+R_N(t,\omega_2)|\Big]\\ &\geq \mathbf{E}_{\omega_1}\Big[\sup_{t\in\mathbb{Q}}\Big|\mathbf{E}_{\omega_2}\Big[S_N(t,\omega_1)+R_N(t,\omega_2)\Big]\Big|\Big] = \mathbf{E}_{\omega_1}\Big[\sup_{t\in\mathbb{Q}}|S_N(t,\omega_1)|\Big].\end{aligned}$$

*Thanks to [Mey73, page 5, Proposition 1] giving a Sidon property of the set of the frequencies of the form $\lambda_n = \sqrt{n(n+1)}$, we get*

$$\mathbf{E}_{\omega_1}\Big[\sup_{t\in\mathbb{Q}}|S_N(t,\omega_1)|\Big] = \mathbf{E}_{\omega_1}\Big[\sup_{t\in\mathbb{R}}|S_N(t,\omega_1)|\Big] \gtrsim \sum_{n=1}^{N}\frac{|a_n|}{\sqrt{\lambda_n}}. \tag{96}$$

*We obtain (95) by making tend $N\to+\infty$.*

[26] The first equality can be proved by modifying the argument in [Ime19, Appendix F].

We now give a simple corollary of Proposition 27 ensuring that the expected bound $f^\omega(t) \lesssim \sqrt{1+|t|}$ is true.

**Corollary 29.** *With the same assumptions as those of Proposition 27, with probability* 1*, for any* $\vartheta > 0$ *there exists a constant* $C > 0$ *such that*

$$\forall t \in \mathbb{R} \qquad \Big| \sum_{n\geq 1} g_n(\omega) \frac{a_n}{\lambda_n^\alpha} e^{i\lambda_n t} \Big| \leq C(1+|t|)^\vartheta.$$

*Proof.* By choosing $\vartheta$ in the countable set $\mathbb{Q}^{+\star}$, it is sufficient to deal with one fixed exponent $\vartheta$. We choose a number $p$ strictly larger than $\frac{1}{\vartheta}$ and we write

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\in\mathbb{R}} \Big|\frac{1}{(1+|t|)^\vartheta} \sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha} e^{i\lambda_n t}\Big|^p\Big] &\leq \sum_{K\in\mathbb{N}} \mathbf{E}\Big[\sup_{K\leq|t|\leq K+1} \Big|\frac{1}{(1+|t|)^\vartheta} \sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha} e^{i\lambda_n t}\Big|^p\Big] \\ &\leq \sum_{K\in\mathbb{N}} \frac{1}{(1+K)^{p\vartheta}} \mathbf{E}\Big[\sup_{|t|\leq K+1} \Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha} e^{i\lambda_n t}\Big|^p\Big].\end{aligned}$$

We now combine (94) and we invoke the Gaussian version of the Kahane-Khintchine inequalities (see [LT91, page 56, Cor 3.2] or [LQ18b, p 256, Cor V.27]) to get the upper bound

$$C_p\Big(\sum_{n\geq 1}|a_n|^2\Big)^{\frac{p}{2}} \sum_{K\in\mathbb{N}} \frac{(\ln(K+2))^{\frac{p}{2}}}{(1+K)^{p\vartheta}} < +\infty.$$

Then, the following inequality almost surely holds true

$$\sup_{t\in\mathbb{R}} \frac{1}{(1+|t|)^\vartheta} \Big|\sum_{n=1}^{+\infty} g_n(\omega)\frac{a_n}{\lambda_n^\alpha} e^{i\lambda_n t}\Big| < +\infty.$$

□

**Proof of Proposition 27.** The proof is completely classical in the spirit of [LT91, MP81]. As already used, we note that the bound (93) easily implies the convergence of $\sum g_n(\omega)\frac{a_n}{\lambda_n^\alpha}e^{i\lambda_n t}$ in $L^1(\Omega, \mathcal{C}^0([-K,K]))$ and thus the almost sure uniform convergence on $[-K,K]$. By making $K$ runs over $\mathbb{N}^\star$, we get the almost sure uniform convergence on any compact subset of $\mathbb{R}$.

**Proof of** (93)**.** The Dudley theorem (see (38) and [LT91, page 346, Th 11.17] or [MP81, page 10, Th 1.3]) allows to get an upper bound with the entropy integral

$$\begin{aligned}\mathbf{E}\Big[\sup_{|t|\leq K}\Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha}e^{i\lambda_n t}\Big|\Big] &= \mathbf{E}\Big[\sup_{|t|\leq 1}\Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha}e^{i\lambda_n K t}\Big|\Big] \\ &\leq \mathbf{E}\Big[\Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha}\Big|\Big] + C\int_0^{+\infty}\sqrt{\ln(N(\delta,\varepsilon))}\mathrm{d}\varepsilon \\ &\leq C\Big(\sum_{n\geq 1}\frac{|a_n|^2}{\lambda_n^{2\alpha}}\Big)^{\frac12} + C\int_0^{+\infty}\sqrt{\ln(N(\delta,\varepsilon))}\mathrm{d}\varepsilon \qquad (97)\end{aligned}$$

in which $N(\delta,\varepsilon)$ is the smallest number of open balls of radius $\varepsilon$ recovering $[-1,1]$ for the pseudo-distance

$$\delta(s,t) = \mathbf{E}\Big[\Big|\sum_{n\geq 1} g_n(\omega)\frac{a_n}{\lambda_n^\alpha}(e^{i\lambda_n K s} - e^{i\lambda_n K t})\Big|^2\Big]^{\frac12} = \sqrt{\sum_{n\geq 1}\frac{|a_n|^2}{\lambda_n^{2\alpha}}|e^{i\lambda_n K s} - e^{i\lambda_n K t}|^2}.$$

The asymptotic $\lambda_n \simeq n$ immediately implies the following inequalities

$$\delta(s,t) \leq \sqrt{\sum_{n\geq 1}\frac{|a_n|^2}{\lambda_n^{2\alpha}}\min(\lambda_n K|s-t|,2)^2} \leq \underbrace{C\sqrt{\sum_{n\geq 1}\frac{|a_n|^2}{n^{2\alpha}}\min(nK|s-t|,1)^2}}_{:=\delta'(s,t)}.$$

As a consequence, we have $N(\delta,\varepsilon) \leq N(\delta',\varepsilon)$ and then

$$\int_0^{+\infty}\sqrt{\ln(N(\delta,\varepsilon))}\mathrm{d}\varepsilon \leq \int_0^{+\infty}\sqrt{\ln(N(\delta',\varepsilon))}\mathrm{d}\varepsilon. \qquad (98)$$

By applying [Ime22, page 763, Corollary 7] with $\varrho = 1$ and $\mathcal{M} = [-1,1]$, we see that the entropy integral of $\delta'$ satisfies the following equivalence (with constants independent of $K$ and $(a_n)$) :

$$\int_0^{+\infty} \sqrt{\ln(N(\delta',\varepsilon))}\mathrm{d}\varepsilon \simeq \int_0^1 \frac{\Upsilon(2Kt)}{t\sqrt{\ln(1/t)}}\mathrm{d}t \qquad \text{for} \quad \Upsilon(t) = \sqrt{\sum_{n\geq 1} \frac{|a_n|^2}{n^{2\alpha}} \min(nt,1)^2}. \tag{99}$$

We now use the assumption $(a_n) \in \ell^2(\mathbb{N}^\star)$ and the simple inequality $\min(t,1) \leq t^\alpha$ to get

$$\begin{aligned} \Upsilon(2Kt) &\leq 2^\alpha \Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2} K^\alpha t^\alpha \qquad (100) \\ \int_0^{+\infty} \sqrt{\ln(N(\delta',\varepsilon))}\mathrm{d}\varepsilon &\leq C\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2} K^\alpha \int_0^1 \frac{1}{t^{1-\alpha}\sqrt{\ln(1/t)}}\mathrm{d}t \\ &\leq C\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2} K^\alpha. \end{aligned}$$

We then get the first general bound (93) thanks to (97) and (98).

**Proof of** (94)**.** For $K \gg 1$, we have the trivial bound

$$\Big(\sum_{n\geq 1} \frac{|a_n|^2}{\lambda_n^{2\alpha}}\Big)^{\frac{1}{2}} \leq \sqrt{\ln(K)}\Big(\sum_{n\geq 1} |a_n|^2\Big)^{\frac{1}{2}}.$$

Hence, by following the last proof (actually (97) and (99)), we merely have to show the following bound for $K \gg 1$

$$\int_0^1 \frac{\Upsilon(2Kt)}{t\sqrt{\ln(1/t)}}\mathrm{d}t \leq C\sqrt{\ln(K)}\Big(\sum_{n\geq 1} |a_n|^2\Big)^{\frac{1}{2}}.$$

We split the integral in two parts.

**Integration on** $t \in \big[0, \frac{1}{K}\big]$**.** We still use the bound (100) to get a term like

$$C\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2} K^\alpha \int_0^{\frac{1}{K}} \frac{1}{t^{1-\alpha}\sqrt{\ln(1/t)}}\mathrm{d}t$$

but for $K \geq 2$, the previous term is directly bounded by

$$C\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2} K^\alpha \int_0^{\frac{1}{K}} \frac{1}{t^{1-\alpha}} \frac{\mathrm{d}t}{\sqrt{\ln(2)}} = \frac{C}{\sqrt{\ln(2)}\alpha}\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2}.$$

**Integration on** $t \in \big[\frac{1}{K}, 1\big]$**.** It actually gives the principal contribution :

$$\begin{aligned} \int_{\frac{1}{K}}^1 \frac{\Upsilon(2Kt)}{t\sqrt{\ln(1/t)}}\mathrm{d}t &\leq \|\Upsilon\|_\infty \int_{\frac{1}{K}}^1 \frac{1}{t\sqrt{\ln(1/t)}}\mathrm{d}t \\ &\leq 2\Big(\sum_{n\geq 1} \frac{|a_n|^2}{n^{2\alpha}}\Big)^{1/2} \sqrt{\ln(K)} \\ &\leq 2\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2} \sqrt{\ln(K)}. \end{aligned}$$

The bound (94) is finally proved. □

In order to finish the proof of Proposition 12, it remains to show that the bounds (92) hold with probability 1.

**First part of the argument for** (92)**.** We fist explain that we can use Proposition 16 with the following objects :

$$\rho_j = \frac{1}{\sqrt{2^j}}, \qquad \mathcal{M}_j = [-2^{-j}, 2^{-j}], \qquad F_j^\omega(t) = \sum_{\lambda_n \geq 2^j} g_n(\omega) \frac{a_n}{\sqrt{\lambda_n}} e^{i\lambda_n t}. \tag{101}$$

Note that the hypothesis **(H-C)** in Section 8 is fulfilled thanks to Proposition 27. Let us check the assumption i) of Proposition 16. We write

$$\sigma_j = \sup_{|t|\leq 2^{-j}} \sqrt{\mathbf{E}\Big[|F_j^\omega(t)|^2\Big]} = \sqrt{\sum_{\lambda_n \geq 2^j} \frac{|a_n|^2}{\lambda_n}}.$$

We claim that $(\sqrt{2^j}\sigma_j)_{j\geq 1}$ belongs to $\ell^2(\mathbb{N}^\star)$ :

$$\sum_{j\geq 1} 2^j \sigma_j^2 = \sum_{j\geq 1}\sum_{\lambda_n \geq 2^j} 2^j \frac{|a_n|^2}{\lambda_n}$$

but since $\lambda_n \simeq n$ (see the gap estimates (17)), there is $j_0 \in \mathbb{N}$ such that

$$\begin{aligned}\sum_{j\geq j_0} 2^j \sigma_j^2 &\leq \sum_{j\geq j_0}\sum_{n\geq 2^{j-j_0}} 2^j \frac{|a_n|^2}{\lambda_n} = \sum_{n\geq 1} \frac{|a_n|^2}{\lambda_n} \sum_{j_0 \leq j \leq j_0 + \log_2(n)} 2^j \\ &\leq 2^{1+j_0} \sum_{n\geq 1} \frac{|a_n|^2}{\lambda_n} n \simeq \sum_{n\geq 1} |a_n|^2 < +\infty.\end{aligned}$$

This fact immediately implies that the series $\sum \exp\Big( -\beta^2 \frac{\rho_j^2}{2\sigma_j^2}\Big) = \sum \exp\Big( -\frac{\beta^2}{2^{j+1}\sigma_j^2}\Big)$ is convergent for any $\beta > 0$, in particular we get the first assumption of Proposition 16 (see Remark 17 with $p = 2$).

For the assumption ii) of Proposition 16, we invoke (93) of Proposition 27 for $K = \frac{1}{2^j}$ and $\alpha = \frac{1}{2}$ :

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\in\mathcal{M}_j} |F_j^\omega(t)|\Big] &\leq C\Big(\sum_{\lambda_n \geq 2^j} \frac{|a_n|^2}{\lambda_n}\Big)^{\frac{1}{2}} + \frac{C}{\sqrt{2^j}}\Big(\sum_{\lambda_n\geq 2^j} |a_n|^2\Big)^{\frac{1}{2}} \\ &\leq C\rho_j \Big(\sum_{n\geq 1} |a_n|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Then the conclusion of Proposition 16 holds and there is $T > 0$ (possibly depending on the fixed sequence $(a_n)$) such that

$$\sum_{j\geq 1} \mathbf{P}\Big( \sup_{|t|\leq 2^{-j}} |F_j^\omega(t)| \geq \frac{T}{\sqrt{2^j}}\Big) < +\infty. \tag{102}$$

**Second part of the argument for** (92)**.** We claim that there is a positive constant $M$ merely depending on the function $\Theta$ (see Section 9) such that, for any $j \in \mathbb{N}^\star$, we have

$$\mathbf{P}\Big(\sup_{k\geq j} \|f^\omega - \Theta(-2^{-2k}\Delta)f^\omega\|_{L^\infty(-2^{-j},2^{-j})} > \frac{MT}{\sqrt{2^j}}\Big) \leq 2\mathbf{P}\Big(\sup_{|t|\leq 2^{-j}} |F_j^\omega(t)| \geq \frac{T}{\sqrt{2^j}}\Big).$$

By looking at (101) and writing

$$f^\omega - \Theta(-2^{-2k}\Delta)f^\omega = \sum_{\lambda_n \geq 2^j} (1 - \Theta(2^{2k}\lambda_n^2))g_n(\omega)\frac{a_n}{\sqrt{\lambda_n}} e^{i\lambda_n t},$$

we see that the expected bound is actually a consequence of Proposition 8 for

$$B = L^\infty([-2^{-j}, 2^{-j}], \mathbb{C}), \qquad u_n(t) = \frac{a_n}{\sqrt{\lambda_n}} e^{i\lambda_n t}\mathbf{1}_{[2^j,+\infty)}(\lambda_n),$$

$$\mathcal{E} = [j, +\infty) \cap \mathbb{N}, \qquad \alpha_{k,n} = 1 - \Theta(2^{2k}\lambda_n^2), \qquad M = 1 + \|\Theta\|_{L^\infty(0,+\infty)} + \|\Theta'\|_{L^1(0,+\infty)},$$

in which this choice of $M$ in (28) is motivated by (25). Thanks to (102), we get

$$\sum_{j\geq 1} \mathbf{P}\Big(\sup_{k\geq j} \|f^\omega - \Theta(-2^{-2k}\Delta)f^\omega\|_{L^\infty(-2^{-j},2^{-j})} \geq \frac{MT}{\sqrt{2^j}}\Big) < +\infty.$$

The Borel-Cantelli Lemma allows us to conclude that

$$\sup_{k\geq j} \|f^\omega - \Theta(-2^{-2k}\Delta)f^\omega\|_{L^\infty(-2^{-j},2^{-j})}$$

is almost surely $\mathcal{O}(2^{-j/2})$. In other words, (92) is finally proved.

## 16 Proof of Theorem 1, iii) $\Rightarrow$ i).

**The zonal case on $\mathbb{S}^d$.** We first explain how to deal with the model of zonal eigenfunctions $(Z_n^{\mathbb{S}^d})_{n\geq 1}$ on the sphere. Let us fix a function $\Theta \in \mathcal{C}_c^\infty([0,+\infty),\mathbb{R})$. All the substantial arguments rely on the following property (see [BGT04, Corollary 2.2]) for the semi-classical multipliers of the Laplace-Beltrami operator $\mathbf{\Delta}$ on $\mathbb{S}^d$ :

$$\sup_{0<h\leq 1} \|\Theta(-h^2\mathbf{\Delta})\|_{L^\infty(\mathbb{S}^d)\to L^\infty(\mathbb{S}^d)} < +\infty. \tag{103}$$

As a consequence of (103), the upper bound $\sup\limits_{h>0} \|\Theta(-h^2\mathbf{\Delta})f^{G,\omega}\|_{L^\infty(\mathbb{S}^d)}$ is almost surely finite. We now recall the spectral definition (following the notations in (10)) :

$$\Theta(-h^2\mathbf{\Delta})f^{G,\omega} = \sum_{k\in\mathbb{N}} \Big(\int_{\mathcal{M}} f^{G,\omega}(x)\phi_k(x)\mathrm{d}x\Big)\Theta(h^2\mu_k^2)\phi_k.$$

We may choose a Hilbert basis $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{S}^d)$ that completes $(Z_n^{\mathbb{S}^d})_{n\geq 1}$. Then, the weak convergence assumed in the assertion iii) with $\psi = Z_n^{\mathbb{S}^d}$, directly shows the equality $g_n(\omega)c_n = \displaystyle\int_{\mathcal{M}} f^{G,\omega}(x)Z_n^{\mathbb{S}^d}(x)\mathrm{d}x$ which, combined with (3), implies the following finite sum

$$\Theta(-h^2\mathbf{\Delta})f^{G,\omega} = \sum_{n=1}^{+\infty} g_n(\omega)c_n\Theta(h^2n(n+d-1))Z_n^{\mathbb{S}^d}. \tag{104}$$

For instance, we may assume that $\Theta \equiv 1$ on $[0,1]$ and that $\Theta$ vanishes on $[2,+\infty)$. We now focus on the point of concentration $P = (1,0,\dots,0)$ to get

$$\sup_{0<h\leq 1} \Big| \sum_{\substack{n\in\mathbb{N}^\star \\ n(n+d-1)\leq \frac{2}{h^2}}} g_n(\omega)c_n\Theta\big(h^2n(n+d-1)\big)Z_n^{\mathbb{S}^d}(P)\Big| < +\infty.$$

Now we may use a result in the spirit of the Marcienkiewicz-Zygmund-Kahane theorem (see [LQ18a, page 240, Th II.4]) but it is much simpler to use Lemma 19 (by considering a sequence, for instance $h_j = 2^{-j}$, which tends to $0^+$) :

$$\sup_{j\in\mathbb{N}} \mathbf{E}\Big[\Big| \sum_{\substack{n\in\mathbb{N}^\star \\ n(n+d-1)\leq 2^{2j+1}}} g_n(\omega)c_n\Theta\big(2^{-2j}n(n+d-1)\big)Z_n^{\mathbb{S}^d}(P)\Big|\Big] < +\infty$$

or equivalently

$$\sup_{j\in\mathbb{N}} \sum_{\substack{n\in\mathbb{N}^\star \\ n(n+d-1)\leq 2^{2j+1}}} |c_n|^2\Theta\big(2^{-2j}n(n+d-1)\big)^2|Z_n^{\mathbb{S}^d}(P)|^2 < +\infty.$$

Remembering the condition $\Theta \equiv 1$ on $[0,1]$, we may restrict the previous sum on the integers $n$ satisfying $n(n+d-1) \leq 2^{2j}$ and hence $\Theta\big(2^{-2j}n(n+d-1)\big) = 1$. By using the asymptotic $Z_n^{\mathbb{S}^d}(P) \simeq \sqrt{n}P_n^{\left(\frac{d}{2}-1,\frac{d}{2}-1\right)}(1) \simeq n^{\frac{d-1}{2}}$, we immediately get the convergence of the series $\sum |c_n|^2 n^{d-1}$.

**The radial case on $\mathbb{R}^d$.** Since each $Z_n^{d,\mathrm{Dir}}$ is smooth on the closed ball $B_d(0,1)$ of $\mathbb{R}^d$, we can try to make the same reasoning with the analogue of (104) :

$$\Theta(-h^2\Delta_{\mathrm{Dir}})f^{G,\omega} = \sum_{n=1}^{+\infty} g_n(\omega)c_n\Theta(h^2\lambda_{d,n}^2)Z_n^{d,\mathrm{Dir}}.$$

but we need the following analogue of (103) :

$$\sup_{0<h\leq 1} \|\Theta(-h^2\Delta_{\mathrm{Dir}})\|_{L^\infty(B_d(0,1))\to L^\infty(B_d(0,1))} < +\infty. \tag{105}$$

The proof of (103), developed in [BGT04], is of pseudo-differential nature and avoids potential boundary problems. For boundary domains like the unit ball $B_d(0,1)$, a proof via heat kernel seems to be easier. Actually, by combining [Dav89, page 89, Coro 3.2.8] and [CCO02, Th 4.3], we see that the heat kernel operator of the Laplacian operator

$-\Delta_{\mathrm{Dir}}$ with Dirichlet boundary conditions satisfy the following estimates : for any $\varepsilon > 0$ and any complex number $z \in \mathbb{C}$ with positive real part, we have

$$\|e^{-z\Delta_{\mathrm{Dir}}}\|_{L^\infty(\mathcal{M})\to L^\infty(\mathcal{M})} \leq C_\varepsilon \Big(\frac{|z|}{\mathrm{Re}(z)}\Big)^{\frac{d}{2}+\varepsilon}.$$

Then (105) follows via a known argument involving an integral formula (see details in [IO22, lines (2.3) and (2.10)]).

## 17 Proof of Theorem 2, preliminaries about $\theta$

We recall that we have the freedom to choose any function $\Theta \in \mathcal{C}_c^\infty([0,+\infty),\mathbb{R})$ in Section 9 (under the condition $\Theta \equiv 1$ near $0^+$). We claim that we can choose a function $\Theta$ such that

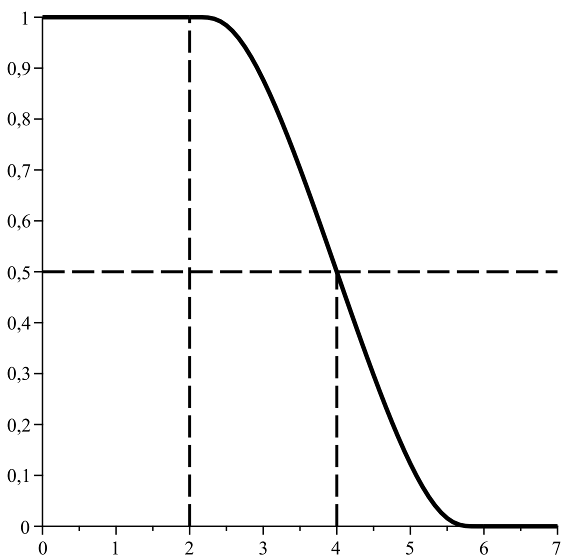
$$\theta(\lambda) := \Theta(\lambda) - \Theta(4\lambda)$$

fulfills the following properties :

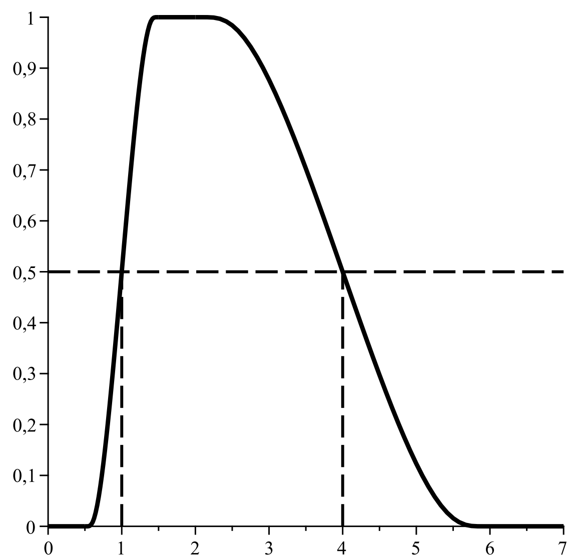
- $\theta$ has support in $[\frac{1}{4}, 16]$ and is bounded by 1,
- $\theta \geq \frac{1}{2}$ on $[1,4]$.

For the choice $\Theta$ given just below (see (106)), we get the following plots :

**Plot of $\Theta$**




**Plot of $\theta$**




We shall choose a function $\Theta$ with support in $[0,6]$, satisfying $\Theta(\lambda) = 1$ for $\lambda \in [0,2]$, which is decreasing on $[2,6]$ and which finally satisfies $\Theta(4) = \frac{1}{2}$. For instance, the following definition is convenient :

$$\begin{aligned}
&\Theta(\lambda) = 1 && \text{for} \quad 0 \leq \lambda \leq 2,\\
&\Theta(\lambda) = \frac{\displaystyle\int_{\frac{\lambda-4}{2}}^{1} \exp\Big(\frac{-1}{1-t^2}\Big)\mathrm{d}t}{\displaystyle\int_{-1}^{1} \exp\Big(\frac{-1}{1-t^2}\Big)\mathrm{d}t} && \text{for} \quad 2 < \lambda \leq 6,\\
&\Theta(\lambda) = 0 && \text{for} \quad \lambda > 6.
\end{aligned} \tag{106}$$

Let us now explain the expected conditions on $\theta(\lambda) = \Theta(\lambda) - \Theta(4\lambda)$. It is clear that $\theta$ has support in $\big[\frac{1}{2}, 6\big] \subset [\frac{1}{4}, 16]$ and is bounded by 1. For the bound $\theta \geq \frac{1}{2}$ on $[1,4]$, we just write

$$\lambda \in [1,2] \quad \Rightarrow \quad \theta(\lambda) = \underbrace{\Theta(\lambda) - \Theta(4)}_{\geq \Theta(2)-\Theta(4)} + \underbrace{\Theta(4) - \Theta(4\lambda)}_{\geq 0} \geq \Theta(2) - \Theta(4) = \frac{1}{2},$$

$$\lambda \in [2,4] \quad \Rightarrow \quad \theta(\lambda) = \Theta(\lambda) \geq \Theta(4) = \frac{1}{2}.$$

# 18 Proof of Theorem 2, i) ⇒ ii)

We first check that the random series $\sum_{n\geq 1} f_n^{G,\omega}$ is almost surely well-defined as an element of $L^2(\mathcal{M})$. Since the functions $f_n^{G,\omega}$ are orthogonal for different $n$, it is sufficient to check that the series $\sum_{n\geq 1} \|f_n^{G,\omega}\|^2_{L^2(\mathcal{M})}$ is almost surely finite. This is indeed a consequence of (12) and (15) which clearly give

$$\mathbf{E}\Big[\sum_{n\geq 1} \|f_n^{G,\omega}\|^2_{L^2(\mathcal{M})}\Big] = \sum_{n\geq 1} \|f_n\|^2_{L^2(\mathcal{M})} < +\infty.$$

Let us now write $f^{G,\omega} = \sum_{n\geq 1} f_n^{G,\omega}$. As for the proof of Theorem 4, we shall again use Proposition 16. Indeed we choose the following parameters :

$$\rho_j = \frac{1}{2^{\alpha j}}, \qquad \mathcal{M}_j = \mathcal{M}, \qquad F_j^\omega(x) = \theta\Big(\frac{-2^{-2j}}{K^2}\boldsymbol{\Delta}\Big) f^{G,\omega}.$$

Actually, if we succeed to check the assumptions of Proposition 16 then its conclusion and the equivalence (61) will show that $f^{G,\omega}$ almost surely belongs to $\mathcal{C}^{0,\alpha}(\mathcal{M})$, namely the assertion ii) of Theorem 2.

In order to check that $F_j^\omega$ is a centered Gaussian process on $\mathcal{M}$, we invoke that $\theta$ is bounded with support in $\big[\frac{1}{4}, 16\big]$ and use the following formula :

$$F_j^\omega(x) = \sum_{n\geq 1} \frac{\|f_n\|_{L^2(\mathcal{M})}}{\sqrt{\dim(E_{(Kn-K,Kn]})}} \sum_{\substack{\mu_k \in (Kn-K,Kn] \\ \frac{2^{-j}\mu_k}{K} \in [\frac{1}{2},4]}} \theta\Big(\frac{2^{-2j}\mu_k^2}{K^2}\Big) g_{n,k}(\omega)\phi_k(x). \tag{107}$$

Since $n$ actually runs over $[2^{j-1}, 2^{j+2}]$ (the other terms are zero), the hypothesis **(H-C)** is true.

**Step 1.** We have to check the assumption i) of Proposition 16 involving the weak variance of $F_j^\omega$. We have

$$\begin{aligned}
\mathbf{E}\Big[|F_j^\omega(x)|^2\Big] \quad &= \quad \sum_{n\geq 1} \frac{\|f_n\|^2_{L^2(\mathcal{M})}}{\dim(E_{(Kn-K,Kn]})} \sum_{\substack{\mu_k \in (Kn-K,Kn] \\ \frac{2^{-j}\mu_k}{K} \in [\frac{1}{2},4]}} \theta\Big(\frac{2^{-2j}\mu_k^2}{K^2}\Big)^2 \phi_k(x)^2 \\
&\leq \quad \sum_{n=2^{j-1}}^{2^{j+2}} \frac{\|f_n\|^2_{L^2(\mathcal{M})}}{\dim(E_{(Kn-K,Kn]})} \sum_{\mu_k \in (Kn-K,Kn]} \phi_k(x)^2.
\end{aligned}$$

We now invoke a famous result about the spectral function on a boundaryless compact Riemannian manifold due to Hörmander (see [Hör68]) which has been used by Burq and Lebeau in [BL13, page 923]. For the exact form we need, we refer to [Ime19, proof of Lemma 8.1] or [Ime22, line (50)] and hence for a constant $K_0$ large enough, we have the following bound for any $K \geq K_0$ :

$$\sup_{x\in\mathcal{M}} \mathbf{E}\Big[|F_j^\omega(x)|^2\Big] \quad \leq \quad C \sum_{n=2^{j-1}}^{2^{j+2}} \|f_n\|^2_{L^2(\mathcal{M})}.$$

For any $j \in \mathbb{N}^\star$, the assumption (15) implies the following upper bound

$$\sup_{x\in\mathcal{M}} \mathbf{E}\Big[|F_j^\omega(x)|^2\Big] \quad \leq \quad \frac{C}{j2^{2\alpha j}}.$$

The left-hand side equals $\sigma_j^2$ (see (49)). Since we have set $\rho_j = 2^{-\alpha j}$, we get $\sigma_j \leq \frac{C}{\sqrt{j}}\rho_j$ which immediately implies the convergence of $\sum_{j\geq 1} \exp\Big(-\beta^2 \frac{\rho_j^2}{2\sigma_j^2}\Big)$ for any fixed constant $\beta > 0$.

**Step 2.** Let us now show that the assumption ii) of Proposition 16 is also true. By using the identity (107) and the contraction principle (see the proof of [LT91, page 98, Th 4.4]), we may get rid of $\theta$ as follows

$$\mathbf{E}\Big[\sup_{x\in\mathcal{M}} |F_j^\omega(x)|\Big] \leq \mathbf{E}\Big[\sup_{x\in\mathcal{M}} \Big| \sum_{n=2^{j-1}}^{2^{j+2}} \frac{\|f_n\|_{L^2(\mathcal{M})}}{\sqrt{\dim(E_{(Kn-K,Kn]})}} \sum_{\mu_k \in (Kn-K,Kn]} g_{n,k}(\omega)\phi_k(x)\Big|\Big]$$

which means (see (12)) :

$$\mathbf{E}\Big[\sup_{x\in\mathcal{M}}|F_j^\omega(x)|\Big]\le \mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=2^{j-1}}^{2^{j+2}} f_n^{G,\omega}(x)\Big|\Big].$$

We now invoke [Ime22, Theorem 3] to get the Gaussian analogue of the upper bound in (13) :

$$\begin{aligned}\mathbf{E}\Big[\sup_{x\in\mathcal{M}}|F_j^\omega(x)|\Big] &\le C\sum_{p=1}^{2^{j+2}}\frac{1}{p\sqrt{\ln(p+1)}}\Big(\sum_{n=\max(p,2^{j-1})}^{2^{j+2}}\|f_n\|^2_{L^2(\mathcal{M})}\Big)^{\frac{1}{2}}\\ &\le C\sqrt{\ln(2^{j+2})}\times\Big(\sum_{n=2^{j-1}}^{2^{j+2}}\|f_n\|^2_{L^2(\mathcal{M})}\Big)^{\frac{1}{2}}.\end{aligned}$$

By using (15), we finally have checked the assumption ii) of Proposition 16 :

$$\mathbf{E}\Big[\sup_{x\in\mathcal{M}}|F_j^\omega(x)|\Big]\le\frac{C}{2^{\alpha j}}.$$

**Remark 30.** *Instead of* (15)*, let us assume the stronger assumption*

$$\sum_{n=2^j}^{2^{j+1}-1}\|f_n\|^2_{L^2(\mathcal{M})}=o\Big(\frac{1}{j2^{\alpha j}}\Big). \tag{108}$$

*Then we claim that the partial sums of the Gaussian random series* $\sum_{n\ge1} f_n^{G,\omega}$ *almost surely converge in* $\mathcal{C}^{0,\alpha}(\mathcal{M})$*. The proof needs a few modifications of the previous argument. Actually, the last proof already shows that* $\sum_{n\ge1} f_n^{G,\omega}$ *belongs to* $\mathcal{C}^{0,\alpha}(\mathcal{M})$ *and it remains to prove that the limit* $\lim_{N\to+\infty}\|\sum_{n=N}^{+\infty} f_n^{G,\omega}\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}$ *equals* $0$*. It is known that it is sufficient to study the subsequence of* $N=2^J$ *as* $J\to+\infty$ *(see [LQ18b, page 132, Th III.5]). Thanks to* (88)*, we have to show that* (108) *implies the following limit with probability* $1$ *:*

$$\Big\|\Theta\Big(\frac{-\mathbf{\Delta}}{K^2}\Big)\sum_{n=2^J}^{+\infty} f_n^{G,\omega}\Big\|_{L^\infty(\mathcal{M})}+\sup_{j\ge1}2^{\alpha j}\Big\|\theta\Big(\frac{-2^{-2j}\mathbf{\Delta}}{K^2}\Big)\sum_{n=2^J}^{+\infty} f_n^{G,\omega}\Big\|_{L^\infty(\mathcal{M})}\xrightarrow{J\to+\infty}0.$$

*Note that the term involving* $\Theta$ *equals* $0$ *for* $J\gg1$*. Let us deal with the term involving* $\theta$*. The same support argument as above allows to make the following reduction*

$$\theta\Big(\frac{-2^{-2j}\mathbf{\Delta}}{K^2}\Big)\sum_{n=2^J}^{+\infty} f_n^{G,\omega}=\theta\Big(\frac{-2^{-2j}\mathbf{\Delta}}{K^2}\Big)\sum_{n=\max(2^J,2^{j-1})}^{2^{j+2}} f_n^{G,\omega}.$$

*In particular,* $j$ *is merely relevant for* $j\ge J-2$*. In order to get rid of* $\theta$*, we used the contraction principle in Step 2 above. But we prefer here to use the uniform boundedness of the semi-classical multipliers* $\theta\big(\frac{-2^{-2j}\mathbf{\Delta}}{K^2}\big)$ *(see [BGT04, Corollary 2.2]). Hence, it is sufficient to prove (with probability* $1$*) :*

$$\sup_{j\ge J-2}2^{\alpha j}\Big\|\sum_{n=\max(2^J,2^{j-1})}^{2^{j+2}} f_n^{G,\omega}\Big\|_{L^\infty(\mathcal{M})}\xrightarrow{J\to+\infty}0.$$

*By separating the cases* $j=J-2$ *and* $j>J-2$*, one checks that the previous limit would be a consequence of several uses of the following ones (with probability* $1$*) :*

$$2^{\alpha j}\|f_{2^j}^{G,\omega}\|_{L^\infty(\mathcal{M})}\xrightarrow{j\to+\infty}0\qquad\text{and}\qquad 2^{\alpha j}\Big\|\sum_{n=2^j}^{2^{j+1}} f_n^{G,\omega}\Big\|_{L^\infty(\mathcal{M})}\xrightarrow{j\to+\infty}0. \tag{109}$$

*We can now repeat exactly the same computations as in Step 1 and Step 2 with the following Gaussian processes*

$$\overline{F_j^\omega}(x):=f_{2^j}^{G,\omega}(x)\qquad\text{and}\qquad\widetilde{F_j^\omega}(x)=\sum_{n=2^j}^{2^{j+1}} f_n^{G,\omega}(x).$$

*For $\rho_j = 2^{-\alpha j}$, we similarly obtain that the analogue weak variances $\overline{\sigma_j}$ and $\widetilde{\sigma}_j$ are $o\Big(\frac{\rho_j}{\sqrt{j}}\Big)$ and hence satisfy*

$$\forall \beta > 0 \qquad \sum_{j\geq 1} \exp\Big(-\beta^2 \frac{\rho_j^2}{2\overline{\sigma_j}^2}\Big) + \exp\Big(-\beta^2 \frac{\rho_j^2}{2\widetilde{\sigma}_j^{\,2}}\Big) < +\infty$$

*but the computations of Step 2 and* (108) *now imply*

$$\mathbf{E}\Big[2^{\alpha j} \sup_{x\in\mathcal{M}} \big|\overline{F_j^\omega}(x)\big| + 2^{\alpha j} \sup_{x\in\mathcal{M}} \big|\widetilde{F_j^\omega}(x)\big|\Big] \xrightarrow{j\to+\infty} 0.$$

*Finally, the conclusion* (51) *proves* (109) *(as explained in Remark 17).*

# 19 Proof of Theorem 2, ii) $\Rightarrow$ iii)

This is not a surprising fact since it seems to be related to the contraction principle already mentioned : for any sequence $(u_n)_{n\in\mathbb{N}}$ in a Banach space $B$, if the Gaussian random series $\sum_{n\in\mathbb{N}} g_n(\omega)u_n$ almost surely converges in $B$ then so does the Rademacher random series $\sum_{n\in\mathbb{N}} \varepsilon_n(\omega)u_n$. For our purpose, a small difficulty is the potential absence of convergence of the Gaussian random series $\sum f_n^{G,\omega}$ in $\mathcal{C}^{0,\alpha}(\mathcal{M})$. Let us explain how to overcome this issue thanks to the characterization (62) whose main interest is to come back to finite random sums.

We refer to the argument[27] of [Ime22, page 787, Step 1] which allows to show the following inequality for any $N \in \mathbb{N}^\star$ and any semi-norm[28] $S$ on the vector space $\mathrm{Span}\{\phi_k,\ k\in\mathbb{N}\}$ :

$$\frac{\sqrt{2}}{\sqrt{\pi}}\mathbf{E}\Big[S\Big(\sum_{n=1}^N f_n^\omega\Big)\Big] \leq \mathbf{E}\Big[S\Big(\sum_{n=1}^N f_n^{G,\omega}\Big)\Big]. \tag{110}$$

We now fix an integer $J \in \mathbb{N}$ and we make the following choice for $S$ :

$$S : f \mapsto \sup_{0\leq j\leq J} \Big\|\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f\Big\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}.$$

We note that $f^\omega = \sum_{n\geq 1} f_n^\omega$ almost surely belongs to $L^2(\mathcal{M})$ (since $\|f_n^\omega\|_{L^2(\mathcal{M})} = \|f_n\|_{L^2(\mathcal{M})}$, see (11)). Since we have constructed $\Theta$ to be in support in $[0,16]$ in Section 17, we deduce that the functions $\Theta\big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\big)f$, for $j\in[0,J]$, are spectrally localized in $[0, K2^{J+2}]$ (with respect to $\sqrt{-\mathbf{\Delta}}$) and do not involve frequencies $\lambda$ satisfying $\lambda > K2^{J+2}$. Remembering that each $f_n^\omega$ and $f_n^{G,\omega}$ is spectrally localized in $(Kn-K, Kn]$, we can choose $N = 2^{J+2}$ in order to transform the inequality (110) into the following one

$$\frac{\sqrt{2}}{\sqrt{\pi}}\mathbf{E}\Big[\sup_{0\leq j\leq J}\Big\|\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^\omega\Big\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}\Big] \leq \mathbf{E}\Big[\sup_{0\leq j\leq J}\Big\|\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^{G,\omega}\Big\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}\Big].$$

By making $J$ tend to $+\infty$, we get

$$\frac{\sqrt{2}}{\sqrt{\pi}}\mathbf{E}\Big[\sup_{j\in\mathbb{N}}\Big\|\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^\omega\Big\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}\Big] \leq \mathbf{E}\Big[\sup_{j\in\mathbb{N}}\Big\|\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^{G,\omega}\Big\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}\Big]. \tag{111}$$

Let us show the finiteness of the upper bound. The assertion ii) of Theorem 2 and the characterization (62) ensure that the sequence of functions $(\Theta(-\frac{2^{-2j}}{K^2}\mathbf{\Delta})f^{G,\omega})_{j\in\mathbb{N}}$ is almost surely bounded in $\mathcal{C}^{0,\alpha}(\mathcal{M})$. We now reformulate this fact as the almost sure boundedness of a specific Gaussian process $(U^\omega(j,x,x'))_{\mathbb{N}\times\mathcal{E}}$ in which $\mathcal{E}$ is a dense countable subset of

$$\{(x,x')\in\mathcal{M}\times\mathcal{M}, \quad x\neq x'\}.$$

[27]The argument does not make play any role to the Banach space $\mathcal{C}^{0,\alpha}(\mathcal{M})$. Actually, the repeatedly used contraction principle can be written in a quite general framework (see [LT91, page 98, Th 4.4]).

[28]We note that $S$ poses no issue of measurability because semi-norms are automatically continuous on finite dimensional spaces and because (110) involves finite dimensions. Moreover, the proof of (110) needs that $S$ is convex on any finite-dimensional vector subspace of $\mathrm{Span}\{\phi_k, k\in\mathbb{N}\}$ and thus satisfies the Jensen inequality (instead of the Hölder inequality used in [Ime22, page 787, Step 1]).

More precisely, due to the definition (14) of the Hölder condition, it is clear that we may consider the following Gaussian process on the countable set $\mathbb{N}\times\mathcal{E}$ :

$$\begin{aligned} U^\omega(j,x,x') &:= \frac{1}{\delta_g(x,x')^\alpha}\Big(\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^{G,\omega}(x)-\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^{G,\omega}(x')\Big)\\ &= \frac{1}{\delta_g(x,x')^\alpha}\sum_{n=1}^{2^{j+2}}\frac{\|f_n\|_{L^2(\mathcal{M})}}{\sqrt{\dim(E_{(Kn-K,Kn]})}}\sum_{Kn-K<\mu_k\le Kn} g_{n,k}(\omega)\Theta\Big(\frac{2^{-2j}\mu_k^2}{K^2}\Big)\Big(\phi_k(x)-\phi_k(x')\Big). \end{aligned}$$

As in the proof of Proposition 11, we again invoke the integrability properties of Gaussian processes (see [Led01, page 134, Th 7.1]) to get

$$\mathbf{E}\Big[\sup_{\mathbb{N}\times\mathcal{E}}|U^\omega(j,x,x')|\Big]<+\infty.$$

In other words, the right-hand side and so the left-hand side of (111) are finite. We immediately infer that, with probability 1, we have

$$\sup_{j\in\mathbb{N}}\Big\|\Theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^\omega\Big\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})}<+\infty.$$

A last use of the characterization (62) shows that $f^\omega$ almost surely belongs to $\mathcal{C}^{0,\alpha}(\mathcal{M})$. The implication ii) $\Rightarrow$ iii) is completely proved.

**Remark 31.** *The formula* (110) *would easily show that if the partial sums of the Gaussian random series* $\sum_{n\ge1} f_n^{G,\omega}$ *almost surely converge in* $\mathcal{C}^{0,\alpha}(\mathcal{M})$ *then so do the partial sums of the random series* $\sum_{n\ge1} f_n^\omega$.

# 20 Proof of Theorem 2, iii) $\Rightarrow$ i)

Let us write $f^\omega=\sum_{n\ge1} f_n^\omega$ which belongs to $\mathcal{C}^{0,\alpha}(\mathcal{M})$ with probability 1 and let us define[29]

$$X_j(\omega):=2^{\alpha j}\Big\|\theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^\omega\Big\|_{L^\infty(\mathcal{M})}. \tag{112}$$

Thanks to Proposition 22 applied to $\lambda\mapsto\theta\big(\frac{\lambda}{K^2}\big)$ instead of $\theta$, we know that the sequence $(X_j)$ is almost surely bounded. The goal of Step 1 will be to show that the following bound almost surely holds

$$\sup_{j\in\mathbb{N}}\mathbf{E}[|X_j|]<+\infty. \tag{113}$$

The goal of Step 2 will be to get (15) from (113).

**Step 1.** To get the bound (113), we will use Lemma 19 and it remains to check (55). By exploiting that the support of $\theta$ is included in $\big[\frac14,16\big]$ (see Section 17), we may develop the following finite sum

$$2^{\alpha j}\theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^\omega(x)=2^{\alpha j}\sum_{n=2^{j-1}}^{2^{j+2}}\|f_n\|_{L^2(\mathcal{M})}\sum_{\mu_k\in(Kn-K,Kn]}\theta\Big(\frac{2^{-2j}\mu_k^2}{K^2}\Big)U_{n,k}(\omega)\phi_k(x). \tag{114}$$

Let us explain why (55) is a simple consequence of the generalization of the standard Kahane-Khintchine inequalities by Marcus and Pisier (see [MP81, line (2.1) of page 81 and line (2.14) of page 91]). Let us focus on the definition (11) of $f_n^\omega$. If one denotes by $\mathcal{E}_n:\Omega\to O_{D_n}(\mathbb{R})$ a random matrix with $D_n=\dim(E_{(Kn-K,Kn]})$ and whose distribution is the normalized Haar measure on the orthogonal group $O_{D_n}(\mathbb{R})$, we recall that the distribution of the first column of $\mathcal{E}_n$ is exactly the uniform distribution of the unit sphere $\mathbb{S}^{D_n-1}\subset\mathbb{R}^{D_n}$. Hence we may assume that the first column of $\mathcal{E}_n$ is exactly the random vector $U_n=(U_{n,k})$ with $\mu_k\in(Kn-K,Kn]$ as in (11). The inequalities (55) will be a direct consequence of the result of Marcus and Pisier once we will provide a matrix $M_n\in\mathcal{M}_{D_n}(L^\infty(\mathcal{M}))$ such that

$$\operatorname{tr}(\mathcal{E}_n(\omega)M_n(x))=\sum_{\mu_k\in(Kn-K,Kn]}\theta\Big(\frac{2^{-2j}\mu_k^2}{K^2}\Big)U_{n,k}(\omega)\phi_k(x). \tag{115}$$

[29] As for **(H-C)** in Section 8, there is no issue in the measurability of $X_j$ since $x\in\mathcal{M}\mapsto F_j^\omega(x)\in\mathbb{C}$ is continuous and thus $X_j$ may be reduced to a bound on a dense countable subset of $\mathcal{M}$.

It suffices to consider the matrix $M_n(x)$ whose first row is exactly $\left(\theta\left(\frac{2^{-2j}\mu_k^2}{K^2}\right)\phi_k(x)\right)_{\mu_k\in(Kn-K,Kn]}$ and other coefficients are 0 so that the coefficients of $\mathcal{E}_n(\omega)M_n(x)$ are given by

$$\mathcal{E}_n(\omega)M_n(x) = \left(\theta\Big(\frac{2^{-2j}\mu_{k'}^2}{K^2}\Big)U_{n,k}(\omega)\phi_{k'}(x)\right)_{(k,k')}. \tag{116}$$

**Step 2.** It is time to exploit the condition $\theta \geq \frac{1}{2}$ on $[1,4]$ (see Section 17) which actually implies, for any integer $n \in [1+2^j, 2^{j+1}]$ and any $\mu_k \in (Kn-K, Kn]$, the inequality

$$\theta\Big(\frac{2^{-2j}\mu_k^2}{K^2}\Big) \geq \frac{1}{2}.$$

We now define the following matrix $T_n \in \mathcal{M}_{D_n}(\mathbb{R})$ as follows :

$$\begin{aligned} n \notin [1+2^j, 2^{j+1}] \quad &\Rightarrow \quad T_n = 0, \\ n \in [1+2^j, 2^{j+1}] \quad &\Rightarrow \quad T_n \text{ is diagonal with the } D_n \text{ eigenvalues } \frac{1}{\theta\big(\frac{2^{-2j}\mu_k^2}{K^2}\big)}. \end{aligned}$$

Due to the formula (116), for each integer $n \in [1+2^j, 2^{j+1}]$, we have the equality

$$\mathrm{tr}(T_n\mathcal{E}_n(\omega)M_n(x)) = \sum_{\mu_k\in(Kn-K,Kn]} U_{n,k}(\omega)\phi_k(x).$$

Note moreover that the norm-operator of $T_n$ in $\mathbb{R}^{D_n}$ (with its canonical Euclidean structure) is bounded by 2. The formula (115) and the multidimensional version of the contraction principle[30] proved by Marcus and Pisier (see [MP81, page 82, Proposition 2.1, line (2.4)]) give the inequalities

$$\begin{aligned} &\mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=1+2^j}^{2^{j+1}}\|f_n\|_{L^2(\mathcal{M})}\sum_{\mu_k\in(Kn-K,Kn]}U_{n,k}(\omega)\phi_k(x)\Big|\Big] \\ &\quad= \mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=2^{j-1}}^{2^{j+2}}\|f_n\|_{L^2(\mathcal{M})}\,\mathrm{tr}\left(T_n\mathcal{E}_n(\omega)M_n(x)\right)\Big|\Big] \\ &\quad\leq 2\mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=2^{j-1}}^{2^{j+2}}\|f_n\|_{L^2(\mathcal{M})}\,\mathrm{tr}\left(\mathcal{E}_n(\omega)M_n(x)\right)\Big|\Big] \\ &\quad\leq 2\mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=2^{j-1}}^{2^{j+2}}\|f_n\|_{L^2(\mathcal{M})}\sum_{\mu_k\in(Kn-K,Kn]}\theta\Big(\frac{2^{-2j}\mu_k^2}{K^2}\Big)U_{n,k}(\omega)\phi_k(x)\Big|\Big]. \end{aligned}$$

Then (114) allows to reformulate the last upper bound in order to get

$$\mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=1+2^j}^{2^{j+1}}\|f_n\|_{L^2(\mathcal{M})}\sum_{\mu_k\in(Kn-K,Kn]}U_{n,k}(\omega)\phi_k(x)\Big|\Big] \leq 2\mathbf{E}\Big[\Big\|\theta\Big(\frac{-2^{-2j}}{K^2}\mathbf{\Delta}\Big)f^\omega\Big\|_{L^\infty(\mathcal{M})}\Big].$$

Looking at (11), (112), and (113), we get

$$\sup_{j\in\mathbb{N}} 2^{\alpha j}\mathbf{E}\Big[\sup_{x\in\mathcal{M}}\Big|\sum_{n=1+2^j}^{2^{j+1}} f_n^\omega(x)\Big|\Big] < +\infty.$$

Point i) of Theorem 2 is easily proved thanks to the lower bound in (13).

[30] In the book [MP81], the notation of the norm-operator of $T_n$ is written $\|T_n\|_\infty$ (see at page 78) and is used in the contraction principle.

**Remark 32.** *Let us replace the assumption iii) with the stronger one : the partial sums of the random series* $\sum_{n=1}^{+\infty} f_n^\omega$ *almost surely converge in* $\mathcal{C}^{0,\alpha}(\mathcal{M})$*. We then claim that we have for* $j \to +\infty$

$$\sum_{n=1+2^j}^{2^{j+1}} \|f_n\|^2_{L^2(\mathcal{M})} = o\Big(\frac{1}{j2^{\alpha j}}\Big). \tag{117}$$

*To get such an asymptotic, we note that* $\Big\| \sum_{n=2^{J-1}}^{2^{J+2}} f_n^\omega \Big\|_{\mathcal{C}^{0,\alpha}(\mathcal{M})} \xrightarrow{J\to+\infty} 0$ *with probability* 1*. Thanks to* (88)*, we infer the following almost sure limit*

$$2^{\alpha J}\Big\|\theta\Big(\frac{-2^{-2J}\mathbf{\Delta}}{K^2}\Big) \sum_{n=2^{J-1}}^{2^{J+2}} f_n^\omega\Big\|_{L^\infty(\mathcal{M})} \xrightarrow{J\to+\infty} 0.$$

*We then can repeat the strategy of Step 1 (based on Lemma 19) and Step 2 to get* (117)*. Let us mention that we can replace in Step 1 the use of Lemma 19 with the following fact : the partial sums of the random series* $\sum_{n=1}^{+\infty} f_n^\omega$ *converge in* $L^1(\Omega, \mathcal{C}^{0,\alpha}(\mathcal{M}))$*. Such a convergence can be seen via the formula* (115) *and by invoking a result*[31] *of Marcus and Pisier.*

# A Appendix : the deterministic case $d = 3$ via a result of Hardy and Littlewood

We recall that the sequence of Jacobi polynomials $P_n^{(\frac{1}{2},\frac{1}{2})}$ are, up to a multiplicative constant, Chebyshev polynomials of the second kind. Any $x \in \mathbb{S}^3$ can be written as $x = (\cos(\theta), \star, \star, \star)$ where $\theta \in [0,\pi]$ stands for the geodesic distance in $\mathbb{S}^3$ between $x$ and the pole $P = (1,0,0,0)$. Then, one may express the zonal eigenfunctions $Z_n^{\mathbb{S}^3}$ as follows

$$Z_n^{\mathbb{S}^3}(x) = c'_{3,n} P_n^{(\frac{1}{2},\frac{1}{2})}(\cos(\theta)) = \kappa \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

for a suitable universal constant $\kappa > 0$ ensuring $\int_{\mathbb{S}^3} |Z_n^{\mathbb{S}^3}(x)|^2 \mathrm{d}x = 1$. Note that the pole $P$ corresponds to $\theta = 0$ and thus $Z_n^{\mathbb{S}^3}(P) = \kappa(n+1)$.

In the proof of [HL24, page 256], Hardy and Littlewood constructed a real sequence $(a_n)_{n\geq 1}$ satisfying the following statements :

a) the asymptotic $a_n = \mathcal{O}\big(\frac{\ln(n)}{n}\big)$ holds,

b) the series $\sum a_n$ is convergent,

c) the limit $\lim_{\theta\to 0} \sum_{n\geq 1} a_n \frac{\sin(n\theta)}{n\theta}$ does not exist.

For $n \geq 1$, we set $c_n = \frac{a_{n+1}}{n+1}$ and hence we get the conditions a), b) and c) written in the introduction for the zonal function $\sum_{n\geq 1} c_n Z_n^{\mathbb{S}^3}$.

# B Appendix : Hölder regularity of random trigonometric series on the torus

We just merely give a few complements of [Kah68, page 89]. At the light of Proposition 22, it is easy to see that for any $\alpha \in (0,1)$ and any bounded sequence $(\varepsilon_n)$, the function $\sum_{j\in\mathbb{N}} \frac{\varepsilon_j}{2^{j\alpha}} e^{i2^j x}$ belongs to $\mathcal{C}^{0,\alpha}(\mathbb{T})$. However, we shall see

[31] See the book [MP81], Theorem 2.14 at page 92 and the definition of $\varepsilon_i$ at page 75.

that the Gaussian random function $x \mapsto \sum_{j\in\mathbb{N}} \frac{g_j(\omega)}{2^{j\alpha}} e^{i2^j x}$ belongs to $\mathcal{C}^{0,\alpha'}(\mathbb{T})$ for any $\alpha' \in (0,\alpha)$ but does not belong to $\mathcal{C}^{0,\alpha}(\mathbb{T})$. We indeed immediately see that i) of Proposition 33 below is false.

For any sequence of coefficients $(c_n)_{n\in\mathbb{N}}$ and any $j \in \mathbb{N}$, we use the following notation

$$s_j = \Big(\sum_{n=2^j}^{2^{j+1}-1} |c_n|^2\Big)^{1/2}. \tag{118}$$

We refer to Remark 34 for explanations about the condition ii) below.

**Proposition 33.** *For any $\alpha \in (0,1)$ and any $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, the Gaussian random function $f^{G,\omega} : x \mapsto \sum_{n\in\mathbb{N}} g_n(\omega) c_n e^{inx}$ defines an element of $\mathcal{C}^{0,\alpha}(\mathbb{T})$ if and only if the following two assertions are fulfilled :*

*i) there exists $\beta > 0$ such that $\sum_{j\in\mathbb{N}} \exp\big(-\frac{\beta^2}{(2^{\alpha j}s_j)^2}\big) < +\infty$,*

*ii) the numbers $2^{\alpha j}\mathbf{E}\Big[\sup_{x\in[-\pi,\pi]}\Big|\sum_{n=2^j}^{2^{j+1}-1} g_n(\omega)c_n e^{inx}\Big|\Big]$ are uniformly bounded with respect to $j \in \mathbb{N}$.*

**Remark 34.** *The expectations in Point ii) may appear to be abstract (since the most general reformulation is given by the Dudley entropy integral). Here is however a more concrete sufficient condition coming from a result of Salem and Zygmund (see [MP81, pages 122-123]) :*

$$\begin{aligned} 2^{\alpha j}\mathbf{E}\Big[\sup_{x\in[-\pi,\pi]}\Big|\sum_{n=2^j}^{2^{j+1}-1} g_n(\omega)c_n e^{inx}\Big|\Big] &= 2^{\alpha j}\mathbf{E}\Big[\sup_{x\in[-\pi,\pi]}\Big|\sum_{n=1}^{2^j} g_n(\omega)c_{n+2^j-1} e^{inx}\Big|\Big] \\ &\lesssim 2^{\alpha j}\sum_{p=1}^{2^j}\frac{1}{p\sqrt{\ln(p+1)}}\Big(\sum_{n=p+2^j-1}^{2^{j+1}-1}|c_n|^2\Big)^{1/2}. \end{aligned} \tag{119}$$

*Moreover, if $(|c_n|)$ is non-increasing then a result of Marcus shows that (119) is an equivalence (see [Mar75] and [MP81, page 129, line (1.27) ]).*

Proof of Proposition 33. Thanks to the Littlewood-Paley result given by Proposition 22, we have to study whether or not the following condition holds true with probability 1 :

$$\sup_{j\geq 1} 2^{j\alpha}\|\theta(-2^{-2j}\mathbf{\Delta})f^{G,\omega}\|_{L^\infty(\mathbb{T})} < \infty. \tag{120}$$

As in Section 17, we may assume that $\theta$ has support in $\big[\frac{1}{4}, 16\big]$, is bounded by 1 and satisfies $\theta \geq \frac{1}{2}$ on $[1,4]$. Hence (120) becomes

$$\sup_{j\geq 1} 2^{j\alpha}\sup_{|x|\leq\pi}\Big|\sum_{n=2^{j-1}+1}^{2^{j+2}-1} g_n(\omega)c_n\theta(2^{-2j}n^2)e^{inx}\Big| < \infty.$$

**Proof of the sufficiency of i) and ii).** The strategy is totally similar to that used in Section 18 consisting in using Proposition 16 with

$$\rho_j = 2^{-\alpha j}, \qquad \mathcal{M} = [-\pi,\pi], \qquad F_j^\omega(x) = \sum_{n=2^{j-1}+1}^{2^{j+2}-1} g_n(\omega)c_n\theta(2^{-2j}n^2)e^{inx}. \tag{121}$$

If ii) holds true then the contraction principle shows the hypothesis ii) of Proposition 16 as follows :

$$\begin{aligned} \mathbf{E}\Big[\sup_{|x|\leq\pi}|F_j^\omega(x)|\Big] &\leq \mathbf{E}\Big[\sup_{|x|\leq\pi}\Big|\sum_{n=2^{j-1}}^{2^{j+2}-1} g_n(\omega)c_n e^{inx}\Big|\Big] \\ &\leq \sum_{a\in\{-1,0,1\}}\mathbf{E}\Big[\sup_{|x|\leq\pi}\Big|\sum_{n=2^{j+a}}^{2^{j+a+1}-1} g_n(\omega)c_n e^{inx}\Big|\Big] = \mathcal{O}\big(2^{-\alpha j}\big) \end{aligned}$$

Let us now assume i). By looking at (118), for any $j \in \mathbb{N}^\star$, we see that the weak variance (49) of $F_j^\omega$ satisfies $\sigma_j^2 \leq s_{j-1}^2 + s_j^2 + s_{j+1}^2$ and hence

$$\forall \gamma > 0 \qquad \exp\Big(-\frac{3\gamma}{\sigma_j^2}\Big) \leq \exp\Big(-\frac{\gamma}{s_{j-1}^2}\Big) + \exp\Big(-\frac{\gamma}{s_j^2}\Big) + \exp\Big(-\frac{\gamma}{s_{j+1}^2}\Big).$$

We then get

$$\begin{aligned}\sum_{j\in\mathbb{N}^\star} \exp\Big(-(\sqrt{6}\beta 2^\alpha)^2 \frac{\rho_j^2}{2\sigma_j^2}\Big) &= \sum_{j\in\mathbb{N}^\star} \exp\Big(-\frac{\beta^2}{(2^{\alpha(j-1)})^2} \times \frac{3}{\sigma_j^2}\Big) \\ &\leq \sum_{a\in\{-1,0,1\}} \sum_{j\in\mathbb{N}^\star} \exp\Big(-\frac{\beta^2}{(2^{\alpha(j-1)})^2} \times \frac{1}{s_{j+a}^2}\Big).\end{aligned}$$

And we obtain the assumption i) of Proposition 16 by bounding $j-1 \leq j+a$ :

$$\begin{aligned}\sum_{j\in\mathbb{N}^\star} \exp\Big(-(\sqrt{6}\beta 2^\alpha)^2 \frac{\rho_j^2}{2\sigma_j^2}\Big) &\leq \sum_{a\in\{-1,0,1\}} \sum_{j\in\mathbb{N}^\star} \exp\Big(\frac{-\beta^2}{(2^{\alpha(j+a)} s_{j+a})^2}\Big) \\ &\leq 3\sum_{j\in\mathbb{N}} \exp\Big(\frac{-\beta^2}{(2^{\alpha j} s_j)^2}\Big) < +\infty.\end{aligned}$$

**Proof of the necessity of i) and ii).** We shall use the reverse result given by Proposition 18. We assume that $(2^{\alpha j} F_j^\omega)_{j\in\mathbb{N}}$ is almost surely bounded on $[-\pi,\pi]$. But the same parameters as in (121) does not seem to be relevant because the Gaussian processes $(2^{\alpha j} F_j^\omega)_{j\in\mathbb{N}^\star}$ may not be independent. Let us fix $r \in \{1,2,3\}$, we note that the Gaussian processes $(2^{\alpha(3j+3)} F_{3j+r}^\omega)_{j\in\mathbb{N}}$ are independent. Hence we can apply Proposition 18 with $\rho_j = 2^{-\alpha(3j+r)}$ (instead of $2^{-\alpha j}$) and whose conclusion involves the weak variance of $F_{3j+r}^\omega$ and the expectation $\mathbf{E}\Big[\sup_{|x|\leq\pi} |F_{3j+r}^\omega(x)|\Big]$. For the weak variance, we may invoke the inequality $\theta \geq \frac{1}{2}$ on $[1,4]$ to get

$$\sigma_j^2 := \sup_{|x|\leq\pi} \mathbf{E}[|F_{3j+r}^\omega(x)|^2] = \sum_{n=1+2^{(3j+r)-1}}^{-1+2^{(3j+r)+2}} |c_n|^2 |\theta(2^{-2(3j+r)} n^2)|^2 \geq \frac{1}{4} \sum_{n=2^{3j+r}}^{2^{(3j+r)+1}-1} |c_n|^2 = \frac{1}{4} s_{3j+r}^2.$$

Hence, the conclusion of Proposition 18 ensures that the series $\sum_{j\in\mathbb{N}} \exp\Big(-\frac{\beta_r^2}{2(2^{\alpha(3j+r)} s_{3j+r})^2}\Big)$ converges for a suitable constant $\beta_r > 0$. This convergence is true for any $r \in \{1,2,3\}$ and thus we get Point i) of Proposition 33 with $\beta = \frac{1}{\sqrt{2}} \max(\beta_1, \beta_2, \beta_3)$.

To get Point ii), we shall use a last time the contraction principle (with the condition $\theta \geq \frac{1}{2}$ on $[1,4]$) in the same spirit as in Step 2 in Section 20. We define

$$\begin{aligned} n \in [2^{3j+r}, 2^{(3j+r)+1} - 1] \quad &\Rightarrow \quad t_n := \frac{1}{\theta(2^{-2(3j+r)} n^2)}, \\ n \notin [2^{3j+r}, 2^{(3j+r)+1} - 1] \quad &\Rightarrow \quad t_n := 0. \end{aligned}$$

We note that $t_n$ is bounded by 2. Then we may come back to $F_{3j+r}^\omega$ given in (121) :

$$\begin{aligned} &\sup_{j\in\mathbb{N}} \Big(2^{\alpha(3j+r)} \mathbf{E}\Big[\sup_{|x|\leq\pi} \Big| \sum_{n=2^{3j+r}}^{2^{(3j+r)+1}-1} g_n(\omega) c_n e^{inx} \Big|\Big]\Big) \\ &= \sup_{j\in\mathbb{N}} \Big(2^{\alpha(3j+r)} \mathbf{E}\Big[\sup_{|x|\leq\pi} \Big| \sum_{n=1+2^{(3j+r)-1}}^{-1+2^{(3j+r)+2}} g_n(\omega) t_n \theta(2^{-2(3j+r)} n^2) c_n e^{inx} \Big|\Big]\Big) \\ &\leq 2 \sup_{j\in\mathbb{N}} \Big(2^{\alpha(3j+r)} \mathbf{E}\Big[\sup_{|x|\leq\pi} |F_{3j+r}^\omega(x)|\Big]\Big) < +\infty, \end{aligned}$$

where the last bound comes from the conclusion of Proposition 18. As above, this is true for any $r \in \{1,2,3\}$ and we obtain Point ii). □

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