Normal forms for semilinear superquadratic quantum oscillators

Rafik Imekraz[1]

Université de Nantes, Laboratoire de Mathématiques Jean Leray UMR CNRS 6629
2, rue de la Houssinière, 44322 Nantes Cedex 3, France
url : http://perso.crans.org/imekraz

Abstract

On the real line, we consider nonlinear Hamiltonian Schrödinger equations with the superquadratic oscillator $-d^2/dx^2 + x^{2p} + \eta(x) + M$, where p is an integer ≥ 2, η is a polynomial of degree $< 2p$ such that $\inf(x^{2p} + \eta(x)) \geq 0$, and M is a multiplier (i.e. simultaneously diagonalized with $-d^2/dx^2+x^{2p}+\eta(x)$). A previous article ([11]) contains the case $p = 1$ in $\mathbb{R}^d$. Here we deal with $d = 1$ but we authorize any superquadratic potential. Under generic conditions on M related to the nonresonance of the linear part, such a Hamiltonian equation admits, in a neighborhood of the origin, a Birkhoff normal form at any order. Consequently we deduce long time existence for solutions of the above equation with small Cauchy data in the high Sobolev spaces. As spectral analysis (spectrum and eigenfunctions) of the linear part is not explicit, we use Helffer-Robert and Yajima-Zhang's results ([13, 21]) to understand asymptotic behavior of both spectrum and eigenfunctions.

Keywords: Nonlinear Schrödinger, Normal form, Superquadratic potential, Hamiltonian

1. Introduction

We are interested in understanding dynamical behavior of the solution of the nonlinear Hamiltonian Schrödinger PDE :

$$\begin{cases} i\partial_t\psi = (-\frac{d}{dx^2} + V(x) + M_k)\psi + \partial_2 g(\psi, \overline{\psi}) & (t,x) \in \mathbb{R}\times\mathbb{R} \\ \psi_{|t=0} = \psi_0 \in \widehat{H}^s \end{cases} \tag{1}$$

where we define $\overline{\mathbb{N}} := \mathbb{N}\backslash\{0\}$, $\overline{\mathbb{Z}} := \mathbb{Z}\backslash\{0\}$ and the following notations :

(A) $V(x)$ is a superquadratic potential, i.e a positive polynomial of degree $2p \geq 4$. We denote $T := -\frac{d}{dx^2} + V(x)$, $(\phi_j)_{j\geq 1}$ the eigenfunctions of T and $(\lambda_j)_{j\geq 1}$ the positive increasing eigenvalues.

[1]Present address : Laboratoire de Mathématiques AGM (Analyse Géométrie Modélisation), Université de Cergy-Pontoise, Site de Saint Martin, 2 avenue Adolphe Chauvin 95302 Cergy-Pontoise Cedex France

(B) The natural Sobolev spaces $\widehat{H}^s := \mathrm{Dom}(T^{s/2}) = \{f \in H^s(\mathbb{R}), x^{ps}f(x) \in L^2(\mathbb{R})\}$ based on T are endowed with the norms $||\cdot||_{\widehat{H}^s}$ (see Section 2.1).

(C) $k \geq 1$ is integer and $(m_j)_{j\geq 1}$ is a sequence which takes values in $[-\frac{1}{2}, \frac{1}{2}]$. Let $M_k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unique bounded linear operator such that $M_k\phi_j = \frac{m_j}{j^k}\phi_j$.

(D) The holomorphic map $g : \mathbb{C}^2 \to \mathbb{C}$ vanishes in $(0,0)$ with order ≥ 3, and $g(\xi, \overline{\xi})$ is real for all $\xi \in \mathbb{C}$. For instance, if $g(\xi_1, \xi_2) = \frac{1}{2}\xi_1^2\xi_2^2$ then $\partial_2 g(\xi, \overline{\xi}) = \xi|\xi|^2$.

(E) The product space $\left[-\frac{1}{2}, \frac{1}{2}\right]^{\overline{\mathbb{N}}}$ is endowed with the canonical product measure when each $[-\frac{1}{2}, \frac{1}{2}]$ is itself endowed with the Lebesgue measure.

Our main result concerns almost global existence in high Sobolev spaces :

Theorem 1.0.1. *For any an integer $k \geq 1$, there is a full measure set $F_k \subset [-\frac{1}{2}, \frac{1}{2}]^{\overline{\mathbb{N}}}$ such that if $m \in F_k$ (the generic case), for all $r \geq 3$ and $s \gg 1$ large enough, if $\varepsilon := ||\psi_0||_{\widehat{H}^s} \ll 1$ then the PDE (1) admits one and only one solution $\psi(t,x) = \sum_{j\geq 1} z_j(t)\phi_j(x)$ in the space $\mathcal{C}^0((-C\varepsilon^{-r}, +C\varepsilon^{-r}), \widehat{H}^s)$. Furthermore, we control for $|t| \leq C\varepsilon^{-r}$*

$$||\psi(t,\cdot)||_{\widehat{H}^s} \leq 2\varepsilon, \quad \textit{and} \quad \sum_{j\geq 1} \lambda_j^s ||z_j(t)|^2 - |z_j(0)|^2| \leq C\varepsilon^3$$

Remark 1.0.2. *Dynamically, the last inequality means $\psi(t,\cdot)$ stays near an infinite torus $|z_j| = |z_j(0)|$.*

Remark 1.0.3. *We can obtain the same conclusion if g is only holomorphic in a neighborhood of $(0,0)$. In fact, the only thing we have to check is that $\partial_2 g(\psi(t,x), \overline{\psi(t,x)})$ is defined if $|t| \leq C\varepsilon^{-r}$. Indeed, when s is large, this is a consequence of the inequality $||\psi(t,\cdot)||_{L^\infty} \leq C(s)||\psi(t,\cdot)||_{\widehat{H}^s}$ which ensures that $\partial_2 g(\psi, \overline{\psi})$ is defined.*

Our method has been developed by Bambusi ([5]), Bambusi-Grébert ([3, 4]) and Faou-Grébert ([10]) for PDEs on torus, by Bambusi-Delort-Grébert-Szeftel ([2]) for the semilinear Klein-Gordon equation on S^d, and by Grébert-Imekraz-Paturel ([11]) for the semilinear quantum harmonic oscillator on $\mathbb{R}^N$, see also a previous work of Bourgain ([8]). In all these previous situations, spectral analysis is important. As in [11], we deal with PDE on noncompact manifold. As the potential grows at infinity, spectrum of linear part is pure point.

In Theorem 1.0.1, it is important to remark that we control behavior of solution when the initial condition is regular (s large). Furthermore, the class perturbation $\partial_2 g(\psi, \overline{\psi})$ is very large. For instance, we may choose focusing and defocusing cubic perturbation $\pm|\psi|^2\psi$. Remark also in the defocusing situation and $V(x) = x^{2p}$, it is well known that there is a global solution in $\widehat{H}^1$ because the following energy is bounded

$$||\psi(t,\cdot)||^2_{\widehat{H}^1} = \int_{\mathbb{R}} |d_x\psi(t,x)|^2 + |x^p\psi(t,x)|^2 dx$$

The set F_k is created to avoid resonances of linear part.

Let us explain the abstract model. The main idea is to transfer the PDE (1) to the space $\mathcal{C}^0(\mathbb{R}, \ell_s(\overline{\mathbb{Z}}))$, where

$$\ell_s(\overline{\mathbb{Z}}) := \left\{ z \in \mathbb{C}^{\overline{\mathbb{Z}}}, \quad ||z||_s := \sqrt{\sum_j \lambda_j^s |z_j|^2} < \infty \right\}$$

with the help of the following isomorphism

$$\begin{array}{rccll} \Gamma_s : & \widehat{H}^s & \to & \ell_s(\overline{\mathbb{N}}) & \subset \ell_s(\overline{\mathbb{Z}}) \\ & \sum_{j\geq 1} u_j \phi_j & \mapsto & (u_j)_{j\geq 1} & = ((\overline{u_{-j}})_{j\leq 1}, (u_j)_{j\geq 1}) \end{array} \tag{2}$$

In other words, we define

$$\psi(t,x) = \sum_{j\geq 1} z_j(t)\phi_j(x) \qquad \forall j \geq 1 \quad \overline{z_{-j}}(t) = z_j(t)$$

The PDE (1) becomes

$$\forall j \in \overline{\mathbb{N}} \quad \left\{ \begin{array}{rclcl} z_j' & = & -i\frac{\partial}{\partial z_{-j}}(H_0 + P) & = & -i\omega_j z_j - i\frac{\partial P}{\partial z_{-j}} \\ z_{-j}' & = & i\frac{\partial}{\partial z_j}(H_0 + P) & = & i\omega_j z_{-j} + i\frac{\partial P}{\partial z_j} \end{array} \right. \tag{3}$$

where the free Hamiltonian and the nonlinear perturbation read

$$H_0(z) = \sum_{j>0} \omega_j z_j z_{-j}, \qquad P(z) = \int_{\mathbb{R}} g\left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j}\phi_j(x) \right) dx \tag{4}$$

Actually, the two differential equations (3) will be redundant because of the assumption (D). With a natural symplectic structure on $\ell_s(\overline{\mathbb{Z}})$, (3) becomes

$$z'(t) = iX_{H_0+P}(z) = iX_{H_0}(z(t)) + iX_P(z(t))$$

where X_{H_0+P} is the symplectic gradient of $H_0 + P$. Notice that $(H_0+P)(z)$ is conserved. As H_0 is quadratic, X_{H_0} is linear, thus we prefer to see the last equation as

$$z(t) = \exp(itX_{H_0})z(0) + \int_0^t \exp(i(t-t')X_{H_0})iX_P(z(t'))dt'$$

The crucial fact is that the flow $\exp(itX_{H_0})$ stabilizes $\ell_s(\overline{\mathbb{Z}})$ and X_P takes values in $\ell_s(\overline{\mathbb{Z}})$. A well known fixed-point argument shows local existence in the space $\mathcal{C}^0((-T,T), \overline{B(z(0),\varepsilon)})$. To prove long time existence and dynamical consequences, we use a Birkhoff normal form procedure. Precisely, we prove that there is a symplectic transformation τ on a neighborhood of $0 \in \ell_s(\overline{\mathbb{Z}})$ such that $(H_0 + P) \circ \tau = H_0 + Z + R$, where Z will be in normal form (i.e. only depends on the actions $z_j z_{-j}$, see Section 3.3) and $||X_R(z)||_s \leq C||z||_s^r$. Here the following is important : X_Z maps $\ell_s(\overline{\mathbb{Z}})$ to $\ell_s(\overline{\mathbb{Z}})$.

The Birkhoff normal form procedure needs to create a model of perturbation which contains of course P and some polynomials in normal form.

When expanding g with its Taylor series in (4), the integrals-products of eigenfunctions appear. In our model, the following estimate

$$\left|\int_{\mathbb{R}} \phi_{j_1}(x)\cdots\phi_{j_k}(x)dx\right| \leq C_N j_3^{\nu}\left(\frac{\sqrt{\lambda_{j_2}\lambda_{j_3}}}{\sqrt{\lambda_{j_2}\lambda_{j_3}}+\lambda_{j_1}-\lambda_{j_2}}\right)^N \tag{5}$$

plays an essential role for arbitrary N and $j_1 \geq \cdots \geq j_k$. In some sense, these integrals explain how different modes interact together via the nonlinear term. For instance, those integrals of products are also present in [11] which deals with the harmonic oscillator on $\mathbb{R}^N$:

$$i\partial_t = (-\Delta + ||x||^2 + M)\psi + \partial_2 g(\psi, \overline{\psi}) \tag{6}$$

In that case, the eigenfunctions ϕ_j are Hermite functions. Analogue estimations occur in [19], where Wei-Min Wang shows the stability under the time dependent perturbation V (quasi-periodic in t) and small δ :

$$-i\partial_t = \frac{1}{2}\left(-\Delta + x^2\right) + \delta V(t,x)$$

The eigenfunctions of harmonic oscillator $-\frac{d^2}{dx^2} + x^2$, i.e. the Hermite functions, are very nice because it is possible to compute exact values of integrals (see [20]) and we know that the eigenvalues are exactly the odd integers. In our case, eigenfunctions and eigenvalues are not explicit.

To obtain estimate of integrals (5), we use a commutator lemma (a bit sharper than in [5] because we need to control degree of some polynomials, see Lemma 2.3.1), and spectral information of T. Indeed, for $p, r \geq 2$ we have the following asymptotic for some $\sigma(r,p) \in \mathbb{R}$ (see [21] or Theorem 2.4.1 in the present article)

$$||\phi_j||_{L^r} \simeq j^{\sigma(r,p)}$$

To compare with the Lebesgue norms of Hermite functions, we use in [11] that one has $\sigma(\infty, 1) < 0$ [14, 18]. Furthermore, we have

$$c|j_1^{2p/(p+1)} - j_2^{2p/(p+1)}| \leq |\lambda_{j_1} - \lambda_{j_2}| \leq C|j_1^{2p/(p+1)} - j_2^{2p/(p+1)}| \tag{7}$$

That means essentially the differences of two eigenvalues do not accumulate to zero. Notice that (7) is not a simple consequence of the Weyl formula $\lambda_j \simeq j^{2p/p+1}$, indeed we need a precise asymptotic behavior

$$\lambda_j = \beta_0 j^{2p/p+1} + \beta_1 j^{(2p-1)/p+1)} + \cdots + \beta_{2p-1} j^{1/(p+1)} + \beta_{2p} + o(1) \tag{8}$$

This has been obtained for large class of 1d differential operator in [13]. The estimate (7) is also helpful to ensure the existence of the full measure set F_k, this set is defined by a nonresonance condition.

This article is organized as follows : in Section 2, we develop all the spectral information that we need. In Section 3 we define the abstract model, i.e. all the polynomial classes and prove estimates on them in view to make work a Birkhoff normal form procedure. In Section 3, we just check that the abstract model is adequate for our purpose.

To finish this introduction, we want to notice the analogue problem seems to be hard for the multi-dimensional case because we do not have a similar asymptotic (8). For instance, we do not know if the following equations in $\mathbb{R}^2$ admit almost global existence in the high Sobolev spaces and for generic bounded operators M:

$$\begin{aligned} i\partial_t \psi &= (-\Delta + (x_1^2 + x_2^2)^2 + M)\psi + |\psi|^2\psi \\ i\partial_t \psi &= (-\Delta + x_1^4 + x_2^4 + M)\psi + |\psi|^2\psi \end{aligned}$$

Whereas the eigenvalues of the multi-dimensional harmonic oscillator are just explicit sums of finite odd integers (see [11] for almost global existence).

2. Spectral analysis

2.1. Sobolev spaces and distribution of eigenvalues

Let $\mathcal{S}(\mathbb{R})$ be the usual Schwartz class. We define the Sobolev spaces which are naturally based on the differential operator $T = -\frac{d}{dx^2} + V(x)$. First of all, the next results are well known (see for example [6] chapter 2.3, Theorems 3.1, 3.3 and Corollary 1)

Theorem 2.1.1. *The differential operator $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is essentially self-adjoint. Its spectrum is an increasing real sequence $(\lambda_j)_{j\geq 1}$ which tends to $+\infty$ and $\lambda_1 > 0$. Furthermore, there is an orthonormal basis $(\phi_j)_{j\geq 1}$ of $L^2(\mathbb{R})$ such that*

a) $\overline{\phi_j} = \phi_j$,

b) $\phi_j \in \mathcal{S}(\mathbb{R})$,

c) $T\phi_j = \lambda_j \phi_j$,

d) each eigenvalue λ_j is simple.

Notice that d) holds because we consider a 1-dimensional case. Now, we define the Sobolev spaces based on T. In fact, all spectral data we need are **asymptotic**. Recall that we can directly define the operator $T^{s/2}$ by $T^{s/2}\phi_j = \lambda_j^{s/2}\phi_j$.

Definition 2.1.2. *For all $s \geq 0$, we define*

$$\widehat{H}^s := Dom(T^{s/2}) = \left\{ f = \sum_{j\geq 1} \alpha_j \phi_j \in L^2(\mathbb{R}), \quad \sum \lambda_j^s |\alpha_j|^2 < +\infty \right\}$$

$$\forall f = \sum_{j\geq 1} \alpha_j \phi_j \in \widehat{H}^s \qquad ||f||_{\widehat{H}^s} = \left(\sum_{j\geq 1} \lambda_j^s |\alpha_j|^2 \right)^{1/2}$$

Remark 2.1.3. *As each ϕ_j lives in $\mathcal{S}(\mathbb{R})$, the Schwartz class is dense in $\widehat{H}^s$.*

Denote $D = d/dx$ and $\langle x \rangle = \sqrt{1+x^2}$, let us define :

$$\langle -iD \rangle^s u(x) = (\mathrm{Id} - \Delta)^{s/2} u(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi}(1+\xi^2)^{s/2} \widehat{u}(\xi) d\xi$$

Thanks to [22], we have the useful theorem :

Theorem 2.1.4. *For all $s \geq 0$, the following norms are equivalent on $\widehat{H}^s$:*

a) $u = \sum_{j\geq 1}^{\infty} \alpha_j \phi_j \mapsto \left(\sum_j |\alpha_j|^2 (1+|\lambda_j|)^s \right)^{1/2}$,

b) $||\langle iD \rangle^s u||_{L^2} + ||\langle x \rangle^{ps} u||_{L^2}$,

c) $||u(x)||_{L^2} + ||x^s \widehat{u}(x)||_{L^2} + ||x^{ps} u(x)||_{L^2}$,

d) $||u||_{H^s} + ||x^{ps} u(x)||_{L^2}$.

For convenience, we may call $|| \cdot ||_{\widehat{H}^s}$ any previous norm.

Proof. The a) and b) norms are equivalent because of Lemma 2.4 of [22]. The equivalence of b), c) and d) norms is clear because of the usual Sobolev space. □

Remark 2.1.5. *The space $\widehat{H}^s$ is a Hilbert space.*

2.2. Asymptotic distribution of eigenvalues

We need a precise behavior description of the differences $\lambda_{j_1} - \lambda_{j_2}$ when j_1 and j_2 tend to $+\infty$. In fact we have

Proposition 2.2.1. *Denote $\widehat{p} = \frac{p}{p+1}$, there are $c, C > 0$ such that*

$$\forall j_1 > j_2 \geq 1 \qquad c(j_1^{2\widehat{p}} - j_2^{2\widehat{p}}) \leq \lambda_{j1} - \lambda_{j2} \leq C(j_1^{2\widehat{p}} - j_2^{2\widehat{p}})$$

The usual Weyl formula (see [17, Theorem XIII.81]) gives us

$$N(E) := \mathrm{Card}(j \geq 1, \lambda_j \leq E) \simeq \frac{2}{2\pi} \int_{\{x, V(x) \leq E\}} \sqrt{E - V(x)} dx$$

A scaling $x \mapsto xE^{1/2p}$ shows $N(E) \simeq cE^{1/2\widehat{p}}$. Consequently, we have

$$\lambda_j \simeq c j^{2\widehat{p}} \tag{9}$$

Unfortunately this is not sufficient to prove Proposition 2.2.1. That is why we have to know an asymptotic expansion of the sequence λ_j when j goes to infinity.

Theorem 2.2.2. *(Helffer-Robert) Consider $k, p \in \overline{\mathbb{N}}$, and V a real polynomial of degree $2p$ which satisfies $\lim\limits_{\pm\infty} V(x) = +\infty$, if $(\lambda_j)_{j\geq 1}$ is the eigenvalues-sequence of the differential operator $-d^{2k}/dx^{2k} + V(x)$ on $L^2(\mathbb{R})$, then there is a sequence $(b_i)_{i\geq 0}$ with $b_0 > 0$ such that one has the following asymptotic expansion*

$$\lambda_j \simeq (j+\sigma)^{(2kp)/(p+k)} \sum_{i\geq 0} b_i (j+\sigma)^{-i/(p+k)}$$

Proof. Theorem (2-2) of [13] (page 858) reads

$$\lambda_j^{(p+k)/(2kp)} \simeq (j+\sigma)\sum_{i\geq 0} b'_i (j+\sigma)^{-i/(p+k)}$$

with some sequences $(b'_i)_{i\geq 0}$ and $b'_0 > 0$. Then our asymptotic expansion is obtained by composition with the function $x \mapsto x^{2kp/(p+k)}$ around b'_0. □

We prove Proposition 2.2.1 by choosing $k = 1$ in the last theorem, hence

$$\lambda_j = b_0 j^{\widehat{p}} + \beta_1 j^{(2p-1)/(p+1)} + \cdots + \beta_{2p-1} j^{1/(p+1)} + \beta_{2p} + o(1)$$

Consequently, for all $j_1, j_2 \geq 1$, the difference $|\lambda_{j1} - \lambda_{j2}|$ is greater than

$$b_0(j_1^{2\widehat{p}} - j_2^{2\widehat{p}}) - \sum_{i=1}^{2p-1} |\beta_i||j_1^{(2p-i)/(p+1)} - j_2^{(2p-i)/(p+1)}| - |R(j_1) - R(j_2)|$$

where $\lim_{j\to+\infty} R(j) = 0$. For $j_1 > j_2$ large enough, we have

$$b_0(j_1^{2\widehat{p}} - j_2^{2\widehat{p}}) > c > |R(j_1) - R(j_2)|$$

So we just have to control the other terms. Recall that, for all $\omega \in (0,1)$, the map $x \mapsto x^{1/\omega}$ is convex on $(0,+\infty)$, thus if $j_1 > j_2$ then

$$\frac{|j_1^{2\widehat{p}} - j_2^{2\widehat{p}}|}{|j_1^{2\widehat{p}\omega} - j_2^{2\widehat{p}\omega}|} \geq \frac{d(x^{1/\omega})}{dx}(j_2^{2\widehat{p}\omega}) = \frac{j_2^{2\widehat{p}(1-\omega)}}{\omega}$$

Let us choose ω such that $2\widehat{p}\omega$ lives in $\{(2p-i)/(p+1), i \in [[1, 2p-1]]\}$, and we understand that there is some $J \geq 1$ such that the following holds :

$$\forall j_1 > j_2 \geq J \qquad \lambda_{j1} - \lambda_{j2} \geq c(j_1^{2\widehat{p}} - j_2^{2\widehat{p}})$$

Recall that $\lambda_j \simeq j^{2\widehat{p}}$ and Proposition (2.2.1) comes with the two facts

$$\inf_{j_1 > J \geq j_2} \frac{\lambda_{j1} - \lambda_{j2}}{(j_1^{2\widehat{p}} - j_2^{2\widehat{p}})} > 0, \qquad \inf_{J \geq j_1 > j_2} \frac{\lambda_{j1} - \lambda_{j2}}{(j_1^{2\widehat{p}} - j_2^{2\widehat{p}})} > 0$$

The same proof shows

$$\lambda_{j1} - \lambda_{j2} \leq C(j_1^{2\widehat{p}} - j_2^{2\widehat{p}})$$

2.3. Commutator Lemma

Like in [5], we will show a commutator lemma. In this part, the map $u : \mathbb{N}^3 \to \mathbb{R}$ satisfies the next conditions, for all $n \in \mathbb{N}, \alpha \in [0, n]$ and $\beta \in [0, 2n - \alpha]$:

i) $u(0, 0, 0) = 0$,

ii) $u(n, \alpha, \beta) \leq u(n + 1, \alpha, \beta)$,

iii) $u(n, \alpha, \beta) \leq u(n + 1, \alpha, \beta + 2)$,

iv) $u(n, \alpha, \beta) \leq u(n+1, \alpha, \beta+1)$,

v) $u(n, \alpha, \beta) \leq u(n+1, \alpha+1, \beta+1)$,

vi) $u(n, \alpha, \beta) \leq u(n+1, \alpha+1, \beta)$,

vii) if $1 \leq k \leq \alpha$ then $u(n, \alpha, \beta) + 1 \leq u(n+1, \alpha-k, \beta)$.

The condition vii) shows u must not be zero. For instance, we may choose $u(n, \alpha, \beta) = \frac{1}{2}(n-\alpha)$. But for our purpose we will use

$$u(n, \alpha, \beta) = \frac{2n - \alpha - \beta}{2} \tag{10}$$

It would be relevant to find the minimal u which satisfies the above mysterious conditions because of the following lemma :

Lemma 2.3.1. *Let $T = -\Delta + V$ be a differential operator on $\mathcal{S}(\mathbb{R})$, where V is a polynomial of degree $d \in \overline{\mathbb{N}}$, and $a \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that*

$$\forall n \in \mathbb{N} \quad \exists c > 0 \qquad |a^{(n)}(x)| \leq c(1+|x|)^c$$

The operator $A_0 : f \mapsto af$ is well defined on $\mathcal{S}(\mathbb{R})$. By induction, we define the operator

$$\forall n \in \mathbb{N} \qquad A_{n+1} = A_n T - T A_n$$

Then we have

a) A_n is a differential operator of order $\leq n$, and precisely

$$A_n = \sum_{\alpha=0}^{n} \left(\sum_{\beta=0}^{2n-\alpha} V_{\alpha,\beta,n} a^{(\beta)} \right) D^\alpha \tag{11}$$

and $V_{\alpha,\beta,n}$ is a polynomial of degree $\leq (d-1)u(n, \alpha, \beta)$. Furthermore the coefficients of $V_{\alpha,\beta,n}$ depend only on α, β, n and V.

b) If ϕ and $\psi \in \mathcal{S}(\mathbb{R})$ satisfy $T\phi = \lambda\phi$, $T\psi = \mu\psi$ and $\lambda \neq \mu$, then

$$\left| \int_{\mathbb{R}} a(x)\phi(x)\psi(x)dx \right| \leq \frac{1}{|\lambda - \mu|^n} \left| \int_{\mathbb{R}} (A_n\phi)(x)\psi(x)dx \right|$$

PROOF.

a) The assumption on the derivatives of a proves that A_0 is well defined on $\mathcal{S}(\mathbb{R})$. Let us see the first differential operators, for all $f \in \mathcal{S}(\mathbb{R})$, we have $T(f) = -f^{''} + Vf$, then

$$A_1(f) = aT(f) - T(af) = a^{(2)}f + 2a'f'$$

$$A_2(f) = (2V'a' + a^{(4)})f + 4a^{(3)}f' + 4a^{(2)}f^{(2)}$$

$$A_3(f) = (2V^{(3)}a' + 8V^{(2)}a^{(2)} + 6V'a^{(3)} + a^{(6)})f + (4V^{(2)}a' + 12V'a^{(2)} + 6a^{(5)})f' + 12a^{(4)}f^{(2)} + 8a^{(3)}f^{(3)}$$

$$
\begin{aligned}
A_4(f) \quad = \quad & (4V'V^{(2)}a' + 2V^{(5)}a' + 12V^{(4)}a^{(2)} + 12(V')^2a^{(2)} + \\
& 32V^{(3)}a^{(3)} + 32V^{(2)}a^{(4)} + 12V'a^{(5)} + a^{(8)})f \\
& +(8V^{(4)}a' + 40V^{(3)}a^{(2)} + 80V^{(2)}a^{(3)} + 48V'a^{(4)} + 8a^{(7)})f' \\
& +(8V^{(3)}a' + 32V^{(2)}a^{(2)} + 48V'a^{(3)} + 24a^{(6)})f^{(2)} \\
& +32a^{(5)}f^{(3)} + 16a^{(4)}f^{(4)}
\end{aligned}
$$

For instance, we check that the degree $\deg V_{\alpha,\beta,n}$ is less than $\frac{(d-1)}{2}(2n-\alpha-\beta)$ for the first computations. Let us see the general case by an induction.

We have $A_0 = a$, the polynomials are constant, their degrees are less than $(d-1)u(0,0,0) = 0$.

Let us assume that A_n has the form (11). We have to compute A_{n+1} :

$$A_{n+1} = -A_n \circ \Delta + \Delta \circ A_n + A_n V - V A_n$$

The part $-A_n \circ \Delta + \Delta \circ A_n$ will contribute to rise the derivation orders but will not increase polynomial degrees, whereas the other part $A_nV - VA_n$ does the contrary.

Let us begin with $-A_n\Delta + \Delta A_n$, for all $f \in \mathcal{S}(\mathbb{R})$ we have

$$-A_n(f^{(2)}) + (A_n(f))^{(2)} = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{2n-\alpha} \left(V_{\alpha,\beta,n}a^{(\beta)}f^{(\alpha)}\right)^{(2)} - V_{\alpha,\beta,n}a^{(\beta)}f^{(\alpha+2)}$$

The summand is linear combination of the five terms

$V^{(2)}_{\alpha,\beta,n}a^{(\beta)}f^{(\alpha)}, V_{\alpha,\beta,n}a^{(\beta+2)}f^{(\alpha)}, V'_{\alpha,\beta,n}a^{(\beta+1)}f^{(\alpha)}, V_{\alpha,\beta,n}a^{(\beta+1)}f^{(\alpha+1)}, V'_{\alpha,\beta,n}a^{(\beta)}f^{(\alpha+1)}$

The conditions $0 \leq \alpha \leq n$ and $0 \leq \beta \leq 2n-\alpha$ prove that $-A_n(f^{(2)}) + (A_n(f))^{(2)}$ is indeed a polynomial combination of $a^{(\beta)}f^{(\alpha)}$ with $0 \leq \alpha \leq n+1$ and $0 \leq \beta \leq 2n-\alpha+2 = 2(n+1)-\alpha$. Furthermore, we must check on the five terms that the degree polynomial coefficient of $a^{(\beta)}f^{(\alpha)}$ is less than $(d-1)u(n+1,\alpha,\beta)$. In fact, for each term we have

$$
\begin{aligned}
V^{(2)}_{\alpha,\beta,n}a^{(\beta)}f^{(\alpha)} \quad &\Rightarrow \quad \deg(V^{(2)}_{\alpha,\beta,n}) \leq \deg(V_{\alpha,\beta,n}) \leq (d-1)u(n,\alpha,\beta) \leq (d-1)u(n+1,\alpha,\beta) \\
V_{\alpha,\beta,n}a^{(\beta+2)}f^{(\alpha)} \quad &\Rightarrow \quad \deg(V_{\alpha,\beta,n}) \leq (d-1)u(n,\alpha,\beta) \leq (d-1)u(n+1,\alpha,\beta+2) \\
V'_{\alpha,\beta,n}a^{(\beta+1)}f^{(\alpha)} \quad &\Rightarrow \quad \deg(V'_{\alpha,\beta,n}) \leq \deg(V_{\alpha,\beta,n}) \leq (d-1)u(n,\alpha,\beta) \leq (d-1)u(n+1,\alpha,\beta+1) \\
V_{\alpha,\beta,n}a^{(\beta+1)}f^{(\alpha+1)} \quad &\Rightarrow \quad \deg(V_{\alpha,\beta,n}) \leq (d-1)u(n,\alpha,\beta) \leq (d-1)u(n+1,\alpha+1,\beta+1) \\
V'_{\alpha,\beta,n}a^{(\beta)}f^{(\alpha+1)} \quad &\Rightarrow \quad \deg(V_{\alpha,\beta,n}) \leq (d-1)u(n,\alpha,\beta) \leq (d-1)u(n+1,\alpha+1,\beta)
\end{aligned}
$$

By that way, we can see that the new polynomials $V_{\alpha,\beta,n+1}$ depend only on the last polynomials $V_{\alpha,\beta,n}$ and V. Now, let us compute $A_nV - VA_n$.

$$
\begin{aligned}
A_n(Vf) - VA_n(f) \quad &= \quad \sum_{\alpha=0}^{n} \sum_{\beta=0}^{2n-\alpha} V_{\alpha,\beta,n}a^{(\beta)} \left((Vf)^{(\alpha)} - Vf^{(\alpha)}\right) \\
&= \quad \sum_{\alpha=1}^{n} \sum_{\beta=0}^{2n-\alpha} V_{\alpha,\beta,n}a^{(\beta)} \left((Vf)^{(\alpha)} - Vf^{(\alpha)}\right)
\end{aligned}
$$

This time, the summand is a linear combination of $V_{\alpha,\beta,n}V^{(k)}a^{(\beta)}f^{(\alpha-k)}$ for $k \in [1,\alpha]$. Hence, $A_n(Vf) - VA_n(f)$ is polynomial combination of $a^{(\beta)}f^{(\alpha)}$ with $\alpha \in [0,n] \subset [0,n+1]$ and $\beta \in [0,2n-\alpha] \subset [0,2n+2-\alpha]$. As above, we have to check in $A_n(Vf) - VA_n(f)$ that the polynomial coefficient of $a^{(\beta)}f^{(\alpha)}$ has a degree less than $(d-1)u(n+1,\alpha,\beta)$. It is sufficient to look at the term $V_{\alpha,\beta,n}V^{(k)}a^{(\beta)}f^{(\alpha-k)}$ if $k \in [\![1,\alpha]\!]$:

$$\begin{aligned} \deg\left(V_{\alpha,\beta,n}V^{(k)}\right) &\leq (d-1)u(n,\alpha,\beta) + (d-1) \\ &\leq (d-1)u(n,\alpha-k,\beta) \end{aligned}$$

Again, the new polynomials depend only on the last polynomials and derivatives of V.

b) The operator T is clearly self-adjoint, and then

$$\int_{\mathbb{R}} (A_{n+1}\phi)\psi = \int_{\mathbb{R}} (A_n T\phi)\psi - \int_{\mathbb{R}} (A_n\phi)(T\psi) = (\lambda - \mu)\int_{\mathbb{R}} (A_n\phi)\psi$$

The conclusion comes with an easy induction.

□

2.4. Some bounds of eigenfunctions and product functions

Recall that $V(x)$ has degree $2p \geq 4$. Theorem 1.5 of [21] (page 576) explains the asymptotic behavior of eigenfunctions in the Lebesgue spaces :

Theorem 2.4.1. *For all $r \in [2,\infty]$ and $p \geq 2$ there is $\sigma(r,p) \geq \frac{-1}{8p}$ such that*

$$||\phi_j||_{L^r} \simeq C_{r,p} j^{\sigma(r,p)}$$

with

$$\begin{aligned} 2 \leq r < 4 &\Rightarrow \sigma(r,p) = \tfrac{1}{2p}\left(\tfrac{1}{r} - \tfrac{1}{2}\right) \leq 0 \\ 4 < r \leq \tfrac{4p-2}{p-2} &\Rightarrow \sigma(r,p) = \tfrac{1}{3}\left(1 - \tfrac{1}{r}\right)\left(1 - \tfrac{1}{2p}\right) - \tfrac{1}{4} \leq 0 \\ \tfrac{4p-2}{p-2} \leq r \leq +\infty &\Rightarrow \sigma(r,p) = \tfrac{1}{3}\left(1 - \tfrac{1}{r}\right)\left(1 - \tfrac{1}{2p}\right) - \tfrac{1}{4} \geq 0 \end{aligned}$$

In particular, we have $\sigma(r,p) \leq \frac{1}{3}\left(1 - \frac{1}{2p}\right) - \frac{1}{4}$.

The Hölder inequality leads to the following easy corollary.

Corollary 2.4.2. *For each $p \geq 2$, $k \geq 3$, there is $\gamma(k,p) > 0$ such that for all $j_1 \geq \cdots j_k$ we have*

$$\left|\int_{\mathbb{R}} \phi_{j_1}(x)\cdots\phi_{j_k}(x)dx\right| \leq C j_3^{\gamma} \tag{12}$$

Proof. Choose $r_1 = r_2 < \frac{4p-2}{p-2}$ and $r_3, \ldots, r_k \geq 2$ such that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \cdots + \frac{1}{r_k} = 1$. Consider $\gamma := (k-2)\left(\frac{1}{3}\left(1 - \frac{1}{2p}\right) - \frac{1}{4}\right)$ and apply Hölder's inequality to dominate the left of (12) by

$$||\phi_{j_1}||_{L^{r_1}}||\phi_{j_2}||_{L^{r_2}}\prod_{i=3}^{k}||\phi_{j_i}||_{L^{r_i}} \leq C j_3^{\gamma}$$

□

Lemma 2.4.3. *For all $f \in \mathcal{S}(\mathbb{R})$ and $a, b \in \mathbb{N}, s \geq 0$ we have*

$$||x^a f^{(b)}||_{\widehat{H}^s} \leq C(s,a,b)||f||_{\widehat{H}^{s+a/p+b}}$$

Proof. With the help of the c) norm of Theorem 2.1.4, we can show that the case $s = 0$ is sufficient. This particular case is an easy consequence of Weyl calculus. The pseudo-differential operator $x^a \dfrac{d^b}{dx^b} \circ T^{-a/2p-b/2}$ is bounded on $L^2(\mathbb{R})$ because of Calderón-Vaillancourt Theorem ([7]): the following map has bounded derivatives

$$(x,\xi) \mapsto \frac{x^a \xi^b}{(1+\xi^2+V(x))^{a/2p+b/2}} < +\infty$$

Hence, there is $C(a,b) > 0$ such that

$$||x^a f^{(b)}||_{L^2} \leq C(a,b)||T^{a/2p+b/2} f||_{L^2} \leq C(a,b)||f||_{\widehat{H}^{a/p+b}}$$

$\square$

Proposition 2.4.4. *For all $s > \frac{1}{2}$ and $f, g \in \widehat{H}^s$ we have $f, g \in L^\infty(\mathbb{R})$ and*

$$||fg||_{\widehat{H}^s} \leq C(s)(||f||_{\widehat{H}^s}||g||_\infty + ||f||_{\widehat{H}^s}||g||_\infty)$$

In other words, $\widehat{H}^s$ is stable by product.

Proof. We have $\widehat{H}^s \subset H^s \subset L^\infty$ and $\widehat{H}^s \subset L^2(\mathbb{R})$, thus the following inequality holds (see [1, page 98, chapter 2, Proposition 2.1.1])

$$||fg||_{H^s} \leq C(s)(||f||_{H^s}||g||_\infty + ||f||_\infty||g||_{H^s})$$

And finally,

$$||fg||_{\widehat{H}^s} = ||fg||_{H^s} + ||x^{ps} f(x)g(x)||_{L^2} \leq C(s)(||f||_{H^s}||g||_\infty + ||f||_{H^s}||g||_\infty) + ||f||_{\widehat{H^s}}||g||_\infty$$

$\square$

Theorem 2.1.4 and the estimate (9) let us understand $||\phi_j||_{\widehat{H}^s} \leq Cj^{s\widehat{p}}$. Theorem 2.4.1, last proposition and an easy induction lead to

Corollary 2.4.5. *There is some $\eta(p) > 0$ which depends only on p, such that for every $j_3 \geq \cdots \geq j_k \in \overline{\mathbb{N}}$, we have*

$$||\phi_{j_3} \cdots \phi_{j_k}||_{\widehat{H}^s} \leq kC(s) j_3^{s\widehat{p}+\eta(p)} \tag{13}$$

Now we check that a holomorphic functional calculus is possible on $\widehat{H}^s$.

Proposition 2.4.6. *Consider $s > \frac{1}{2}$, $f, g \in \widehat{H}^s$ and $K : \mathbb{C}^2 \to \mathbb{C}$ a real analytic function which vanishes in $(0,0)$:*

$$K(\xi_1,\xi_2) = \sum_{(n_1,\ldots,n_4)\in\mathbb{N}^4\setminus\{(0,0,0,0)\}} \alpha(n_1,\ldots,n_4)\xi_1^{n_1}\overline{\xi_1}^{n_1}\xi_2^{n_3}\overline{\xi_2}^{n_4}$$

Then, $K(f,g)$ is well defined in $\widehat{H}^s$ and equals the map $x \mapsto K(f(x),g(x))$.

Proof. By bilinearity and Proposition 2.4.4, there is $C(s) > 0$ such that

$$\forall f, g \in \widehat{H}^s \qquad ||fg||_{\widehat{H}^s} \leq C(s)||f||_{\widehat{H}^s}||g||_{\widehat{H}^s}$$

Theorem 2.1.4 shows that $\widehat{H}^s$ is invariant by $f \mapsto \overline{f}$ and $||f||_{\widehat{H}^s} = ||\overline{f}||_{\widehat{H}^s}$. Hence, the following series converges uniformly on bounded subsets of $\widehat{H}^s \times \widehat{H}^s$ for the sub-multiplicative norm $C(s)||\cdot||_{\widehat{H}^s}$:

$$K(f,g) := \sum_{(n_1,\ldots,n_4)\in\mathbb{N}^4\setminus\{(0,0,0,0)\}} \alpha(n_1,\ldots,n_4) f^{n_1}\overline{f}^{n_1} g^{n_3}\overline{g}^{n_4}$$

As $s > \frac{1}{2}$, the convergence in $\widehat{H}^s$ implies the convergence in $L^\infty(\mathbb{R})$. That means $K(f,g)$ equals the map

$$x \mapsto \sum_{(n_1,\ldots,n_4)\in\mathbb{N}^4\setminus\{(0,0,0,0)\}} \alpha(n_1,\ldots,n_4) f(x)^{n_1}\overline{f(x)}^{n_1} g(x)^{n_3}\overline{g(x)}^{n_4} = K(f(x),g(x))$$

□

2.5. Integrals of eigenfunctions products

Recall that $\widehat{p} = \frac{p}{p+1} \in \left(\frac{2}{3}, 1\right)$. We can now give an estimation of eigenfunctions products.

Proposition 2.5.1. *For all $p \geq 2, k \geq 3, N \geq 1$, there are $\nu = \nu(p,k) > 0$ and $C(k,N,p) > 0$ such that*

$$\forall j \in \overline{\mathbb{N}}^k \qquad \left|\int_{\mathbb{R}} \phi_{j1}(x)\cdots\phi_{jk}(x)dx\right| \leq C(k,N,p) j_3^\nu A(j)^N \tag{14}$$

where $j_1 \geq j_2 \geq ... \geq j_k$ and

$$A(j) := \frac{(j_2 j_3)^{\widehat{p}}}{(j_2 j_3)^{\widehat{p}} + j_1^{2\widehat{p}} - j_2^{2\widehat{p}}}$$

Remark 2.5.2. *This estimate generalizes the one we obtained in [11] for Hermite product integrals.*

Remark 2.5.3. *Thanks to Proposition 2.2.1, it is not hard to see*

$$cA(j) \leq \frac{\sqrt{\lambda_{j_2}\lambda_{j_3}}}{\sqrt{\lambda_{j_2}\lambda_{j_3}} + \lambda_{j_1} - \lambda_{j_2}} \leq CA(j)$$

Proof. For convenience, p will not appear in the constants. We now consider two cases.

- $j_1^{2\widehat{p}} - j_2^{2\widehat{p}} \leq (j_2 j_3)^{\widehat{p}}$. It is clear because $A(j) \geq \frac{1}{2}$ and Corollary 2.4.2.

- $(j_2 j_3)^{\widehat{p}} \leq j_1^{2\widehat{p}} - j_2^{2\widehat{p}}$. In particular $j_1 > j_2$.

We introduce the operator $A : f \mapsto af$ where $a = \phi_{j_3} \cdots \phi_{j_k}$. Lemma 2.3.1 let us see

$$\begin{aligned}
\left|\int_{\mathbb{R}} \phi_{j1} \cdots \phi_{jk} dx\right| &= \left|\int_{\mathbb{R}} (A\phi_{j_2})\phi_{j_1}\right| \\
&\leq \frac{1}{|\lambda_{j1} - \lambda_{j2}|^N} ||A_N \phi_{j2}||_{L^2} \\
&\leq \frac{1}{|\lambda_{j1} - \lambda_{j2}|^N} \sum_{\alpha=0}^{N} \sum_{\beta=0}^{2N-\alpha} ||V_{\alpha,\beta,N}(D^\beta a)(D^\alpha \phi_{j2})||_{L^2} \\
&\leq \frac{1}{|\lambda_{j1} - \lambda_{j2}|^N} \sum_{\alpha=0}^{N} \sum_{\beta=0}^{2N-\alpha} ||V_{\alpha,\beta,N} D^\beta a||_{L^\infty} ||D^\alpha \phi_{j2}||_{L^2} \\
&\leq \frac{C}{|\lambda_{j1} - \lambda_{j2}|^N} \sum_{\alpha=0}^{N} \sum_{\beta=0}^{2N-\alpha} ||V_{\alpha,\beta,N} D^\beta a||_{H^1} ||D^\alpha \phi_{j2}||_{L^2}
\end{aligned}$$

Lemma 2.4.3 and the inclusion $H^1(\mathbb{R}) \subset \widehat{H}^1(\mathbb{R})$ give us

$$\begin{aligned}
\left|\int_{\mathbb{R}} \phi_{j1} \cdots \phi_{jk} dx\right| &\leq \frac{C}{|\lambda_{j1} - \lambda_{j2}|^N} \sum_{\alpha=0}^{N} \sum_{\beta=0}^{2N-\alpha} ||a||_{\widehat{H}^{1+\beta+(1/p)\deg V_{\alpha,\beta,N}}} ||\phi_{j2}||_{\widehat{H}^\alpha} \\
&\leq \frac{C(N)}{|\lambda_{j1} - \lambda_{j2}|^N} \max_{\alpha \in [\![0,N]\!], \beta \in [\![0,2N-\alpha]\!]} ||a||_{\widehat{H}^{1+\beta+(1/p)\deg V_{\alpha,\beta,N}}} ||\phi_{j2}||_{\widehat{H}^\alpha}
\end{aligned}$$

Now, remember Proposition 2.2.1, estimate (13) and $||\phi_j||_{\widehat{H}^\alpha} \leq C j^{\widehat{p}\alpha}$, hence

$$\left|\int_{\mathbb{R}} \phi_{j1} \cdots \phi_{jk} dx\right| \leq \frac{C(k,N) j_3^{\eta(p)}}{(j_1^{2\widehat{p}} - j_2^{2\widehat{p}})^N} \max_{\alpha \in [\![0,N]\!], \beta \in [\![0,2N-\alpha]\!]} j_3^{\widehat{p}(1+\beta)+(\deg V_{\alpha,\beta,N})/(p+1)} j_2^{\alpha\widehat{p}} \tag{15}$$

Lemma 2.3.1 implies $\deg V_{\alpha,\beta,N} \leq \frac{2p-1}{2}(2N - \alpha - \beta)$. Considering now $\nu = \eta(p) + \widehat{p}$, we get

$$\begin{aligned}
(j_1^{2\widehat{p}} - j_2^{2\widehat{p}})^N \left|\int_{\mathbb{R}} \phi_{j1} \cdots \phi_{jk} dx\right| &\leq C(k,N) j_3^\nu \max_{\alpha,\beta} j_2^{\widehat{p}\alpha} j_3^{\widehat{p}\beta+(2N-\alpha-\beta)(2p-1)/(2p+2)} \\
&\leq C(k,N) j_3^\nu \max_{\alpha,\beta} j_2^{\widehat{p}\alpha} j_3^{\beta/(2p+2)+(2N-\alpha)(2p-1)/(2p+2)} \\
&\leq C(k,N) j_3^\nu \max_{\alpha} j_2^{\widehat{p}\alpha} j_3^{(2N-\alpha)2p/(2p+2)} \\
&\leq C(k,N) j_3^{\nu+2\widehat{p}N} \max_{\alpha} \left(\frac{j_2}{j_3}\right)^{\widehat{p}\alpha} \\
&\leq C(k,N) j_3^\nu (j_2 j_3)^{\widehat{p}N}
\end{aligned}$$

Recall that $(j_2 j_3)^{\widehat{p}} + j_1^{2\widehat{p}} - j_2^{2\widehat{p}} \leq 2(j_1^{2\widehat{p}} - {j_2}^{2\widehat{p}})$, hence we get (14).

□

3. Abstract Model

3.1. Discretization of PDE

For convenience, we recall that $\overline{\mathbb{N}} = \mathbb{N}\backslash\{0\}$ and $\overline{\mathbb{Z}} = \mathbb{Z}\backslash\{0\}$.

Definition 3.1.1. *We endow* $\left[-\frac{1}{2}, \frac{1}{2}\right]^{\overline{\mathbb{N}}}$ *with the natural product Lebesgue measure.*

Hence, for $(m_j)_{j\geq 1} \in \left[-\frac{1}{2}, \frac{1}{2}\right]^{\overline{\mathbb{N}}}$, we consider M_k the unique bounded operator of $L^2(\mathbb{R})$ such that $M_k\phi_j = j^{-k}m_j\phi_j$. Thus, the spectrum of $T+M_k$ is of course $\omega_j := \lambda_j + j^{-k}m_j$. Let us introduce the space on which we will transfer the PDE (1).

Definition 3.1.2. *We define* $\ell_s(\overline{\mathbb{Z}})$ *the space of sequence* $(z_j)_{j\in\overline{\mathbb{Z}}}$ *such that*

$$||z||_s := \left(\sum_{j\in\overline{\mathbb{Z}}} |j|^{2s\widehat{p}}|z_j|^2\right)^{1/2} < +\infty$$

We define by the same way $\ell_s(\overline{\mathbb{N}})$.

Definition 3.1.3. *A sequence which has the form* $(\overline{z}, z)$ *with* $z \in \ell_s(\overline{\mathbb{N}})$ *is called real. Thus, we identify* $\ell_s(\overline{\mathbb{N}})$ *to a subset of* $\ell_s(\overline{\mathbb{Z}})$.

For each $s \geq 0$, Theorem 2.1.4 and $\lambda_j \simeq j^{2\widehat{p}}$ show the map (2) is a bounded isomorphism, in particular it is a $\mathcal{C}^\infty$-diffeomorphism. The method to solve the PDE (1) in the space $\mathcal{C}^0((-T,+T),\widehat{H}^s)$ is to transfer it in the space $\mathcal{C}^0((-T,T),\ell_s(\overline{\mathbb{Z}}))$ with the help of $z(t) = \Gamma_s(\psi(t,\cdot))$, in other words

$$\psi(t,x) = \sum_{j\geq 1} z_j(t)\phi_j(x) \qquad \forall j \geq 1 \quad \overline{z_{-j}}(t) = z_j(t)$$

The PDE (1) becomes

$$i\sum_{j\geq 1} z_j'(t)\phi_j(x) = \sum_{j\geq 1} \omega_j z_j(t)\phi_j(x) + \partial_2 g\left(\sum_{j\geq 1} z_j(t)\phi_j(x), \sum_{j\geq 1} z_{-j}(t)\phi_j(x)\right)$$

Consider now the two functions on $\ell_s(\overline{\mathbb{Z}})$ defined by (4). In fact, it is easy to check that H_0 is $\mathcal{C}^\infty$ regular on $\ell_s(\overline{\mathbb{Z}})$ for s large because ω_j is polynomially bounded. Proposition 4.0.4 will show that P is also $\mathcal{C}^\infty$ for s large, and especially for each $j \geq 1$ we have

$$\partial_1 g\left(\sum_{j\geq 1} z_j(t)\phi_j(x), \sum_{j\geq 1} z_{-j}(t)\phi_j(x)\right) = \sum_{j\geq 1} \phi_j(x)\frac{\partial P}{\partial z_j}$$

$$\partial_2 g\left(\sum_{j\geq 1} z_j(t)\phi_j(x), \sum_{j\geq 1} z_{-j}(t)\phi_j(x)\right) = \sum_{j\geq 1} \phi_j(x)\frac{\partial P}{\partial z_{-j}}$$

Consequently, (1) is equivalent to the ordinary differential equations

$$\forall j \geq 1 \qquad iz'_j = \omega_j z_j + \frac{\partial P}{\partial z_{-j}}$$

But we must solve it in the space $\ell_s(\overline{\mathbb{N}})$. In fact, as g is holomorphic and satisfies $g(\xi,\overline{\xi}) \in \mathbb{R}$, we can show the following condition (see the proof of Lemma 4.0.6)

$$\forall \xi \in \mathbb{C} \quad \partial_2 g(\xi,\overline{\xi}) = \overline{\partial_1 g(\xi,\overline{\xi})}$$

Finally, the previous remarks prove that solving (1) in the space $\mathcal{C}^0((-T,+T),\widehat{H}^s)$ is equivalent to solve the following Hamiltonian system in the space $\mathcal{C}^0((-T,T),\ell^2_s(\overline{\mathbb{Z}}))$

$$\forall j \in \overline{\mathbb{N}} \quad \begin{cases} \dot{z_j} & = & -i\frac{\partial}{\partial z_{-j}}(H_0+P) & = & -i\omega_j z_j - i\frac{\partial P}{\partial z_{-j}} \\ \dot{z_{-j}} & = & i\frac{\partial}{\partial z_j}(H_0+P) & = & i\omega_j z_{-j} + i\frac{\partial P}{\partial z_j} \end{cases} \tag{16}$$

Notice that the two equations are conjugate if $z(0) \in \ell_s(\overline{\mathbb{N}})$. Local existence is easy (see the next subsection). The main objective of the rest of the article is to develop an abstract model to solve (16) for long time.

3.2. Symplectic structure and Poisson bracket

We will always have $s > 0$, so $\ell_s(\overline{\mathbb{Z}}) \subset \ell_0(\overline{\mathbb{Z}})$. Now, let us recall some usual definitions, we start with the canonical symplectic structure on $\ell_s(\overline{\mathbb{Z}})$ which is given by the following automorphism of $\ell_s(\overline{\mathbb{Z}})$

$$J : ((z_j)_{j<0},(z_j)_{j>0}) \mapsto ((-z_{-j})_{j<0},(z_{-j})_{j>0})$$

Thus $J^2 = -Id$ and $J^\star = J$ if $\ell_s(\overline{\mathbb{Z}})$ is endowed with the canonical duality :

$$\langle z, z'\rangle = \sum_{j\in\overline{\mathbb{Z}}} z_j z'_j$$

If $f : \ell_s \to \mathbb{C}$ is a regular map, then we define its gradient by the formula

$$\forall x \in \ell_s(\overline{\mathbb{Z}}) \quad \forall h \in \ell^2_s(\overline{\mathbb{Z}}) \quad \langle \nabla f_x, h\rangle = \sum_{j\in\overline{\mathbb{Z}}} \left(\frac{\partial f}{\partial z_j}\right)_x h_j$$

Remark that without any condition on f, the gradient ∇f lies in $\ell_{-s}(\overline{\mathbb{Z}})$. The symplectic gradient is just $X_f = iJ\nabla_f$. Notice that (16) becomes

$$z'(t) = iX_{H_0+P}(z(t)) = iX_{H_0}(z(t)) + iX_P(z(t)) \tag{17}$$

As we said in the Introduction, we prefer to reformulate the last equation as

$$z(t) = \exp(itX_{H_0})z(t) + \int_0^t \exp(i(t-t')X_{H_0})iX_P(z(t'))dt' \tag{18}$$

Proposition 4.0.4 will show that X_P takes values on $\ell_s(\overline{\mathbb{Z}})$. As $(\exp(itX_{H_0}))_{t\in\mathbb{R}}$ is a unitary group of $\ell_s(\overline{\mathbb{Z}})$ (solve (16) if $P = 0$), thus (1) has local existence by a classic fixed-point argument.

When it is possible, we define Poisson bracket for two regular functions f and $g \in \mathcal{C}^1(\ell_s(\overline{\mathbb{Z}}), \mathbb{C}))$:

$$\{f, g\} = i\langle \nabla_f, J\nabla_g\rangle = i\sum_{j\geq 1} \frac{\partial f}{\partial z_j}\frac{\partial g}{\partial z_{-j}} - \frac{\partial f}{\partial z_{-j}}\frac{\partial g}{\partial z_j} \tag{19}$$

For instance, if X_f or X_g takes values in $\ell_s(\overline{\mathbb{Z}})$. For any solution $z(t)$ of (16) and regular test function $\phi : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ we have

$$\frac{d}{dt}\phi(z(t)) = \langle \nabla_{z(t)}\phi, z'(t)\rangle = -i\{H_0 + P, \phi\}(z(t))$$

As we see in (18), the map P must have a gradient which takes values in $\ell_s(\overline{\mathbb{Z}})$, that is why we give the following definition :

Definition 3.2.1. *For each $s > 0$, $\mathcal{H}^s$ is the class of functions $F : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ holomorphic on a neighborhood of 0 which satisfy*

$$X_F \in \mathcal{C}^\infty(\ell_s, \ell_s)$$

and if $(F_k)_{k\geq 0}$ is the F-polynomial Taylor sequence in 0, then

$$F_0 = F_1 = F_2 = 0$$

$$F_k \in \mathcal{C}^\infty(\ell_s, \mathbb{C}) \qquad X_{F_k} \in \mathcal{C}^\infty(\ell_s, \ell_s)$$

Finally, we recall a usual condition for regularity polynomial. A homogeneous polynomial P on $\ell_s(\overline{\mathbb{Z}})$ is by definition given by

$$P = \sum_{k=0}^{n} \phi_k(z, \dots, z)$$

where ϕ_k is a continuous k-linear form on $\ell_s(\overline{\mathbb{Z}})$.

Proposition 3.2.2. *Consider $P : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ a homogeneous polynomial of degree $k \geq 1$ which satisfies*

$$\forall z \in \ell_s(\overline{\mathbb{Z}}) \qquad |P(z)| \leq C||z||_s^k$$

then $P : \ell_s(\mathbb{Z}) \to \mathbb{C}$ is of class $\mathcal{C}^\infty$. If furthermore P satisfies $\quad ||X_P(z)||_s \leq C||z||_s^{k-1}$ *then $X_P : \ell_s(\overline{\mathbb{Z}}) \to \ell_s(\overline{\mathbb{Z}})$ is a $\mathcal{C}^\infty$ regular map.*

In fact, in the last proposition P is a holomorphic map (see [9] for polynomial regularity or [15] for holomorphy).

3.3. The polynomial classes

Definition 3.3.1. *A formal polynomial P on $\ell^2_s(\overline{\mathbb{Z}})$ is in the class $T_{k,\nu}$ if it is homogeneous of degree k and can be written :*

$$P(z) = \sum_{(j_1,\cdots,j_k)\in\overline{\mathbb{Z}}^k} a_j z_{j_1}\cdots z_{j_k}$$

such that for all $N > 0$ we have

$$|a_j| \le C(N)\mu(j)^\nu A(j)^N$$

where we order $(j_1,\cdots,j_k)$ in $(j_1^\star,\cdots,j_k^\star)$ such that $|j_1^\star| \ge \cdots \ge |j_k^\star|$ and define

$$\mu(j) = |j_3^\star| \qquad A(j) = \frac{(|j_2^\star j_3|^\star)^{\widehat{p}}}{(|j_2^\star j_3^\star|)^{\widehat{p}} + |j_1^\star|^{2\widehat{p}} - |j_2^\star|^{2\widehat{p}}}$$

In fact, with this definition the formal polynomials of class $T_{k,\nu}$ are regular.

Proposition 3.3.2. *For each $P \in T_{k,\nu}$ and $s > (\nu+1/2)/\widehat{p}$, we have*

$$|P(z)| \le C(P)||z||_s^k$$

Consequently, $P : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ is of class $\mathcal{C}^\infty$ (and even holomorphic).

Proof. Like $0 \le A(j) \le 1$ holds, we deduce

$$\begin{aligned}
|P(z)| &\le C(N)\sum_{j\in\overline{\mathbb{Z}}^k} \mu(j)^\nu |z_{j1}|\cdots|z_{jk}| \\
&\le C(N)\sum_{j\in\overline{\mathbb{Z}}^k} \frac{\mu(j)^\nu}{\prod_{i=1}^k |j_i|^s} \prod_{i=1}^k |j_i|^s |z_{ji}| \\
&\le C(N)\sum_{j\in\overline{\mathbb{Z}}^k} \frac{1}{\prod_{i=1}^k |j_i|^{s-\nu}} \prod_{i=1}^k |j_i|^s |z_{ji}| \\
&\le C(N)\left(\sum_{j\in\overline{\mathbb{Z}}^k} \frac{1}{\prod_{i=1}^k |j_i|^{2s-2\nu}}\right)^{1/2} \left(\sum_{j\in\overline{\mathbb{Z}}^k} \prod_{i=1}^k |j_i|^{2s}|z_{ji}|^2\right)^{1/2} \\
&\le C(N)\left(\sum_{j\in\overline{\mathbb{Z}}} \frac{1}{|j|^{2s-2\nu}}\right)^{k/2} ||z||_{s\widehat{p}}^k
\end{aligned}$$

The conclusion comes with Proposition 3.2.2. □

Unfortunately it does not seem that the condition $P \in T_{k,\nu}$ implies that X_P takes values in $\ell_s(\overline{\mathbb{Z}})$. Furthermore, the classes $(T_{k,\nu})_{k\ge 3}$ does not seem to be invariant by the Poisson bracket. That is why we introduce a new class of polynomials $T^+_{k,\nu}$.

Definition 3.3.3. *A formal polynomial P on $\ell_s(\overline{\mathbb{Z}})$ is in the class $T^+_{k,\nu}$ if it is homogeneous of degree k of the form :*

$$P(z) = \sum_{(j_1,\cdots,j_k)\in\overline{\mathbb{Z}}^k} a_j z_{j_1}\cdots z_{j_k}$$

and if for each $N > 0$, we have :

$$|a_j| \le C(N)\frac{\mu(j)^\nu A(j)^N}{1+S(j)}, \quad S(j) = |j_1^\star|^{2\widehat{p}} - |j_2^\star|^{2\widehat{p}}$$

Lemma 3.3.4. *If j,l are two multi-indexes, we call $A(j,l) := A((j,l))$ the number obtained with the multi-index (j,l). Here, if $l \in \overline{\mathbb{Z}}$, we have*

$$|l|A(j,l) \le C|j_1^\star|$$

Proof. If $|l| \le 2|j_1^\star|$, then it is clear because $A(j,l) \le 1$. If $|l| > 2|j_1^\star| > |j_1| \ge |j_2^\star|$, we have

$$|l|A(j,l) = |l|\frac{(|j_1^\star||j_2^\star|)^{\widehat{p}}}{(|j_1^\star||j_2^\star|)^{\widehat{p}} + |l|^{2\widehat{p}} - |j_1^\star|^{2\widehat{p}}} \le |l|\frac{|j_1^\star|^{2\widehat{p}}}{C|l|^{2\widehat{p}}} \le C\frac{|j_1^\star|^{2\widehat{p}}}{|l|^{2\widehat{p}-1}} \le C|j_1^\star|$$

□

The Cauchy-Schwarz inequality leads to the easy following lemma :

Lemma 3.3.5. *For each $s \ge 0$, $z \in \ell_{s+s_0}(\overline{\mathbb{Z}})$ we have*

$$\sum_{k\in\overline{\mathbb{Z}}} |j|^s|z_j| \le C||z||_{(s+1)/\widehat{p}}$$

For this new class, there is no loss of regularity.

Proposition 3.3.6. *Consider $k \ge 3$, $\nu > 0$, $s > (\nu+3)/\widehat{p}$ and $P \in T^+_{k,\nu}$. We have*

i) P is $\mathcal{C}^\infty$ regular;

ii) The map X_P is regular from $\ell_s(\mathbb{C})$ to $\ell_s(\mathbb{C})$, precisely for each $z \in \ell_s(\overline{\mathbb{Z}})$ we have

$$||X_P(z)||_s \le C||z||_s^{k-1}$$

Proof. Point i) comes from the inclusion $T^+_{k,\nu} \subset T_{k,\nu}$ (see Proposition 3.3.2). Let us prove ii). We only consider the case $k \ge 4$, but the same method gives the case $k = 3$. We choose $N = \widehat{p}s + 1$ in the $T^+_{k,\nu}$ definition, a computation gives us

$$\begin{aligned} \left|\frac{\partial P}{\partial z_l}\right| &\le kC\sum_{j\in\overline{\mathbb{Z}}^{k-1}} \frac{\mu(j,l)^\nu A(j,l)^N}{1+S(j,l)}|z_{j_1}\cdots z_{j_{k-1}}| \\ &\le k\times(k-1)!C\sum_{|j_1|\ge\cdots\ge|j_{k-1}|} \frac{\mu(j,l)^\nu A(j,l)^N}{1+S(j,l)}|z_{j_1}\cdots z_{j_{k-1}}| \end{aligned}$$

Now recall Lemma 3.3.5

$$\begin{aligned}\left|\frac{\partial P}{\partial z_l}\right| &\leq C\left(\sum_{j\in\overline{\mathbb{Z}}}|z_j|\right)^{k-4}\left(\sum_{|j_1|\geq|j_2|\geq|j_3|}\frac{\mu(j,l)^\nu A(j,l)^N}{1+S(j,l)}|z_{j_1}z_{j_2}z_{j_3}|\right)\\ &\leq C||z||_s^{k-4}\sum_{|j_1|\geq|j_2|\geq|j_3|}\frac{\mu(j,l)^\nu A(j,l)^N}{1+S(j,l)}|z_{j_1}z_{j_2}z_{j_3}|\end{aligned}$$

Thus

$$\begin{aligned}||X_P(z)||_s^2 &= \sum_{l\in\overline{\mathbb{Z}}}|l|^{2\widehat{p}s}\left|\frac{\partial P}{\partial z_l}\right|^2\\ &\leq C||z||_s^{2(k-4)}\sum_{l\in\overline{\mathbb{Z}}}|l|^{2\widehat{p}s}\left(\sum_{|j_1|\geq|j_2|\geq|j_3|}\frac{\mu(j,l)^\nu A(j,l)^N}{1+S(j,l)}|z_{j_1}z_{j_2}z_{j_3}|\right)^2\end{aligned}$$

Define the following sets and maps for each $l\in\overline{\mathbb{Z}}$

$$\begin{aligned}\Omega_1(l) &:= \{(j_1,j_2,j_3)\in\overline{\mathbb{Z}}^3, |j_1|\geq|j_2|\geq|j_3|, |j_2|\geq|l|\}\\ \Omega_2(l) &:= \{(j_1,j_2,j_3)\in\overline{\mathbb{Z}}^3, |j_1|\geq|j_2|\geq|j_3|, |l|>|j_2|\}\\ T_i(l) &:= |l|^{\widehat{p}s}\sum_{\Omega_i(l)}\frac{\mu(j,l)^\nu A(j,l)^N}{1+S(j,l)}|z_{j_1}||z_{j_2}||z_{j_3}|\end{aligned}$$

Consequently, we aim to prove

$$\sum_{l\in\overline{\mathbb{Z}}}T_i(l)^2\leq C||z||_s^6,\qquad i\in\{0,1\}\tag{20}$$

First, we deal with the case $i=1$. We use Lemma 3.3.4 and the fact $A(j,l)\leq 1\leq|j_2|/|l|$:

$$\begin{aligned}T_1(l) &\leq C\sum_{\Omega_1(l)}|l|^{\widehat{p}s}|j_2|^\nu A(j,l)^{s\widehat{p}+1}|z_{j_1}z_{j_2}z_{j_3}|\\ &\leq C\sum_{\Omega_1(l)}|l||j_1|^{s\widehat{p}-1}|j_2|^\nu A(j,l)^2|z_{j_1}z_{j_2}z_{j_3}|\\ &\leq C|l|^{-1}\sum_{\Omega_1(l)}|j_1|^{s\widehat{p}-1}|j_2|^{\nu+2}|z_{j_1}z_{j_2}z_{j_3}|\\ &\leq C|l|^{-1}\sum_{j_1\overline{\mathbb{Z}}}|j_1|^{\widehat{p}s-1}|z_{j_1}|\sum_{j_2\in\overline{\mathbb{Z}}}|j_2|^{\nu+2}\sum_{j_3\in\overline{\mathbb{Z}}}|z_{j_3}|\end{aligned}$$

Now call Lemma 3.3.5 to get (20) for $i=1$:

$$T_1(l)\leq C|l|^{-1}||z||_s^3$$

Let us deal with the estimate of $T_2(l)$. Remark that $A(j,l)^N \leq A(j,l)^{\widehat{p}s} \leq C|j_1|^{\widehat{p}s}|l|^{-\widehat{p}s}$ and that j_3 does not appear in $A(j,l)$, we get :

$$\begin{aligned}
T_2(l) &\leq C|l|^{\widehat{p}s}||z||_s \sum_{j_1,j_2} \frac{|j_2|^\nu A(j,l)^N}{1+S(j,l)}|z_{j_1}||z_{j_2}| \\
&\leq C||z||_s \sum_{j_1,j_2} \frac{|j_1|^{\widehat{p}s}|z_{j_1}||z_{j_2}||j_2|^\nu}{1+||j_1|^{2\widehat{p}}-|l|^{2\widehat{p}}|} \\
&\leq C||z||_s \sum_{j_2} |z_{j_2}||j_2|^\nu \sum_{j_1} \frac{|j_1|^s|z_{j_1}|}{1+||j_1|-|l||^{2\widehat{p}}} \\
&\leq C||z||_s^2 \sum_{j_1} \frac{|j_1|^{\widehat{p}s}|z_{j_1}|}{1+||j_1|-|l||^{2\widehat{p}}} \\
\sum_{l\in\overline{\mathbb{Z}}} T_2(l)^2 &\leq C||z||_s^4 \left(\sum_{j_1} \frac{|j_1|^{\widehat{p}s}|z_{j_1}|}{1+||j_1|-|l||^{2\widehat{p}}}\right)^2
\end{aligned}$$

Now, if we decompose $(l,j_1) \in \overline{\mathbb{Z}}^2$ on the different subsets $\pm\overline{\mathbb{N}} \times \pm\overline{\mathbb{N}}$, we see four convolution products (or discrete Fubini) of $(|j_1|^{s\widehat{p}}z_{j_1}) \in \ell^2$ with $((1+|j_1|)^{-2\widehat{p}})) \in \ell^1$, thus (20) holds for $i=2$. □

Finally, we define now the normal form polynomials class :

Definition 3.3.7. *For all $j \in \overline{\mathbb{N}}$, we call the j-th action $I_j : \ell_s \mapsto z_j z_{-j}$.*

For each even integer $k=2m$, a homogeneous polynomial Z on $\ell_s(\overline{\mathbb{Z}})$ is said to be in normal form of degree k if we have

$$Z(z) = \sum_{j\in\mathbb{N}\setminus\{0\}^m} b_j I_{j_1}\cdots I_{j_m}$$

Notice that for each map $f : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ we have $\{I_j, f\} = i\left(z_j\partial_{z_j} - z_{-j}\partial_{z_{-j}}\right)f$, in particular if f is a polynomial in normal form then $\{I_j, f\} = 0$. Following [11] (Proposition 2.13,iv), we have the following crucial fact.

Proposition 3.3.8. *Consider $k \geq 3$, $\nu > 0$, there is $\nu = \nu(s)$ such that if $s > \nu$ and if $Z \in T_{k,\nu}$ is in normal form. The map X_Z is $\mathcal{C}^\infty$ from $\ell_s(\mathbb{C})$ to $\ell_s(\mathbb{C})$.*

3.4. Poisson bracket estimate

Thus, the same proof as [11] (Lemmas 2.19 and 2.20) gives us the following lemma because the analogue of $A(j)$ is exactly

$$\frac{\sqrt{|j_2^* j_3^*|}}{\sqrt{|j_2^* j_3^*|}+|j_1^\star|-|j_2^\star|}$$

Lemma 3.4.1. *For each $i \in \mathbb{Z}^{k_1}, j \in \mathbb{Z}^{k_2}, l \in \mathbb{Z}$, we have*

$$A(j,l)^2 A(i,l)^2 \leq CA(i,j)$$

$$\max\left(\mu(j,l)A(i,l)^{1/\widehat{p}}, \mu(i,l)A(j,l)^{1/\widehat{p}}\right) \leq C\mu(i,j)^{1/\widehat{p}}$$

Proposition 3.4.2. *Consider $k_1, k_2 \geq 2$, $\nu_1, \nu_2 \geq 0$. There is $\nu := \nu(\nu_1, \nu_2) > 0$ such that the Poisson bracket $(P, Q) \mapsto \{P, Q\}$ is well defined from $T^+_{k_1+1,\nu_1} \times T_{k_2+1,\nu_2}$ to $T_{k_1+k_2,\nu}$.*

Proof. Consider $M > 0$, $N := 2M + \frac{\nu_2}{\widehat{p}}$ and $N' := 2M + 1 + \frac{\nu_1}{\widehat{p}}$. We assume

$$P = \sum_{j \in \overline{\mathbb{Z}}^{k_1+1}} a_j z^{j_1} \cdots z^{j_{k_1+1}}, \qquad |a_j| \leq C(N) \frac{\mu(j)^{\nu_1} A(j)^N}{1 + S(j)}$$

$$Q = \sum_{j \in \overline{\mathbb{Z}}^{k_2+1}} b_j z^{j_1} \cdots z^{j_{k_2+1}}, \qquad |b_j| \leq C(N') \mu(j)^{\nu_2} A(j)^{N'}$$

We have

$$\{P, Q\} = \sum_{(i,j) \in \overline{\mathbb{Z}}^{k_1+k_2}} c_{i,j} z_{i_1} \cdots z_{i_{k_1}} z_{j_1} \cdots z_{j_{k_2}}$$

$$|c_{i,j}| \leq \sum_{l \in \overline{\mathbb{Z}}} \frac{\mu(j,l)^{\nu_1}}{1 + S(j,l)} A(j,l)^N \mu(i,l)^{\nu_2} A(i,l)^{N'}$$

Now, just write

$$\begin{aligned} |c_{i,j}| &\leq \sum_{l \in \overline{\mathbb{Z}}} \frac{A(i,l)}{1 + S(j,l)} \left(\mu(j,l) A(i,l)^{1/\widehat{p}}\right)^{\nu_1} \left(\mu(i,l) A(j,l)^{1/\widehat{p}}\right)^{\nu_2} \left(A(j,l) A(i,l)\right)^{2M} \\ &\leq C \left(\sum_{l \in \overline{\mathbb{Z}}} \frac{A(i,l)}{1 + S(j,l)}\right) \mu(i,j)^{(\nu_1+\nu_2)/\widehat{p}} A(i,j)^M \end{aligned}$$

The conclusion will come with the following inequality

$$\sum_{l \in \overline{\mathbb{Z}}} \frac{A(i,l)}{1 + S(j,l)} \leq C \mu(i,j)^{2\widehat{p}}$$

First, we have the obvious case

$$\sum_{|l| > |j_2^\star|} \frac{A(i,l)}{1 + S(j,l)} \leq \sum_{|l| \geq |j_2^\star|} \frac{1}{1 + ||j_1| - |l||^{2\widehat{p}}} \leq \sum_{l \in \mathbb{Z}} \frac{1}{1 + |l|^{2\widehat{p}}} < +\infty$$

The second case will come with the inequality and two sub-cases :

$$\sum_{|l| \leq |j_2^\star|} \frac{A(i,l)}{1 + S(j,l)} \leq \sum_{|l| \leq |j_2^\star|} A(i,l)$$

a) $|j_2^\star| \leq \mu(i,j)$. Like $A(i,l) \leq 1$, we have $\sum_{|l| \leq |j_2|} A(i,l) \leq \mu(i,j) \leq \mu(i,j)^{2\widehat{p}}$.

b) $|j_2^\star| > \mu(i,j)$. Obviously, we have $|j_1^\star| \geq |j_2^\star| > \mu(i,j) \geq |i_1^\star|$. Thus :

$$\sum_{|l| \leq |j_2^\star|} A(i,l) = \sum_{|l| \leq |i_1^\star|} A(i,l) + \sum_{|i_1^\star| < |l| \leq |j_2^\star|} A(i,l) \leq \mu(i,j) + \sum_{|i_1^\star| < |l| \leq |j_2^\star|} A(i,l)$$

In the last summand, we have

$$A(i,l) = \frac{(|i_1^\star i_2^\star|)^{\widehat{p}}}{(|i_1^\star i_2^\star|)^{\widehat{p}} + |l|^{2\widehat{p}} - |i_1^\star|^{2\widehat{p}}} \leq \frac{\mu(i,j)^{2\widehat{p}}}{1+(|l|-|i_1^\star|)^{2\widehat{p}}}$$

$$\sum_{|i_1^\star|<|l|\leq|j_2^\star|} A(i,l) \leq \mu(i,j)^{2\widehat{p}} \sum_{|i_1^\star|<|l|} \frac{1}{1+(|l|-|i_1^\star|)^{2\widehat{p}}} \leq C\mu(i,j)^{2\widehat{p}}$$

□

3.5. Lie transform of $T^+_{k,\nu}$

Definition 3.5.1. *A map $f : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ is in th class T_ν if there is some $s_0 > 0$ such that*

- *for each $s \geq s_0$, f is analytic on a neighborhood $U_s \subset \ell_s(\overline{\mathbb{Z}})$ of 0,*
- *0 is a triple zero of f,*
- *for each $k \geq 3$, the k-th Taylor polynomial lives in $T_{k,\nu}$.*

Let χ be in $T^+_{l,\delta}$, we know that χ lives in $\mathcal{C}^\infty(\ell_s,\mathbb{C})$ if s is sufficiently large (see Proposition 3.3.6). In particular, we can introduce the symplectic flow of χ :

$$\forall z \in \ell_s \quad \frac{d}{dt}\Phi^t(z) = X_\chi(\Phi^t(z))$$

As $l \geq 3$, a bootstrap argument shows that $\Phi^t(z)$ is well defined if $t \in [0,1]$ and $||z||_s < \varepsilon$ (for ε small). And in fact, $\Phi^t(z)$ is analytic in z.

We say that the Lie transform $\phi := \Phi^1$ of χ is well defined. The map ϕ is relevant because it is a symplectic map : the Poisson brackets are conserved. In other words, for every maps A and B of class $\mathcal{C}^\infty$, we have

$$\{A \circ \phi, B \circ \phi\} = \{A, B\} \circ \phi$$

Definition 3.5.2. *If $F : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ satisfies $F(z,\overline{z}) \in \mathbb{R}$ in a neighborhood of 0, we say that F is real.*

Proposition 3.5.3. *Let χ be a homogeneous real polynomial $\in T^+_{l,\delta}$ with $\delta \geq 0$,$l \geq 3$. And consider s sufficiently large.*

i) The Lie transform of χ is well defined and analytic in a ball $B_\varepsilon = \{z \in \ell_s(\overline{\mathbb{Z}}), ||z|| < \varepsilon\}$ and takes values in $B_{2\varepsilon}$. Furthermore, we have

$$\forall z \in B_\varepsilon \quad ||\phi(z) - z||_s \leq C_s ||z||_s^{l-1} \leq C_s||z||_s^2$$

ii) For each $F \in \mathcal{H}^s$ with $s > s_1$, $F \circ \phi \in \mathcal{H}^s$. If χ is real then $F \circ \phi$ is also real.

iii) Considering $P \in T_{n,\nu} \cap \mathcal{H}^s$ with $\nu \geq 0$, $n \geq 3$ and $r \geq n$, we have

$P \circ \phi = Q_r + R_r$,

- *Q_r is a polynomial (not necessarily homogeneous) of degree $\leq r$ and lives in $\mathcal{H}^s$ and $T_{\nu'}$ for some $\nu' > 0$,*
- *R_r is a map who lives in $\mathcal{H}^s \cap T_{\nu''}$, for some $\nu'' > 0$, and admits 0 as a zero of order $\geq r+1$.*

Proof. i) For $\varepsilon > 0$ small, Proposition 3.3.6 gives us $\sup_{||z||_s<\varepsilon} ||X_\chi(z)||_s \leq C\varepsilon^2$. Recall that $\Phi^0(z) = z$ and $||z|| < \varepsilon$. A bootstrap argument let us show that $\Phi^t(z)$ is defined when $|t| \leq \frac{\varepsilon}{C\varepsilon^2}$, which implies $|t| \leq 1$ when ε is rather small.

ii) The map F is $\mathcal{C}^\infty$ on a neighborhood of ℓ_s (see Definition 3.2.1), so is $F\circ\phi$. Again, $X_F = iJ\nabla F \in \mathcal{C}^\infty(\ell_s, \ell_s)$ and $d\phi$ is $\mathcal{C}^\infty$ from ℓ_s to the space of linear bounded maps. We check for z near 0

$$X_{F\circ\phi}(z) = iJ\nabla_{F\circ\phi}(z) = iJd\phi_z^\star(\nabla F(\phi(z))) = -id\phi_z^\star(J\nabla F(\phi(z))) = -id\phi_z^\star(X_F(\phi(z)))$$

Thus $X_{F\circ\phi}$ is $\mathcal{C}^\infty$ regular on a neighborhood of $0 \in \ell_s$ and takes values to ℓ_s.

If χ is real, Φ^t transports the real part of $\ell_s(\overline{\mathbb{Z}})$, i.e. $\{(z,\overline{z}), z \in \ell_s(\overline{\mathbb{N}})\}$ in itself.

We want to get the Taylor polynomials of $h(t) = F\circ\Phi^t$ around $t=0$. We pose $F^{[0]} = F$ and $F^{[k+1]} = \{F^{[k]},\chi\}$ by induction. We can write

$$\frac{d}{dt}\Phi^t(z) = X_\chi(\Phi^t(z)) \qquad \Phi^0(z) = z \qquad \Phi^t(0) = 0$$

$$h'(t) = \langle\nabla_{\Phi^t(z)}F, X_\chi(\Phi^t(z))\rangle = \{F,\chi\}_{\Phi^t(z)}$$

Thus, an induction gives $h^{(k)}(t) = F^{[k]}(\Phi^t(z))$. The Taylor formula leads to

$$h(t) = \sum_{k=0}^{n} h^{(k)}(0)\frac{t^k}{k!} + \frac{t^{n+1}}{n!}\int_0^1 (1-u)^n h^{n+1}(tu)du$$

$$F\circ\phi(z) = \sum_{k=0}^{n} F^{[k]}(z)\frac{1}{k!} + \frac{1}{n!}\int_0^1 (1-u)^n F^{[n+1]}\circ\Phi^u(z)du$$

Define F_k the k-th polynomial Taylor of F in 0 and recall that $\deg\{P,\chi\} = (\deg P + \deg\chi) - 2$. Hence,

$$F\circ\phi(z) = \sum_{k=0}^{n}\frac{1}{k!}\sum_{j+j'(l-2)=k} F_j^{[j']}(z) + \frac{1}{n!}\int_0^1 (1-u)^n F^{[n+1]}\circ\Phi^u(z)du$$

The last part is $O(||z||^{n+1})$, so it does not contribute in degree. We end as in [11] (Proposition 2.21)

iii) Let K be the integer part of $\frac{r-n}{l-2}$, we decompose $P\circ\phi = Q_r + R_r$

$$Q_r = \sum_{k=0}^{K} P^{[k]}(z)\frac{1}{k!}, \qquad R_r = \frac{1}{n!}\int_0^1 (1-u)^n P^{[n+1]}\circ\Phi^u(z)du$$

We have $P^{[0]} = P \in T_{n,\nu}$. By induction, Proposition 3.4.2 proves that $P^{[k]} \in T_{n+k(l-2),\nu'}$. As $n+K(l-2) \leq r \leq r+1 \leq n+(K+1)(l-2)$, we understand that Q_r has degree $\leq r$ and lives in $T_{\nu'}$. Furthermore, $P^{[k+1]} \in T_{n+(k+1)(l-2),\nu''}$, for some $\nu'' > 0$, R_r admits zero of order $\geq r+1$ and lives in $T_{\nu''}$. Consequently, $R_r = P\circ\phi - Q_r \in \mathcal{H}^s$. □

3.6. The normal form theorem

Now, we introduce the nonresonance definition.

Definition 3.6.1. *A vector frequencies $(\omega_j)_{j\in\mathbb{N}}$ is nonresonant if for all $r \in \overline{\mathbb{N}}$, there is $\gamma, \delta > 0$ such that for all $j \in \overline{\mathbb{N}}^r$ and $i \in [[1, r]]$ we have*

$$|\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}| \geq \frac{\gamma(1 + S(j))}{\mu(j)^\delta} \tag{21}$$

except if $\{j_1, \ldots, j_i\} = \{j_{i+1}, \ldots, j_r\}$. Here, $S(j)$ is the same that in definition (3.3.3).

Now, we claim that the following holds

Theorem 3.6.2. *For each $k \geq 1$, there is a full measure set $\mathcal{F}_k \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^{\mathbb{N}}$ such that for every $(m_j) \in \mathcal{F}_k$, the vector frequencies $\left(\lambda_j + \dfrac{m_j}{j^k}\right)_{j\geq 1}$ is nonresonant.*

The proof of the previous theorem is the same as Theorem 5.7 of [12], we do not repeat it in this article, in fact it only needs growth condition on the sequence $(\lambda_j)_{j\geq 1}$:

i) $(\lambda_j)_{j\geq 1}$ is positive and increasing,

ii) there are $C > 1$ and $\theta \geq 1$ such that $\frac{1}{C} j^\theta \leq \lambda_j \leq C j^\theta$,

iii) the set $\Lambda := \{\lambda_j - \lambda_{j'}, (j, j') \in \mathbb{N}^2\}$ satisfies for some $\sigma > 0$

$$\forall t \geq 1 \qquad \mathrm{Card}(\Lambda \cap [0, t]) \leq C t^\sigma$$

Point iii) means the differences $\lambda_j - \lambda_{j'}$ do not accumulate to zero and is a consequence of Proposition 2.2.1. We also claim that the crucial normal form theorem holds :

Theorem 3.6.3. *Let P be in $\mathcal{H}^s \cap T_\nu$ for s large and let $H_0 = \sum_{j\geq 1} \omega_j I_j$ be a Hamiltonian with nonresonant frequencies ω. The perturbation reads $H := H_0 + P$. Consider $r \geq 3$, there are two neighborhoods U_s and W_s of $0 \in \ell_s^2(\overline{\mathbb{Z}})$ and $\tau_s : U_s \to W_s$ a real symplectic diffeomorphism such that $H \circ \tau = H_0 + Z + R$ with*

i) Z is polynomial of degree $\leq r$, lives in $\mathcal{H}^s$ and depends only on actions I_j,

ii) $R \in \mathcal{H}^s$ and $||X_R(z)||_s \leq C(s, r)||z||_s^r$ for each $z \in U_s$,

iii) $||\tau(z) - z||_s \leq C_s ||z||_s^2$ for each $z \in U_s$.

The proof of Theorem 2.23 is the same as this of [11], and the reality of τ_s means $\tau_s(\ell_s(\overline{\mathbb{N}})) \subset \ell_s(\overline{\mathbb{N}})$. Let us recall formally the idea of the proof for $r = 3$. Write $P = P_3 + P_4$ where P_3 is the Taylor polynomial of degree 3 of P. First of all, we need to solve the so-called homological, i.e. find some homogeneous polynomial

$$\chi := \sum_{j_1, j_2, j_3 \in \overline{\mathbb{Z}}} q(j_1, j_2, j_3) z_{j_1} z_{j_2} z_{j_3}$$

of degree 3 such that X_χ lives in $\ell_s(\overline{\mathbb{Z}})$ and $Z := \{H_0, \chi\} + P_3$ is in normal form (so depends only on the actions). Computing the Poisson bracket gives us

$$\{H_0, \chi\} = i \sum_{j_1, j_2, j_3 \in \overline{\mathbb{Z}}} (\mathrm{sg}(j_1)\omega_{|j_1|} + \mathrm{sg}(j_2)\omega_{|j_2|} + \mathrm{sg}(j_3)\omega_{|j_3|}) q(j_1, j_2, j_3) z_{j_1} z_{j_2} z_{j_3}$$

Where $\mathrm{sg}(k)$ equals $+1$ if $k > 0$ and -1 if $k < 0$. As (ω_j) is nonresonant, we can choose

$$q(j_1, j_2, j_3) = \frac{ip(j_1, j_2, j_3)}{\mathrm{sg}(j_1)\omega_{|j_1|} + \mathrm{sg}(j_2)\omega_{|j_2|} + \mathrm{sg}(j_3)\omega_{|j_3|}}$$

where of course $P_3 = \sum p(j_1, j_2, j_3) z_{j_1} z_{j_2} z_{j_3} \in T_{3,\nu}$. The nonresonance condition (21) implies that χ lives in $T^+_{3,\nu+\delta}$. As 3 is odd, we are able to eliminate all the terms of P_3, thus $Z := \{H_0, \chi\} + P_3 = 0$, but other terms could appear if we would have started with $r = 4$, for instance $z_{j_1} z_{j_2} z_{-j_1} z_{-j_2}$. As χ lives in $T^+_{3,\nu+\delta}$, we can define the Lie transform ϕ of χ and we have

$$\begin{aligned} (H_0 + P) \circ \phi(z) &= H_0 + Z + (H_0 \circ \phi - H_0 - \{H_0, \chi\}) + \\ &\quad (P_3 \circ \phi - P_3) + P_4 \circ \phi \end{aligned}$$

Point i) of Proposition 3.5.3 shows that ϕ is near the identity map, and consequently the three terms $(H_0 \circ \phi - H_0 - \{H_0, \chi\})$, $(P_3 \circ \phi - P_3)$ and $P_4 \circ \phi$ are less than $||z||_s^4$ near 0. In the general case $r \geq 4$, τ is constructed by composition of canonical transformations ϕ for various real polynomials $\chi \in T^+_{k,\nu}$ which solve homological equations.

4. Perturbation regularity and conclusion

Theorem 1.0.1 is a consequence of Theorem 3.6.3, Lemma 4.0.6 and the fact that the following nonlinear perturbation lives in $\mathcal{H}^s \cap T_\nu$ for s and ν large.

$$\forall z \in \ell_s(\overline{\mathbb{Z}}) \qquad P(z) = \int_{\mathbb{R}} g\left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j} \phi_j(x)\right) dx$$

Proposition 4.0.4. *For s large, the map $P : \ell_s(\overline{\mathbb{Z}}) \to \mathbb{C}$ is $\mathcal{C}^\infty$ regular and its gradient takes values in $\ell^2_s(\mathbb{C})$.*

Proof. ♦ The map P can be written

$$\begin{array}{ccccccc} \ell_{2s/3}(\overline{\mathbb{Z}}) & \longrightarrow & \widehat{H}^s \times \widehat{H}^s & \longrightarrow & \widehat{H}^s & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \left(\sum_{j>0} z_j \phi_j, \sum_{j>0} z_{-j} \phi_j\right) & \longmapsto & g\left(\sum_{j>0} z_j \phi_j, \sum_{j>0} z_{-j} \phi_j\right) & \longmapsto & P(z) \end{array}$$

The first arrow is just Γ_s, thus is a diffeomorphism. The third arrow is regular because $\widehat{H}^s \subset L^1(\mathbb{R})$ for s large. Let us prove the second arrow is regular. The analytic condition on g leads to the expression

$$g(\xi_1, \xi_2) = \sum_{m+n \geq 3} \alpha_{m,n} \xi_1^m \xi_2^n$$

Remembering the proof of Proposition 2.4.6, we understand the second arrow is uniform limit of polynomials on each bounded set of $\widehat{H}^s \times \widehat{H}^s$:

$$P_N : (f_1, f_2) \mapsto \sum_{m+n \le N} \alpha_{m,n} f_1^m f_2^n$$

Consequently, $(f_1, f_2) \mapsto g(f_1, f_2)$ is holomorphic, hence $\mathcal{C}^\infty$, on $\widehat{H}^s \times \widehat{H}^s$ (see [15] § 6 Proposition 4 and § 7 Proposition 3, or [16] appendix A Theorems 1 and 2).

⧫ Let us deal with the gradient. As first and third arrows are linear, the gradient of P comes naturally by linearization of the second arrow around a point $(f_1, f_2) \in \widehat{H}^s \times \widehat{H}^s$. First, Taylor formula gives us three holomorphic maps G_1, G_{12} and G_2 on $\mathbb{C}^4$ such that for each $(\xi_1, \chi_1, \xi_2, \chi_2) \in \mathbb{C}^4$ we have

$$g(\xi_1 + \chi_1, \xi_2 + \chi_2) - g(\xi_1, \xi_2) = \chi_1 \partial_1 g(\xi_1, \xi_2) + \chi_2 \partial_2 g(\xi_1, \xi_2) +$$

$$\chi_1^2 G_1(\xi_1, \chi_1, \xi_2, \chi_2) + \chi_1 \chi_2 G_{12}(\xi_1, \chi_1, \xi_2, \chi_2) + \chi_2^2 G_2(\xi_1, \chi_1, \xi_2, \chi_2)$$

Hence, for $\widetilde{h_1}, \widetilde{h_2} \in \widehat{H}^s \times \widehat{H}^s$, Proposition 2.4.6 leads to

$$g(f_1 + \widetilde{h_1}, f_2 + \widetilde{h_2}) - g(f_1, f_2) = \widetilde{h_1} \partial_1 g(f_1, f_2) + \widetilde{h_2} \partial_2 g(f_1, f_2) + O(||\widetilde{h_1}, \widetilde{h_1}||^2_{\widehat{H}^s \times \widehat{H}^s})$$

Coming back to P, for all z and $h \in \ell_s(\overline{\mathbb{Z}})$ we have

$$D_z P(h) = \sum_{j>0} h_j \int_{\mathbb{R}} \phi_j(x) \partial_1 g \left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j} \phi_j(x) \right) dx + \tag{22}$$

$$+ \sum_{j<0} h_j \int_{\mathbb{R}} \phi_j(x) \partial_2 g \left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j} \phi_j(x) \right) dx$$

Hence

$$||\nabla_z P||^2_{\widehat{H}^s} = \sum_{j>0} j^{2s\widehat{p}} \left| \int_{\mathbb{R}} \phi_j(x) \partial_1 g \left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j} \phi_j(x) \right) dx \right|^2 +$$

$$\sum_{j<0} |j|^{2s\widehat{p}} \left| \int_{\mathbb{R}} \phi_{-j}(x) \partial_2 g \left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j} \phi_j(x) \right) dx \right|^2$$

With the help of Γ_s, $\overline{\phi_j} = \phi_j$, and Proposition 2.4.6, we conclude

$$||\nabla_z P||^2_{\widehat{H}^s} \simeq \left\| \partial_1 g \left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j} \phi_j(x) \right) \right\|^2_{\widehat{H}^s} + \left\| \partial_2 g \left(\sum_{j>0} z_j \phi_j(x), \sum_{j>0} z_{-j} \phi_j(x) \right) \right\|^2_{\widehat{H}^s} < +\infty$$

□

Now, we check the following

Proposition 4.0.5. *For s and ν large, the map P lives in $\mathcal{H}^s \cap T_\nu$.*

PROOF.
Let us check that $P \in \mathcal{H}^s$. We call $P = \sum_{k\geq 3} P_k$ the Taylor decomposition of P. As P is holomorphic, each P_k is $\mathcal{C}^\infty$. Thus, $X_P = \sum X_{P_k}$ is holomorphic from $\ell_s(\mathbb{C})$ in itself, so is each X_{P_k}. Now, let us check Definition 3.5.1. Like g is holomorphic, for each $k \geq 3$ there are holomorphic maps $G_1, \cdots, G_k$ on $\mathbb{C}^2$ such that

$$\forall \xi_1, \xi_2 \in \mathbb{C} \quad g(\xi_1, \xi_2) = \sum_{r=3}^{k} \frac{1}{r!} \sum_{\ell=0}^{r} (\xi_1^\ell \xi_2^{r-\ell}) \partial_1^\ell \partial_2^{r-\ell} g(0,0) + \sum_{\ell=0}^{k+1} \xi_1^\ell \xi_2^{k+1-\ell} G_\ell(\xi_1, \xi_2)$$

Remember Proposition 2.4.6 and the proof of Proposition 4.0.4, the Taylor polynomial P_r of P around $(0,0)$ appears

$$P_r(z) = \frac{1}{r!} \sum_{\ell=0}^{r} \partial_1^\ell \partial_2^{r-\ell} g(0,0) \int_{\mathbb{R}} \left(\sum_{j>0} z_j \phi_j(x) \right)^\ell \left(\sum_{j<0} z_{-j} \phi_j(x) \right)^{r-\ell} dx$$

$$= \frac{1}{r!} \sum_{\ell=0}^{r} \partial_1^\ell \partial_2^{r-\ell} g(0,0) \sum_{j \in \overline{\mathbb{N}}^r} z_{j_1} \cdots z_{j_\ell} z_{-j_{\ell+1}} \cdots z_{-j_r} \int_{\mathbb{R}} \phi_{j_1}(x) \cdots \phi_{j_r}(x) dx$$

Finally, we get that P_r lives in $T_{r,\nu}$ for large ν thanks to Proposition 2.5.1. □

To finish, as said in Section 3.1, the property $g(z, \overline{z}) \in \mathbb{R}$ ensures the final lemma

Lemma 4.0.6. *A solution $z(t)$ of (16) is real, i.e. $\overline{z_j(t)} = z_{-j}(t)$, if and only if its initial condition $z(0)$ is real.*

PROOF. Let $\mathcal{D}$ be the complex line $\{(\xi, \overline{\xi}), \xi \in \mathbb{C}\}$. The holomorphic map g satisfies $g(\mathcal{D}) \subset \mathbb{R}$. A differentiation gives us

$$\forall z, \xi \in \mathbb{C} \quad \xi \partial_1 g(z, \overline{z}) + \overline{\xi} \partial_2 g(z, \overline{z}) \in \mathbb{R}$$

Consequently, $\partial_2 g(z, \overline{z}) = \overline{\partial_1 g(z, \overline{z})}$. Thanks to (22), we have $\frac{\partial P}{\partial z_{-j}}(z) = \overline{\frac{\partial P}{\partial z_j}(z)}$. In other words, the two equations of (16) are self-conjugate. □

Let us prove our main theorem. We apply Theorem 3.6.3 for $r+2$, let $(z(t))_{t \in (-T_m, T_M)}$ be a maximal solution of (17) in $\ell_s(\overline{\mathbb{Z}})$ for s large. As τ and τ^{-1} are defined on a neighborhood of $0 \in \ell_s(\overline{\mathbb{Z}})$ for $||\xi(0)||_s < \varepsilon$ we can define $\xi(t) := \tau^{-1}(z(t))$. As τ is real symplectic, we have

$$\xi_{-j}(t) = \overline{\xi_j(t)}, \qquad \xi'(t) = iX_{H_0+Z+R}(\xi(t))$$

Let us define $N(z) = ||z||_s^2 = \sum_{j\geq 1} \lambda_j^s z_j z_{-j}$ for all $z \in \ell_s(\overline{\mathbb{Z}})$. Let T_M^+ be the maximal time in $(0, T_M)$ such that $||\xi(t)||_s \leq \frac{3}{2}\varepsilon$ for $t \in [0, T_M^+]$. We have for $t \in [0, T_M^+]$

$$\begin{aligned}
\left|\tfrac{d}{dt}N(\xi(t))\right| &= |\{N, H_0 + Z + R\}(\xi(t))| = |\{N, R\}|(\xi(t))| \\
&= |\sum_{j\geq 1} \lambda_j^s(\xi_j(t)\partial_{-j}R - \xi_{-j}(t)\partial_{-j}R)| \\
&\leq 2\left(\sum_{j\geq 1}\lambda_j^s|\xi_j(t)|^2\right)^{1/2} ||X_R(\xi(t))||_s \\
&\leq C||\xi(t)||_s^{r+3} \leq C\varepsilon^{r+3} \\
|||\xi(t)||_s^2 - ||\xi(0)||_s^2| &\leq Ct\varepsilon^{r+3}
\end{aligned}$$

For $t = T_M^+$ we get $T_M^+ \geq C\varepsilon^{-1-r}$. Let us check what happens when $t \in [0, C\varepsilon^{-r}]$. By the same estimate seen above, we have

$$\sum_{j\geq 1} \lambda_j^s |\frac{d}{dt}|\xi_j|^2| = \sum_{j\geq 1} \lambda_j^s |\{I_j, R\}(\xi(t))| \leq C\varepsilon^{r+3}$$

Hence,

$$\sum_{j\geq 1} \lambda_j^s ||\xi_j(t)|^2 - |\xi_j(0)|^2| \leq Ct\varepsilon^{r+3} \leq C\varepsilon^3$$

Remember that $||\xi(t)||_s \leq 2\varepsilon$. Point iii) of Theorem 3.6.3 leads to

$$\forall t \in [0, C\varepsilon^{-r}] \qquad ||z(t) - \xi(t)||_s \leq C_s||z(t)||_s^2 \leq C_s\varepsilon^2 \tag{23}$$

Consequently, for small ε we have $||z(t)||_s \leq \frac{3}{2}\varepsilon$. Finally, $\sum_{j\geq 1} \lambda_j^s ||z_j(t)|^2 - |z_j(0)|^2|$ is less than

$$\sum_{j\geq 1} \lambda_j^s ||z_j(t)|^2 - |\xi_j(t)|^2| + \sum_{j\geq 1} \lambda_j^s ||\xi_j(t)|^2 - |\xi_j(0)|^2| + \sum_{j\geq 1} \lambda_j^s ||\xi_j(0)|^2 - |z_j(0)|^2|$$

The first and third sums are less than ε^4 thanks to (23). Theorem 1.0.1 is proved.

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