Explicit examples of eigenfunctions of the Hermite operator saturating some L^p bounds

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Abstract

The goal of the paper is to provide a sequence of eigenfunctions that saturates the L^p bounds obtained by Koch and Tataru for the multidimensional Hermite operator. More precisely, several such sequences of eigenfunctions have already been identified by Koch and Tataru, and we present an example in another range of p.

Keywords : Hermite operator, quantum harmonic oscillator, eigenfunctions, Lebesgue space

Let $(E_n)_{n\in\mathbb{N}}$ denote the sequence of eigenspaces of the Hermite operator (also called the quantum harmonic oscillator) $-\Delta+|x|^2$ on $L^2(\mathbb{R}^d)$:

$$E_n = \ker(-\Delta+|x|^2-2n-d).$$

Each eigenspace E_n is finite-dimensional (more precisely $\dim(E_n)\simeq n^{d-1}$). The classical paper [KT05] by Koch and Tataru gives an almost complete set of optimal $L^2\to L^p$ inequalities for all $p\in[2,+\infty]$ for the eigenspaces E_n. For any fixed dimension $d\geq 2$, the result reads as follows:

$$\forall f\in E_n \qquad \|f\|_{L^p(\mathbb{R}^d)}\leq C(d,p)n^{a(p)}\|f\|_{L^2(\mathbb{R}^d)} \tag{1}$$

where $a(p)$ is an explicit convex function of $\frac{1}{p}$, defined by

$$a(p)=\begin{cases} -\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right) & \text{with} \quad 2\leq p<\frac{2(d+3)}{d+1}, \\ \frac{1}{2}\left(-\frac{1}{3}+\frac{d}{3}\left(\frac{1}{2}-\frac{1}{p}\right)\right) & \text{with} \quad \frac{2(d+3)}{d+1}<p\leq\frac{2d}{d-2}, \\ \frac{1}{2}\left(-1+d\left(\frac{1}{2}-\frac{1}{p}\right)\right) & \text{with} \quad \frac{2d}{d-2}\leq p\leq+\infty. \end{cases}$$

In fact, the case $p=\frac{2(d+3)}{d+1}$ for $d\geq 3$ has recently been completed by [JLR24b], while the case $d=2$ remains open (see also [JLR24a] for a related topic).

The paper [KT05] also provides proofs of sharpness of the $L^2\to L^p$ bounds and it is interesting to see whether one could give explicit examples of sequence of eigenfunctions saturating such bounds. At the end of [KT05], explicit examples are provided for

$$p\in\left[2,\frac{2(d+3)}{d+1}\right)\bigcup\left[\frac{2d}{d-2},+\infty\right].$$

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The aim of this article is to extend this analysis in the middle range $\left(\frac{2(d+3)}{d+1}, \frac{2d}{d-2}\right)$.

Before going into details, we introduce some notations for the one-dimensional eigenfunctions, namely Hermite functions:

$$h_n(x) = \frac{H_n(x)}{\sqrt{n!2^n\sqrt{\pi}}} e^{-x^2/2} \tag{2}$$

where H_n denotes the Hermite polynomial of order n (physicists' convention). For example, the ground state is given by $h_0(x) = \pi^{-\frac{1}{4}} e^{-x^2/2}$. The following equalities are well-known:

$$-h_n''(x) + x^2 h_n(x) = (2n+1)h_n(x) \qquad \text{and} \qquad \|h_n\|_{L^2(\mathbb{R})} = 1. \tag{3}$$

In the multidimensional case, the eigenspace E_n (corresponding to the eigenvalue $2n+d$ of $-\Delta + |x|^2$) can be described using a simple orthonormal basis as follows:

$$E_n = \mathrm{Span}\Big(\Big\{h_{n_1} \otimes \cdots \otimes h_{n_d}, \quad \text{with} \quad \sum_{i=1}^{d} n_i = n\Big\}\Big).$$

For the sequel, it is worthwhile to note, as at the end of [KT05], that if $J \subset \mathbb{N}^d$ is a non-empty set consisting of tuples $(n_1, \dots, n_d)$ satisfying $n_1 + \cdots + n_d = n$, then the following function is an $L^2(\mathbb{R}^d)$-normalized eigenfunction corresponding to the eigenvalue $2n+d$:

$$x \in \mathbb{R}^d \mapsto \frac{1}{\sqrt{\mathrm{Card}(J)}} \sum_{(n_1,\dots,n_d)\in J} h_{n_1}(x_1)\dots h_{n_d}(x_d). \tag{4}$$

With such notations, we now briefly recall the known explicit examples:

- for $p \in \left[2, \frac{2(d+3)}{d+1}\right)$, the L^p norm of the eigenfunction $h_n \otimes h_0 \otimes \cdots \otimes h_0$ (which corresponds to the eigenvalue $2n+d$) satisfies for $n \gg 1$:

$$\begin{aligned}
\|h_n \otimes h_0 \otimes \cdots \otimes h_0\|_{L^p(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \left|\frac{1}{\pi^{\frac{d-1}{4}}} h_n(x_1) e^{-\frac{x_2^2+\cdots+x_d^2}{2}}\right|^p \mathrm{d}x_1 \dots \mathrm{d}x_d\right)^{1/p} \\
&= \frac{1}{\pi^{\frac{d-1}{4}}} \left(\int_{\mathbb{R}} |h_n(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} e^{-px^2/2} \mathrm{d}x\right)^{\frac{d-1}{p}} \\
&\simeq \|h_n\|_{L^p}.
\end{aligned}$$

Moreover, it is known that $\|h_n\|_{L^p(\mathbb{R})} \simeq n^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}$ for $p \in [2,4)$ (see [Tha93, Lemma 1.5.2]). This covers the relevant case since $\frac{2(d+3)}{d+1} < 4$.

- for $p > \frac{2d}{d-2}$ if $d \geq 3$ (set $p = \infty$ if $d = 2$), as shown in [KT05], one may also consider a sequence of eigenfunctions of the form (4). It is proved in [IRT16, Proposition 2.4] that the sequence of radial eigenfunctions of $-\Delta + |x|^2$ also saturates the $L^2 \to L^p$ bounds. Geometrically, both examples concentrate around the origin.

The result of this article shows that one may also construct examples similar to (4) in the middle range $p \in \left(\frac{2(d+3)}{d+1}, \frac{2d}{d-2}\right)$.

Theorem 1. *We assume $d \geq 2$. For any $\alpha \in (0, \frac{1}{6}]$, $\beta \geq 0$ and $n \in \mathbb{N}$, define*

$$J_n^{(\alpha,\beta)} := \left\{(n_1, \dots, n_d) \in \mathbb{N}^d, \quad \forall i \in \{1, \dots, d\} \quad \left|n_i - \frac{n}{d}\right| \leq \alpha\Big(\frac{2n}{d}+1\Big)^{\frac{1}{3}} + \beta \text{ and } \sum_{i=1}^{d} n_i = n\right\}$$

and the following eigenfunction of $-\Delta + |x|^2$ corresponding to the eigenvalue $2n+d$:

$$v_n^{(\alpha,\beta)}(x_1,\dots,x_d) := \frac{1}{\sqrt{\mathrm{Card}(J_n^{(\alpha,\beta)})}} \sum_{(n_1,\dots,n_d)\in J_n^{(\alpha,\beta)}} h_{n_1}(x_1)\dots h_{n_d}(x_d). \tag{5}$$

In particular, we have $\|v_n^{(\alpha,\beta)}\|_{L^2(\mathbb{R}^d)} = 1$. Let $p \geq 1$ be fixed. For any $n \gg 1$ the norm $\|v_n^{(\alpha,\beta)}\|_{L^p(\mathbb{R}^d)}$ saturates the Koch-Tataru $L^2 \to L^p$ estimates in the middle range:

$$\|v_n^{(\alpha,\beta)}\|_{L^p(\mathbb{R}^d)} \gtrsim n^{\frac{1}{2}\left(-\frac{1}{3}+\frac{d}{3}\left(\frac{1}{2}-\frac{1}{p}\right)\right)} = n^{\frac{d}{12}-\frac{1}{6}-\frac{d}{6p}}. \tag{6}$$

Due to (1), the bound from below (6) is actually an equivalence for $p \in \left(\frac{2(d+3)}{d+1}, \frac{2d}{d-2}\right]$.

We finish this introduction by outlining the organization of the paper and the structure of the proof of Theorem 1.

Section 1 is devoted to giving two graphical representations in dimension 2 that illustrate the concentration of the eigenfunctions $v_n^{(\alpha,\beta)}$ for $\alpha = \frac{1}{6}$ and $\beta = 4$. We emphasize that the parameter β plays no essential role from a theoretical point of view in the statement of Theorem 1, it is only introduced to ensure $\mathrm{Card}(J_n^{(\alpha,\beta)}) \gg 1$ for small values of n in graphical representations.

In Section 2, we prove two results (Proposition 3 and Corollary 5) that provide a quantitative statement about the spatial concentration of the one-dimensional Hermite functions (h_n). More precisely, for $n \gg 1$, Corollary 5 will imply that each $h_{n_i}(x_i)$ in (5) is bounded below by $n^{-\frac{1}{12}}$ (up to a multiplicative constant) on the following interval

$$\left[\sqrt{\frac{2n}{d}+1} - \frac{3.89}{(\frac{2n}{d}+1)^{\frac{1}{6}}}, \sqrt{\frac{2n}{d}+1} - \frac{3.72}{(\frac{2n}{d}+1)^{\frac{1}{6}}}\right]. \tag{7}$$

To prove such a concentration property, we use a well-known WKB approximation of the Hermite functions. More precisely, by setting

$$g_n(y) := \frac{(2n+1)}{2}\left(\arccos(y) - y\sqrt{1-y^2}\right) - \frac{\pi}{4}, \qquad \forall y \in (-1,1),$$

the following holds uniformly for $y \in (-1,1)$

$$h_n(y\sqrt{2n+1}) = \frac{\sqrt{2}}{\sqrt{\pi}(2n+1)^{\frac{1}{4}}} \frac{\cos(g_n(y))}{(1-y^2)^{\frac{1}{4}}} + \mathcal{O}\left(\frac{1}{(2n+1)^{\frac{5}{4}}(1-y^2)^{\frac{7}{4}}}\right). \tag{8}$$

Written in this form, this WKB approximation of h_n merely yields unknown constants in (7) (instead of 3.89 and 3.72) and also an unknown upper bound for the constant α in Theorem 1. Without an explicit remainder in (8), the conclusion of Theorem 1 remains true for α small enough, which seems essentially useless for numerical applications. To obtain the explicit condition $\alpha \in \left(0, \frac{1}{6}\right]$, we need to provide an explicit constant in the remainder term of (8). This is the purpose of Proposition 2, whose proof is postponed to Section 6.

In Section 3, we give an elementary proof of the asymptotics of the cardinality of the sets $J_n^{(\alpha,\beta)}$ for $n \gg 1$. For any choice of $\alpha > 0$, Lemma 6 below ensures that the set $J_n^{(\alpha,\beta)}$ has cardinality of order $n^{\frac{d-1}{3}}$.

In Section 4, we explain how the results of the previous sections lead to a straightforward proof of Theorem 1. Due to (7), each $h_{n_1}(x_1)\dots h_{n_d}(x_d)$ in (5) is bounded below by $n^{-d/12}$ (up to a multiplicative constant) on the following hypercube

$$\left[\sqrt{\frac{2n}{d}+1} - \frac{3.89}{(\frac{2n}{d}+1)^{\frac{1}{6}}}, \sqrt{\frac{2n}{d}+1} - \frac{3.72}{(\frac{2n}{d}+1)^{\frac{1}{6}}}\right]^d. \tag{9}$$

Taking into account the cardinality of $J_n^{(\alpha,\beta)}$, we derive (6) from a lower bound of the L^p norm of $v_n^{(\alpha,\beta)}$, since the volume of the hypercube (9) is of order $n^{-\frac{d}{6}}$.

As written above, the last point is to give an explicit constant in the remainder in (8). In Section 5, we establish several explicit inequalities for the Airy functions which do not appear explicitly in the literature but can be deduced from [Olv97]. These explicit inequalities are then used in Section 6, where the constants in the analysis of [Olv97] are followed.

In the sequel, $\mathbb{N}$ denotes the set of non-negative integers $\{0, 1, 2, \dots\}$.

1 Graphical representations in dimension $d = 2$

In dimension $d = 2$, the sets $J_n^{(\alpha,\beta)}$ are easy to describe and we indeed have

$$\begin{aligned} J_n^{(\alpha,\beta)} &= \left\{(k, n-k), \quad \text{with } \left\lceil \frac{n}{2} - \alpha\sqrt[3]{n+1} - \beta \right\rceil \leq k \leq \left\lfloor \frac{n}{2} + \alpha\sqrt[3]{n+1} + \beta \right\rfloor \right\} \\ \mathrm{Card}(J_n^{(\alpha,\beta)}) &= 1 - \left\lceil \frac{n}{2} - \alpha\sqrt[3]{n+1} - \beta \right\rceil + \left\lfloor \frac{n}{2} + \alpha\sqrt[3]{n+1} + \beta \right\rfloor \\ &\underset{n\to+\infty}{\sim} \frac{\sqrt[3]{n}}{3}. \end{aligned}$$

As $n \to +\infty$, the number $\mathrm{Card}(J_n^{(\alpha,\beta)})$ tends to $+\infty$, and it appears graphically that the different eigenfunctions in (5) change signs except if (x_1, x_2) lies in a concentration region. Since our choice of n is somewhat restricted if we want to use (2) with factorials, and since $\frac{1}{6}(n+1)^{\frac{1}{3}} < 1$ for $n \leq 200$, we will use the extra parameter β in order to make $\mathrm{Card}(J_n^{(\alpha,\beta)})$ large enough to expect a cancellation phenomenon outside a concentration region. We choose $\alpha = \frac{1}{6}$, which is the maximal value allowed by Theorem 1, and $\beta = 4$ that provide quite satisfactory graphical representations. Here is the formula for the eigenfunction $v_n^{(\frac{1}{6},4)}$ (corresponding to the eigenvalue $2n + 2$):

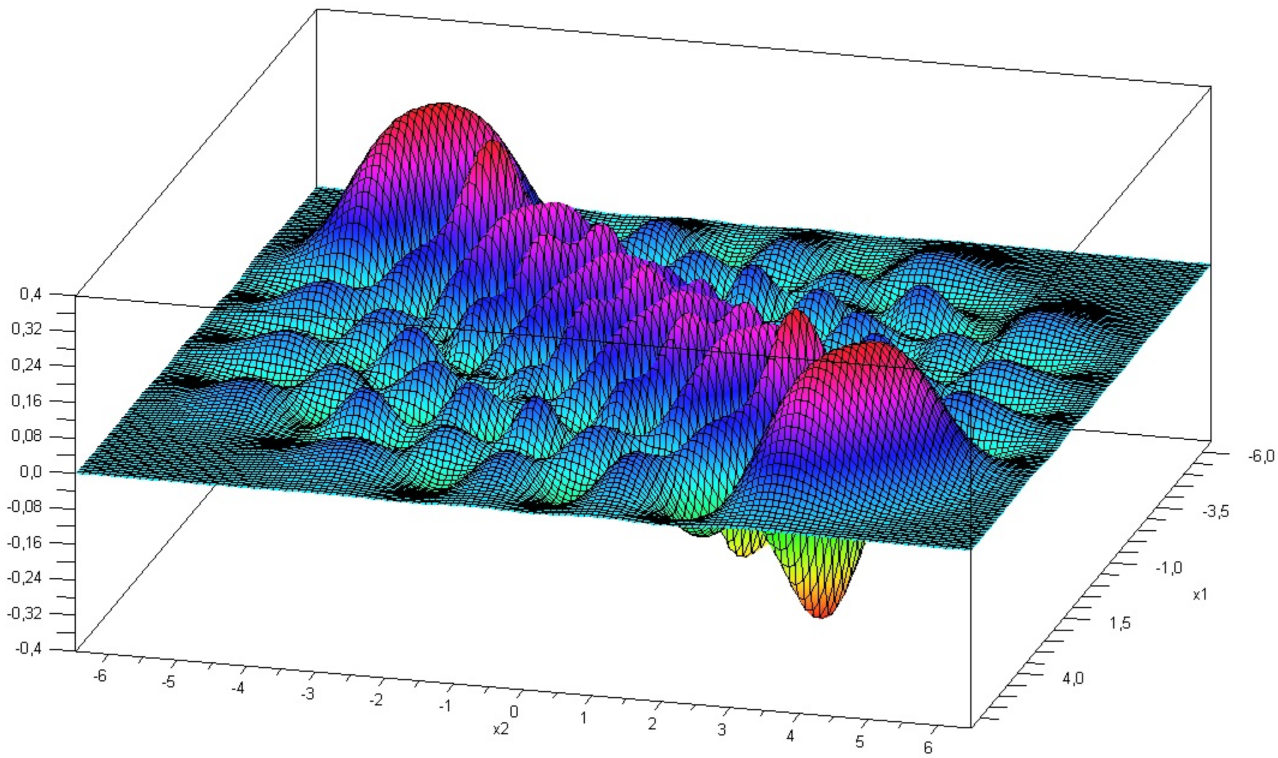
$$v_n^{(\frac{1}{6},4)} : (x_1, x_2) \mapsto \frac{1}{\sqrt{\mathrm{Card}(J_n^{(\frac{1}{6},4)})}} \sum_{k=\lceil \frac{n}{2} - \frac{1}{6}\sqrt[3]{n+1} - 4\rceil}^{\lfloor \frac{n}{2} + \frac{1}{6}\sqrt[3]{n+1} + 4\rfloor} h_k(x_1) h_{n-k}(x_2),$$

Here are two graphical representations on $[-\sqrt{2n+2}, \sqrt{2n+2}]^2$ for $n = 20$ and $n = 80$ of $v_n^{(\frac{1}{6},4)}$ and we check the equalities

$$\begin{aligned} J_{20}^{(\frac{1}{6},4)} &= \{(6,14), (7,13), (8,12), (9,11), (10,10), (11,9), (12,8), (13,7), (14,6)\}, \\ J_{80}^{(\frac{1}{6},4)} &= \{(36,44), (37,43), (38,42), (39,41), (40,40), (41,39), (42,38), (43,37), (44,36). \end{aligned}$$

As mentioned in the introduction, these graphical representations confirm the concentration at least around the two points with coordinates $x_1 = x_2 \simeq \pm\sqrt{n+1}$.

n=20



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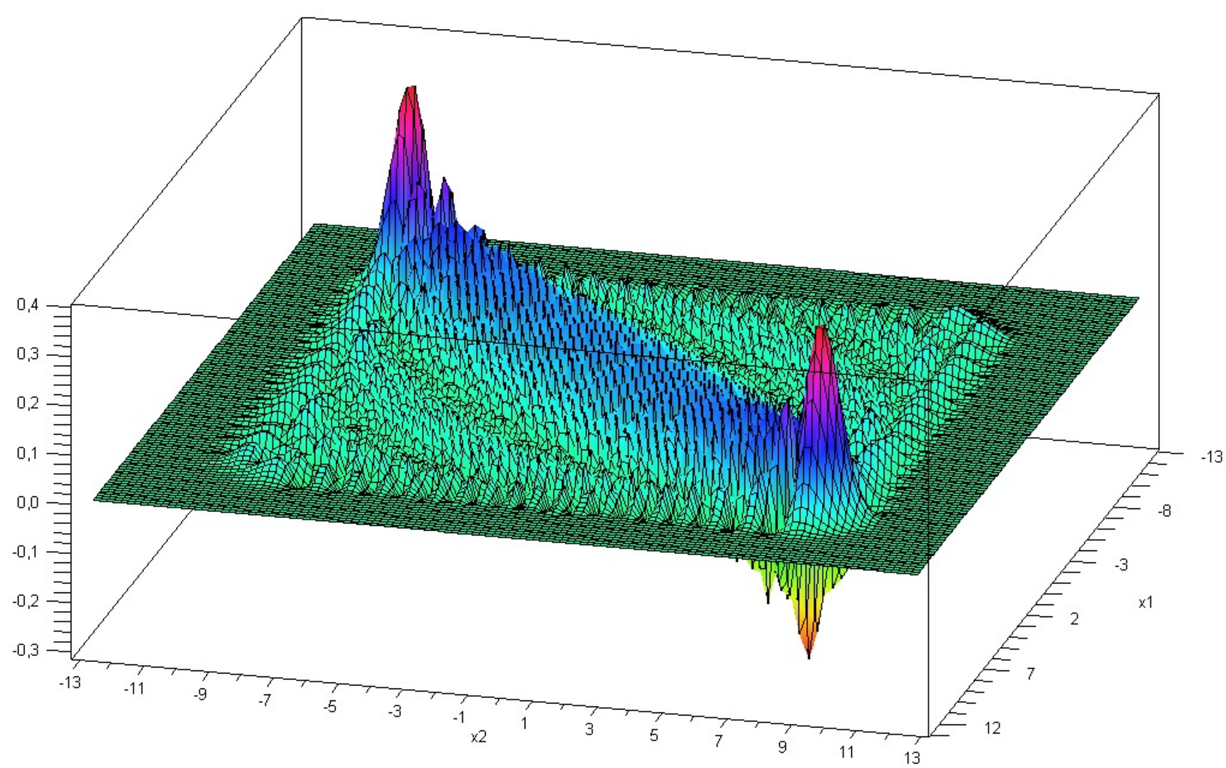
0,32
0,24
0,16
0,08
0,0
-0,08
-0,16
-0,24
-0,32
-0,4
-6
-5
-4
-3
-2
-1
0
x2
1
2
3
4
5
6
-6,0
-3,5
-1,0
x1
1,5
4,0

n=80



0,4
0,3
0,2
0,1
0,0
-0,1
-0,2
-0,3
-13
-11
-9
-7
-5
-3
-1
x2
1
3
5
7
9
11
13
-13
-8
-3
x1
2
7
12

2 Study of the one-dimensional eigenfunctions

For any $y \in (-1,1)$ and any positive integer n, we define:

$$g_n(y) = \frac{(2n+1)}{2}\Big(\arccos(y) - y\sqrt{1-y^2}\Big) - \frac{\pi}{4}. \tag{10}$$

We require an explicit approximation of the one-dimensional Hermite functions in the allowed region $(-\sqrt{2n+1}, \sqrt{2n+1})$, as stated in the following result:

Proposition 2. *For any $n \gg 1$, the following inequality holds uniformly for any $y \in (-1,1)$:*

$$\left| h_n(y\sqrt{2n+1}) - \frac{\sqrt{2}}{\sqrt{\pi}(2n+1)^{\frac{1}{4}}} \frac{\cos(g_n(y))}{(1-y^2)^{\frac{1}{4}}} \right| \leq \frac{0.5}{(2n+1)^{\frac{5}{4}}(1-y^2)^{\frac{7}{4}}}. \tag{11}$$

The previous result is obtained by carefully following the constants in a standard WKB method with explicit remainders (see Section 6 for details of the proof). The simple constant 0.5 has been chosen because it is sufficient for what follows[1] but the proof in Section 6 provides a slightly better constant. Even without the explicit constant 0.5, such an approximation is very well-known (see [Sko59] and [Erd60, pages 22-23]). Proposition 2 is only meaningful when y is not to close to 1 and becomes useful precisely if the upper bound in (11) is controlled by the amplitude of the oscillating term, which gives the condition in [Muc70, (2.4)]:

$$\frac{1}{(2n+1)^{\frac{1}{6}}} \lesssim \sqrt{2n+1} - y\sqrt{2n+1}.$$

For $y \to 1$, a more precise approximation of the eigenfunction h_n is obtained by comparing it with an Airy function (see Section 6).

Proposition 3. *Define the following set*

$$\mathcal{C} := \Big\{ \Big(\frac{3\pi}{\sqrt{2}}\Big)^{\frac{2}{3}} k^{\frac{2}{3}}, \text{ with } k \in \mathbb{N}^\star \Big\}. \tag{12}$$

For any fixed constant $C \in \mathcal{C}$, for any $n \gg 1$ and any x in the interval

$$\left[\sqrt{2n+1} - \frac{C + \frac{1}{\sqrt{C}}}{(2n+1)^{\frac{1}{6}}}, \sqrt{2n+1} - \frac{C}{(2n+1)^{\frac{1}{6}}} \right] \tag{13}$$

the following inequality holds uniformly

$$h_n(x) \geq \frac{0.3}{C^{\frac{1}{4}}(2n+1)^{\frac{1}{12}}}.$$

Remark 4. *In the sequel, we only need the previous result for a single choice of C, for instance the smallest one $C = \big(\frac{3\pi}{\sqrt{2}}\big)^{\frac{2}{3}} \simeq 3.541$. In particular, we note that an immediate corollary is the bound*

$$\|h_n\|_{L^p(\mathbb{R})} \gtrsim n^{-\frac{1}{12}-\frac{1}{6p}}$$

which is known to be optimal for $p \in (4, +\infty]$ (see [Tha93, Lemma 1.5.2]).

[1] See (18).

PROOF. Set $x = y\sqrt{2n+1}$. Then the zone (13) becomes

$$y \in \Big[1 - \frac{C + \frac{1}{\sqrt{C}}}{(2n+1)^{\frac{2}{3}}}, 1 - \frac{C}{(2n+1)^{\frac{2}{3}}}\Big]. \tag{14}$$

Using Proposition 2, we may write

$$h_n(x) = P_n(y)\cos(g_n(y)) + R_n(y) \tag{15}$$

with

$$P_n(y) = \frac{\sqrt{2}}{\sqrt{\pi}(2n+1)^{\frac{1}{4}}(1-y^2)^{\frac{1}{4}}} \quad \text{and} \quad |R_n(y)| \leq \frac{0.5}{(2n+1)^{\frac{5}{4}}(1-y^2)^{\frac{7}{4}}}. \tag{16}$$

We temporarily admit that the condition $C \in \mathcal{C}$ implies the following inequality (see Step 2 for justification):

$$\cos(g_n(y)) \geq \frac{1}{2}. \tag{17}$$

Step 1. We now prove that (17) implies the expected inequality $h_n(x) \geq \frac{1}{4C^{\frac{1}{4}}(2n+1)^{\frac{1}{12}}}$. We first observe that the following holds uniformly for any $y \in \big[0, 1 - \frac{C}{(2n+1)^{\frac{2}{3}}}\big]$ as $n \to +\infty$:

$$|R_n(y)| \leq \frac{0.5}{(2n+1)^{\frac{5}{4}}(1-(1-\frac{C}{(2n+1)^{\frac{2}{3}}})^2)^{\frac{7}{4}}} \underset{n\to+\infty}{\sim} \frac{0.5}{(2C)^{\frac{7}{4}}(2n+1)^{\frac{1}{12}}}.$$

For any $C \in \mathcal{C}$, the inequality $C^{\frac{3}{2}} \geq \frac{3\pi}{\sqrt{2}} = 6.66...$ yields the following rough bound $C + \frac{1}{\sqrt{C}} \leq \frac{7}{6}C$. We then provide a lower bound of the principal term $P_n(y)$ under the condition $1 - \frac{\frac{7}{6}C}{(2n+1)^{\frac{2}{3}}} \leq y$:

$$\frac{P_n(y)}{2} \geq \frac{1}{\sqrt{2\pi}(2n+1)^{\frac{1}{4}}(1-(1-\frac{\frac{7}{6}C}{(2n+1)^{\frac{2}{3}}})^2)^{\frac{1}{4}}} \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{2\pi}\big(\frac{14}{6}\big)^{\frac{1}{4}}C^{\frac{1}{4}}(2n+1)^{\frac{1}{12}}}.$$

We again use the inequality $C^{\frac{3}{2}} \geq \frac{3\pi}{\sqrt{2}}$ to obtain the following numerical lower bound[2]:

$$\frac{1}{\sqrt{2\pi}\big(\frac{14}{6}\big)^{\frac{1}{4}}C^{\frac{1}{4}}} - \frac{0.5}{2^{\frac{7}{4}}C^{7/4}} \geq \frac{1}{C^{\frac{1}{4}}}\Big(\frac{1}{\sqrt{2\pi}\big(\frac{14}{6}\big)^{\frac{1}{4}}} - \frac{0.5\times\sqrt{2}}{2^{\frac{7}{4}}\times 3\pi}\Big) > \frac{0.3}{C^{\frac{1}{4}}}. \tag{18}$$

Combining the above considerations, we obtain the following uniform lower bound for $n \gg 1$:

$$\frac{P_n(y)}{2} - |R_n(y)| \geq \frac{0.3}{C^{\frac{1}{4}}(2n+1)^{\frac{1}{12}}}.$$

Now (15) and (17) finally show the bound from below:

$$h_n(x) \geq \frac{0.3}{C^{\frac{1}{4}}(2n+1)^{\frac{1}{12}}}.$$

[2]We note that if the constant on the right-hand side of (11) is significantly larger than 0.5 then the smallest value of C in (12) cannot be used with the argument in (18).

Step 2 (use of (12) to prove (17)). Since we aim to bound from below $\cos(g_n(y))$, it suffices to ensure that $g_n(y)$ is near $2\pi\mathbb{N}$. For $C \in \mathcal{C}$, we shall use the condition

$$\frac{2\sqrt{2}}{3}C^{\frac{3}{2}} \in 2\pi\mathbb{N}. \tag{19}$$

We first integrate the Taylor expansion at $\omega = 0$ as $a \to 0^+$:

$$\begin{aligned} \arccos(1-a^2) &= \sqrt{2}\int_0^a \frac{1}{\sqrt{1-\frac{\omega^2}{2}}}\mathrm{d}\omega = \sqrt{2}\int_0^a 1 + \frac{\omega^2}{4} + \mathcal{O}(\omega^4)\mathrm{d}\omega \\ &= \sqrt{2}a + \frac{\sqrt{2}}{12}a^3 + \mathcal{O}(a^5). \end{aligned}$$

As $\theta \to 0$, one also has

$$2\theta - \sin(2\theta) = \frac{4}{3}\theta^3 - \frac{4}{15}\theta^5 + \mathcal{O}(\theta^7).$$

By composition with $\theta = \arccos(1-a^2)$, we get the following expansion for $a \to 0^+$:

$$\begin{aligned} 2\arccos(1-a^2) - \sin(2\arccos(1-a^2)) &= 2\theta - \sin(2\theta) \\ &= \frac{8\sqrt{2}}{3}a^3 - \frac{2\sqrt{2}}{\sqrt{5}}a^5 + \mathcal{O}(a^7). \end{aligned}$$

Remembering the inequality $C^{\frac{3}{2}} \geq \frac{3\pi}{\sqrt{2}}$, we may define the following positive constant:

$$K = \frac{8\sqrt{2}}{3} - \frac{\pi}{3C^{\frac{3}{2}}}.$$

By using $K < \frac{8\sqrt{2}}{3}$, we deduce the following two-sided inequalities for $a \to 0^+$:

$$Ka^3 \leq 2\arccos(1-a^2) - \sin(2\arccos(1-a^2)) \leq \frac{8\sqrt{2}}{3}a^3.$$

We now introduce

$$G(y) := 2\arccos(y) - 2y\sqrt{1-y^2} = 2\theta(y) - \sin(2\theta(y)) \quad \text{with} \quad \theta(y) := \arccos(y).$$

For y in the zone (14), we set $a = \sqrt{1-y} \in \Big[\frac{\sqrt{C}}{(2n+1)^{\frac{1}{3}}}, \frac{\sqrt{C+\frac{1}{\sqrt{C}}}}{(2n+1)^{\frac{1}{3}}}\Big]$. We then deduce the following inequalities for $n \gg 1$:

$$\begin{aligned} KC^{\frac{3}{2}} \leq\ & (2n+1)G(y) && \leq \frac{8\sqrt{2}}{3}\Big(C + \frac{1}{\sqrt{C}}\Big)^{\frac{3}{2}} \\ \frac{8\sqrt{2}}{3}C^{\frac{3}{2}} - \frac{\pi}{3} \leq\ & (2n+1)G(y) && \leq \frac{8\sqrt{2}}{3}C^{\frac{3}{2}} + \frac{8\sqrt{2}}{3}\Big[\Big(C + \frac{1}{\sqrt{C}}\Big)^{\frac{3}{2}} - C^{\frac{3}{2}}\Big]. \end{aligned}$$

Since $C^{\frac{3}{2}}$ is larger than 1, we may bound[3]:

$$\Big(C + \frac{1}{\sqrt{C}}\Big)^{\frac{3}{2}} - C^{\frac{3}{2}} = C^{\frac{3}{2}}\Big(\Big(1 + \frac{1}{C^{\frac{3}{2}}}\Big)^{\frac{3}{2}} - 1\Big) \leq \sqrt{8} - 1.$$

[3] We merely use $(1+\varepsilon)^{\frac{3}{2}} \leq 1 + (\sqrt{8}-1)\varepsilon$ for any $\varepsilon \in [0,1]$.

Due to the relation $g_n(y) = \frac{(2n+1)G(y)-\pi}{4}$, we thus obtain

$$\frac{2\sqrt{2}}{3}C^{\frac{3}{2}} - \frac{\pi}{3} \quad \leq g_n(y) \leq \quad \frac{2\sqrt{2}}{3}C^{\frac{3}{2}} + \frac{\frac{8\sqrt{2}}{3}(\sqrt{8}-1)-\pi}{4}.$$

A numerical computation shows that the upper bound is less than $\frac{2\sqrt{2}}{3}C^{\frac{3}{2}}+\frac{\pi}{3}$. Finally, the condition (19) proves that $g_n(y)$ belongs to $\left[\frac{-\pi}{3}, \frac{\pi}{3}\right] + 2\pi\mathbb{Z}$ and hence $\cos(g_n(y)) \geq \frac{1}{2}$. □

For the multidimensional case, the following corollary will be required.

Corollary 5. *Let $C \in \mathcal{C}$ be a constant as in Proposition 3 and fix $\alpha > 0$ satisfying $3\alpha < \frac{1}{\sqrt{C}}$. Moreover, we consider $\beta \geq 0$. Define the following intervals*

$$I_t := \left[\sqrt{2t+1} - \frac{C + \frac{2}{3\sqrt{C}}}{(2t+1)^{\frac{1}{6}}}, \sqrt{2t+1} - \frac{C + \frac{1}{3\sqrt{C}}}{(2t+1)^{\frac{1}{6}}}\right] \qquad \forall t \geq 1. \tag{20}$$

For any $t \gg 1$ and $n \in \mathbb{N}$ satisfying $|n-t| \leq \alpha(2t+1)^{\frac{1}{3}} + \beta$, the following inequality holds

$$\min_{x \in I_t} h_n(x) \geq \frac{1}{4C^{\frac{1}{4}}(2t+1)^{\frac{1}{12}}}. \tag{21}$$

Proof. For simplicity, we use the following notations:

$$C_1 = C, \qquad C_1' = C + \frac{1}{3\sqrt{C}} \qquad C_2' = C + \frac{2}{3\sqrt{C}}, \qquad C_2 := C + \frac{1}{\sqrt{C}}.$$

We will show that if α is chosen small enough then the inequality $|n-t| \leq \alpha(2t+1)^{\frac{1}{3}} + \beta$ implies that the interval I_t is included in the one appearing in (13):

$$\left[\sqrt{2t+1} - \frac{C_2'}{(2t+1)^{\frac{1}{6}}}, \sqrt{2t+1} - \frac{C_1'}{(2t+1)^{\frac{1}{6}}}\right] \subset \left[\sqrt{2n+1} - \frac{C_2}{(2n+1)^{\frac{1}{6}}}, \sqrt{2n+1} - \frac{C_1}{(2n+1)^{\frac{1}{6}}}\right].$$

By invoking Proposition 3, such an inclusion will yield the uniform bound $h_n(x) \geq \frac{0.3}{C^{\frac{1}{4}}(2n+1)^{\frac{1}{12}}}$ for $x \in I_t$, which in turn implies the expected bound (21) because $n \sim t$ for $t \gg 1$.

We first compare the lower bounds of the two intervals and we have to show the inequality

$$\sqrt{2n+1} - \frac{C_2}{(2n+1)^{\frac{1}{6}}} \leq \sqrt{2t+1} - \frac{C_2'}{(2t+1)^{\frac{1}{6}}}.$$

The sequence $n \mapsto \sqrt{2n+1} - \frac{C_2}{(2n+1)^{\frac{1}{6}}}$ is increasing, which allows us to apply the bound

$$n \leq t + \alpha T^{\frac{1}{3}} + \beta \quad \text{with } T = 2t+1,$$

and

$$\begin{aligned}\sqrt{2n+1} - \frac{C_2}{(2n+1)^{\frac{1}{6}}} &\leq \sqrt{T + 2\alpha T^{\frac{1}{3}} + 2\beta} - \frac{C_2}{(T + 2\alpha T^{\frac{1}{3}} + 2\beta)^{\frac{1}{6}}} \\ &\leq \sqrt{T} - \frac{C_2 - \alpha}{T^{\frac{1}{6}}} + \frac{\beta}{\sqrt{T}} + o\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

It suffices to show for $T \gg 1$ that the last upper bound is less or equal to

$$\sqrt{T} - \frac{C_2'}{T^{\frac{1}{6}}}.$$

It is clear that it is sufficient to set $0 < \alpha < C_2 - C_2' = \frac{1}{3\sqrt{C}}$.

For the comparison of the upper bounds of the two intervals, we briefly give the similar arguments. We use $t - \alpha T^{\frac{1}{3}} - \beta \leq n$ so that it is sufficient to prove the inequality

$$\sqrt{T} - \frac{C_1'}{T^{\frac{1}{6}}} \leq \sqrt{T - 2\alpha T^{\frac{1}{3}} - 2\beta} - \frac{C_1}{(T - 2\alpha T^{\frac{1}{3}} - 2\beta)^{\frac{1}{6}}} = \sqrt{T} - \frac{C_1 + \alpha}{T^{\frac{1}{6}}} - \frac{\beta}{\sqrt{T}} + o\Big(\frac{1}{\sqrt{T}}\Big)$$

that would lead to the sufficient condition $0 < \alpha < C_1' - C_1 = \frac{1}{3\sqrt{C}}$. □

3 Cardinality of the set $J_n^{(\alpha,\beta)}$

We require a combinatorial lemma in which we will work under the assumptions $T \to +\infty$ and $T = o(n)$ for $n \to +\infty$.

Lemma 6. *For any integer $d \geq 2$, there exists a constant $C(d) \geq 1$, with the property that, for any $n \geq d(d-1)$ and any real number T satisfying $d - 1 \leq T \leq \frac{n}{d}$, the cardinality of the set Λ defined below is of order T^{d-1}:*

$$\Lambda := \Big\{(n_1, \dots, n_d) \in \mathbb{N}^d, \quad \max_{1 \leq i \leq d} \Big| n_i - \frac{n}{d} \Big| \leq T \quad \textit{and} \quad \sum_{i=1}^{d} n_i = n \Big\},$$

$$\frac{1}{C(d)} T^{d-1} \leq \mathrm{Card}(\Lambda) \leq C(d) T^{d-1}.$$

Proof. The case $d = 2$ follows directly from

$$\mathrm{Card}(\Lambda) = \mathrm{Card}\Big(\mathbb{N} \cap \Big[\frac{n}{d} - T, \frac{n}{d} + T\Big]\Big) \simeq T.$$

In what follows, we assume $d \geq 3$.

Step 1. The upper bound follows from the fact that each n_i belongs to $\left[\frac{n}{d} - T, \frac{n}{d} + T\right]$. Thus $(n_1, \dots, n_{d-1})$ runs over a set of size $\mathcal{O}(T^{d-1})$ and $-n_d = n_1 + \cdots + n_{d-1}$ is determined by the choice of the $d - 1$ first integers.

We now explain the lower bound of $\mathrm{Card}(\Lambda)$.

Step 2. Let $n_1^\star, \dots, n_{d-1}^\star$ be $d - 1$ non-negative integers satisfying for each $i \in \{1, \dots, d-1\}$

$$\Big| \frac{n}{d} - n_i^\star \Big| \leq \frac{1}{2}. \tag{22}$$

The integer $n_d^\star := n - (n_1^\star + \cdots + n_{d-1}^\star) = \frac{n}{d} + \sum_{i=1}^{d-1} (\frac{n}{d} - n_i^\star)$ satisfies

$$\Big| \frac{n}{d} - n_d^\star \Big| \leq \frac{d-1}{2} \qquad \text{and} \qquad n_d^\star \geq \frac{n}{d} - \frac{d-1}{2} \geq \frac{d-1}{2} > 0. \tag{23}$$

We have just identified an element $(n_1^\star, \dots, n_d^\star)$ of Λ.

Step 3. In order to construct other elements of Λ, consider $(\mu_1, \dots, \mu_{d-1}) \in \mathbb{Z}^{d-1}$ satisfying

$$\max_{1 \le i \le d-1} |\mu_i| \le \frac{T}{2(d-1)}.$$

The number of tuples $(\mu_1, \dots, \mu_{d-1})$ is $(1 + 2\lfloor \frac{T}{2(d-1)} \rfloor)^{d-1} \ge \big(\frac{T}{2(d-1)}\big)^d$. For $T \ge d-1$, we will use the following consequence:

$$|\mu_1| + \dots + |\mu_{d-1}| \le \frac{T}{2} \le T - \frac{d-1}{2}. \tag{24}$$

We now introduce the following linearly independent vectors:

$$e_1 := \begin{pmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}, e_2 := \begin{pmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \dots, e_{d-1} := \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

We claim that all the above conditions ensure that the following point belongs to Λ:

$$\begin{pmatrix} n_1^\star \\ \vdots \\ n_{d-1}^\star \\ n_d^\star \end{pmatrix} + \sum_{k=1}^{d-1} \mu_k e_k = \begin{pmatrix} n_1^\star - \mu_1 \\ \vdots \\ n_{d-1}^\star - \mu_{d-1} \\ n_d^\star + \mu_1 + \dots + \mu_{d-1} \end{pmatrix}. \tag{25}$$

- It is clear that the sum of the coordinates equals n.
- We now check

$$\max_{1 \le i \le d-1} \left| n_i^\star - \mu_i - \frac{n}{d} \right| \le T \qquad \text{and} \qquad \left| n_d^\star + \mu_1 + \dots + \mu_{d-1} - \frac{n}{d} \right| \le T. \tag{26}$$

Such inequalities come directly from (22), (23) and (24).

- We finally note that (26) and the assumption $T \le \frac{n}{d}$ show that each coordinate of (25) is non-negative. □

4 Proof of Theorem 1

Let $C = \min(\mathcal{C}) = \big(\frac{3\pi}{\sqrt{2}}\big)^{\frac{2}{3}}$ as in Proposition 3. In particular, $\frac{1}{3\sqrt{C}} > 0.17 > \frac{1}{6}$. We fix $\alpha \in (0, \frac{1}{6}]$ so that we can apply Corollary 5. We then set

$$T = \alpha \Big(\frac{2n}{d} + 1\Big)^{\frac{1}{3}} + \beta.$$

In particular, one has $T = o(n)$. Lemma 6 then ensures that, for $n \gg 1$, the set $J_n^{(\alpha,\beta)}$ has size of order $n^{\frac{d-1}{3}}$. Consider the interval I_t defined in (20) for $t = \frac{n}{d}$. For any $(x_1, \dots, x_d)$ belonging to the subset $I_{\frac{n}{d}} \times \cdots \times I_{\frac{n}{d}}$, the following holds:

$$\begin{aligned} v_n^{(\alpha,\beta)}(x_1,\ldots,x_n) &\gtrsim \frac{1}{\sqrt{J_n^{(\alpha,\beta)}}} \sum_{(n_1,\ldots,n_d)\in J_n^{(\alpha,\beta)}} n^{-\frac{d}{12}} \\ &\gtrsim \sqrt{J_n^{(\alpha,\beta)}} n^{-\frac{d}{12}} \\ &\gtrsim n^{\frac{d}{12}-\frac{1}{6}}. \end{aligned}$$

A numerical computation allows us to restrict $I_{\frac{n}{d}} \times \cdots \times I_{\frac{n}{d}}$ to the following hypercube:

$$\left[\sqrt{\frac{2n}{d}+1} - \frac{3.89}{(\frac{2n}{d}+1)^{\frac{1}{6}}}, \sqrt{\frac{2n}{d}+1} - \frac{3.72}{(\frac{2n}{d}+1)^{\frac{1}{6}}}\right]^d$$

whose volume is of order $n^{-\frac{d}{6}}$. We finally integrate the norm in $L^p(\mathbb{R}^d)$ of the function equaling 1 on this hypercube to get the conclusion:

$$\|v_n^{(\alpha,\beta)}\|_{L^p(\mathbb{R}^d)} \gtrsim n^{\frac{d}{12}-\frac{1}{6}} \times n^{-\frac{d}{6p}}.$$

5 Appendix: explicit inequalities about Airy functions

We recall several notations and write explicit inequalities which are consequences of [Olv97]. The Airy function is defined as the improper integral

$$\mathrm{Ai}(X) = \frac{1}{\pi}\int_0^{+\infty} \cos\Big(\frac{t^3}{3} + Xt\Big)\mathrm{d}t, \qquad \forall X \in \mathbb{R}.$$

As $X \to +\infty$, the following asymptotic is known:

$$\mathrm{Ai}(X) \underset{X\to+\infty}{\sim} \frac{1}{2\sqrt{\pi}X^{\frac{1}{4}}} e^{-\frac{2}{3}X^{\frac{3}{2}}}. \tag{27}$$

We require the following result that gives a quantitative statement of the oscillating behavior on $(-\infty, 0)$.

Lemma 7. *The following holds true for any $X > 0$:*

$$\mathrm{Ai}(-X) = \frac{1}{\sqrt{\pi}X^{\frac{1}{4}}} \cos\Big(\frac{2}{3}X^{\frac{3}{2}} - \frac{\pi}{4}\Big) + \widetilde{\varepsilon}(X) \qquad \textit{with} \qquad |\widetilde{\varepsilon}(X)| \leq \frac{5}{48\sqrt{\pi}X^{\frac{7}{4}}}. \tag{28}$$

Moreover the constant $\frac{5}{48\sqrt{\pi}}$ is optimal.

Proof. Fix $\nu > \frac{-1}{2}$, we shall consider the first Hankel function $H_\nu^{(1)}$ which may be defined by the following formula (see [Wat45, page 168, line (3)] or [Olv97, page 268, lines (13.01) and (13.07)]) for positive x:

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} \frac{e^{i\left(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)}}{\Gamma(\nu+\frac{1}{2})} \int_0^{+\infty} e^{-\tau}\tau^{\nu-\frac{1}{2}} \left(1 + \frac{i\tau}{2x}\right)^{\nu-\frac{1}{2}} \mathrm{d}\tau.$$

Following [Olv97, page 269, Ex 13.1], we introduce two real terms[4] $P(\nu,x)$ and $Q(\nu,x)$ as follows:

$$\begin{aligned} P(\nu,x)+iQ(\nu,x) &= H_\nu^{(1)}(x)\sqrt{\frac{\pi x}{2}}e^{-i\left(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)} \\ &= \frac{1}{\Gamma(\nu+\frac{1}{2})}\int_0^{+\infty} e^{-\tau}\tau^{\nu-\frac{1}{2}}\left(1+\frac{i\tau}{2x}\right)^{\nu-\frac{1}{2}}\mathrm{d}\tau. \end{aligned} \tag{29}$$

Since the following inequality will be needed at the end of this section, we first require $\nu\in\left(\frac{-1}{2},\frac{1}{2}\right]$:

$$\begin{aligned} \sqrt{P(\nu,x)^2+Q(\nu,x)^2} &\le \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{+\infty} e^{-\tau}\tau^{\nu-\frac{1}{2}}\left|1+\frac{i\tau}{2x}\right|^{\nu-\frac{1}{2}}\mathrm{d}\tau \\ &\le \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{+\infty} e^{-\tau}\tau^{\nu-\frac{1}{2}}\left(1+\frac{\tau^2}{4x^2}\right)^{\frac{\nu}{2}-\frac{1}{4}}\mathrm{d}\tau \\ &\le \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{+\infty} e^{-\tau}\tau^{\nu-\frac{1}{2}}\mathrm{d}\tau \\ &\le 1. \end{aligned} \tag{30}$$

For $\nu\in\left(\frac{-1}{2},\frac{3}{2}\right]$, we now claim that the following inequality holds true for any $x>0$:

$$\sqrt{(P(\nu,x)-1)^2+Q(\nu,x)^2}\le\frac{|\nu-\frac{1}{2}|(\nu+\frac{1}{2})}{2x}. \tag{31}$$

The proof follows directly from the previous integral representations:

$$\begin{aligned} P(\nu,x)-1+iQ(\nu,x) &= \frac{1}{\Gamma(\nu+\frac{1}{2})}\int_0^{+\infty} e^{-\tau}\tau^{\nu-\frac{1}{2}}\left[\left(1+\frac{i\tau}{2x}\right)^{\nu-\frac{1}{2}}-1\right]\mathrm{d}\tau \\ \sqrt{(P(\nu,x)-1)^2+Q(\nu,x)^2} &\le \frac{1}{\Gamma(\nu+\frac{1}{2})}\int_0^{+\infty} e^{-\tau}\tau^{\nu-\frac{1}{2}}\left|\left(1+\frac{i\tau}{2x}\right)^{\nu-\frac{1}{2}}-1\right|\mathrm{d}\tau. \end{aligned}$$

By using $\nu\le\frac{3}{2}$, we may invoke the inequality

$$\left|\left(1+\frac{i\tau}{2x}\right)^{\nu-\frac{1}{2}}-1\right|=\left|\int_0^{\frac{\tau}{2x}}\left(\nu-\frac{1}{2}\right)(1+i\omega)^{\nu-\frac{3}{2}}\mathrm{d}\omega\right|\le\frac{|\nu-\frac{1}{2}|\tau}{2x}.$$

[4] For positive x, it turns out that (29) allows to define $P(\nu,x)$ and $Q(\nu,x)$ with the Bessel functions J_ν and Y_ν (that actually are the real and imaginary parts of $H_\nu^{(1)}$):

$$\begin{aligned} P(\nu,x) &= \sqrt{\frac{\pi x}{2}}\Big(\cos\big(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\big)J_\nu(x)+\sin\big(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\big)Y_\nu(x)\Big), \\ Q(\nu,x) &= \sqrt{\frac{\pi x}{2}}\Big(\cos\big(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\big)Y_\nu(x)-\sin\big(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\big)J_\nu(x)\Big). \end{aligned}$$

As a consequence, we continue our computations to prove (31):

$$\begin{aligned}\sqrt{(P(\nu,x)-1)^2+Q(\nu,x)^2} &\leq \frac{|\nu-\frac{1}{2}|}{2x\Gamma(\nu+\frac{1}{2})}\int_0^{+\infty} e^{-\tau}\tau^{\nu+\frac{1}{2}}\mathrm{d}\tau \\ &\leq \frac{|\nu-\frac{1}{2}|\Gamma(\nu+\frac{3}{2})}{2x\Gamma(\nu+\frac{1}{2})} \\ &\leq \frac{|\nu-\frac{1}{2}|(\nu+\frac{1}{2})}{2x}.\end{aligned}$$

For $\nu=\frac{1}{3}$, this inequality reads

$$\sqrt{\Big(P\Big(\frac{1}{3},x\Big)-1\Big)^2+Q\Big(\frac{1}{3},x\Big)^2} \leq \frac{5}{72x}. \tag{32}$$

We use the following identity, which relates the Airy function to the Hankel function $H^{(1)}_{\frac{1}{3}}$ and, consequently, to $P\big(\frac{1}{3},\cdot\big)$ and $Q\big(\frac{1}{3},\cdot\big)$ (see [Olv97, page 392, line (1.05)] and then use (29)):

$$\begin{aligned}\mathrm{Ai}(-X) &= \sqrt{\frac{X}{3}}\Re\Big(e^{i\frac{\pi}{6}}H^{(1)}_{\frac{1}{3}}(\xi)\Big) \qquad \text{with} \quad \xi=\frac{2}{3}X^{\frac{3}{2}} \\ &= \frac{1}{\sqrt{\pi}X^{\frac{1}{4}}}\Re\Big(e^{i(\xi-\frac{\pi}{4})}\Big(P\Big(\frac{1}{3},\xi\Big)+iQ\Big(\frac{1}{3},\xi\Big)\Big)\Big) \\ &= \frac{1}{\sqrt{\pi}X^{\frac{1}{4}}}\Big(\cos\Big(\xi-\frac{\pi}{4}\Big)P\Big(\frac{1}{3},\xi\Big)-\sin\Big(\xi-\frac{\pi}{4}\Big)Q\Big(\frac{1}{3},\xi\Big)\Big).\end{aligned} \tag{33}$$

Then (28) is a straightforward consequence of (32) and the last identity.

The sharpness of (28) for $X\to+\infty$ is due to the direct appearance of the constant $\frac{5}{48\sqrt{\pi}}$ in the classical asymptotics of the Airy function[5]:

$$\mathrm{Ai}(-X)=\frac{1}{\sqrt{\pi}X^{\frac{1}{4}}}\cos\Big(\frac{2}{3}X^{\frac{3}{2}}-\frac{\pi}{4}\Big)+\frac{5}{48\sqrt{\pi}X^{\frac{7}{4}}}\sin\Big(\frac{2}{3}X^{\frac{3}{2}}-\frac{\pi}{4}\Big)+\mathcal{O}\Big(\frac{1}{X^{\frac{13}{4}}}\Big). \tag{34}$$

It is sufficient to make X tend to $+\infty$ under the condition $\Big|\sin\Big(\frac{2}{3}X^{\frac{3}{2}}-\frac{\pi}{4}\Big)\Big|=1$. ☐

Following [Olv97, pages 394-395], one may define three functions as follows.

- The Airy function Bi of the second kind is defined by

$$\mathrm{Bi}(X)=\frac{1}{\pi}\int_0^{+\infty} e^{\frac{-1}{3}t^3+Xt}+\sin\Big(\frac{t^3}{3}+Xt\Big)\mathrm{d}t \qquad \forall X\in\mathbb{R}.$$

Moreover, a formula similar to (33) also holds (see [Olv97, page 394] by setting $\xi=\frac{2}{3}X^{\frac{3}{2}}$ or [Olv97, page 393, line (1.14)] and (29)):

$$\mathrm{Bi}(-X)=\frac{-1}{\sqrt{\pi}X^{\frac{1}{4}}}\Big(\sin\Big(\xi-\frac{\pi}{4}\Big)P\Big(\frac{1}{3},\xi\Big)+\cos\Big(\xi-\frac{\pi}{4}\Big)Q\Big(\frac{1}{3},\xi\Big)\Big) \;\text{ for } X>0. \tag{35}$$

[5] whose proof is indeed a consequence of (33).

- A weight function E has a definition involving

$$c = -0.366...$$

the largest negative solution of $\mathrm{Ai}(c) = \mathrm{Bi}(c)$. We then define

$$E(X) = \begin{cases} 1 & \text{for} \quad X < c, \\ \sqrt{\frac{\mathrm{Bi}(X)}{\mathrm{Ai}(X)}} & \text{for} \quad X \geq c. \end{cases}$$

As explained in [Olv97], the function E is non-decreasing and satisfies

$$E(X) \geq 1, \qquad \forall X \in \mathbb{R}. \tag{36}$$

- A modulus function M is finally given by

$$M(X) = \begin{cases} \sqrt{\mathrm{Ai}(X)^2 + \mathrm{Bi}(X)^2} & \text{for} \quad X < c, \\ \sqrt{2\,\mathrm{Ai}(X)\,\mathrm{Bi}(X)} & \text{for} \quad X \geq c. \end{cases}$$

Note that for any $X > 0$, we may write

$$M(-X) \leq \sqrt{\mathrm{Ai}(-X)^2 + \mathrm{Bi}(-X)^2}.$$

From (33) and (35) and then (30), we infer the following

$$\begin{aligned} M(-X) &\leq \frac{1}{\sqrt{\pi} X^{\frac{1}{4}}} \sqrt{P\Big(\frac{1}{3}, \xi\Big)^2 + Q\Big(\frac{1}{3}, \xi\Big)^2} \\ &\leq \frac{1}{\sqrt{\pi} X^{\frac{1}{4}}}. \end{aligned} \tag{37}$$

We conclude this section with a remark that will be used below. For any $X > 0$ (and thus $X \geq c$), the following identity holds:

$$\frac{M(X)}{E(X)} = \sqrt{2}\,\mathrm{Ai}(X). \tag{38}$$

6 Appendix: proof of Proposition 2

Instead of following the constants in the proof of [Sko59], we prefer to use the results presented in [Olv97, page 403, Ex 4.2 & 4.3] since some explicit computations are already provided.

We aim to study the following function for $n \gg 1$:

$$\mathcal{R}_n : y \in [0,1) \mapsto (2n+1)^{\frac{5}{4}} (1-y^2)^{\frac{7}{4}} \Big(h_n(y\sqrt{2n+1}) - \frac{\sqrt{2}}{\sqrt{\pi}(2n+1)^{\frac{1}{4}}} \frac{\cos(g_n(y))}{(1-y^2)^{\frac{1}{4}}} \Big).$$

While precise statements are given below, we first note that, as $y \to 1$, h_n is approximated by an Airy-type function, in its oscillatory regime, that roughly reads

$$h_n(y\sqrt{2n+1}) \simeq \frac{2^{\frac{1}{3}}}{(2n+1)^{\frac{1}{12}}} \,\mathrm{Ai}\left((2n+1)^{\frac{2}{3}} \zeta(y)\right) \qquad \text{with} \quad \zeta(y) \underset{y \to 1^-}{\sim} -2^{\frac{1}{3}}(1-y). \tag{39}$$

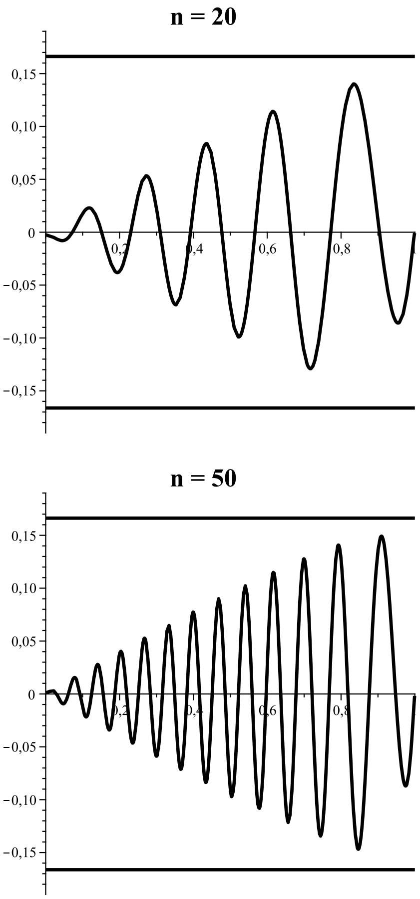
The figure below shows two graphical representations of $\mathcal{R}_n$ for $n \in \{20, 50\}$, compared with approximations $\mathcal{R}_n$ in which an Airy-type function is used instead of h_n (in fact, the principal term of (41) was employed because it is more precise than (39)).

In particular, we note the following observations:

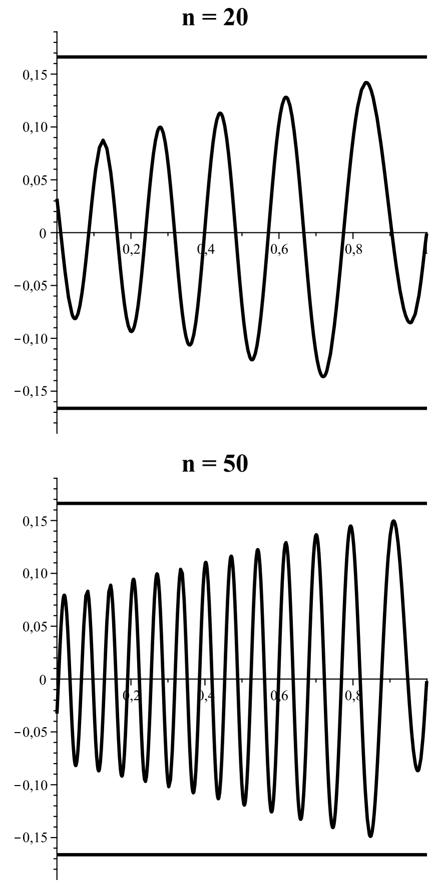
- the two curves are essentially superposed for y near 1, indicating that h_n is well approximated by an Airy-type function near the turning point $\sqrt{2n+1}$,

- if the Airy function is replaced with its classical asymptotics at $-\infty$, see (34), one conjectures that a natural expected bound of $\mathcal{R}_n$ would be $\frac{5\sqrt{8}}{48\sqrt{\pi}} = 0.166...$

Representations of $\mathcal{R}_n$



Representations of approximate $\mathcal{R}_n$ if h_n is replaced with an Airy-type function (see (41))



However, a residual term in the proof yields a larger upper bound 0.5 (see (49)), which is sufficient for our purpose (see (18)). Proposition 2 constitutes a special case of the following result.

Proposition 8. *Given any $K > \frac{5\sqrt{2}}{24\sqrt{\pi}}$, there exists a constant $y_K \in [0,1)$ such that, for all $n \gg 1$, the following inequality holds uniformly for $y \in (-1, y_K] \cup [y_K, 1)$:*

$$\left| h_n(y\sqrt{2n+1}) - \frac{\sqrt{2}}{\sqrt{\pi}(2n+1)^{\frac{1}{4}}} \frac{\cos(g_n(y))}{(1-y^2)^{\frac{1}{4}}} \right| \leq \frac{K}{(2n+1)^{\frac{5}{4}}(1-y^2)^{\frac{7}{4}}}. \tag{40}$$

Moreover, for the constant $K = 0.5$, the value $y_{0.5} = 0$ is suitable.

We now outline the main ideas for obtaining an explicit constant in the upper bound of (40).

From (10), the formulas $g_n(-y) = n\pi - g_n(y)$ and $h_n(-x) = (-1)^n h_n(x)$ ensure that it is sufficient to prove (40) for $y \in [0,1)$.

From the differential equation (3) satisfied by h_n, we deduce that the function $y \mapsto h_n(y\sqrt{2n+1})$ satisfies the following one:

$$-\frac{d^2w}{dy^2} + (2n+1)^2(y^2-1)w(y) = 0 \qquad \forall y \in [0, +\infty).$$

The set of such w forms a two-dimensional vector space. We now invoke [Olv97, page 399, Theorem 3.1] that provides two solutions expressed with the Airy functions Ai and Bi: one solution diverges to $+\infty$ as its argument tend to $+\infty$ whereas the other tends to 0. It follows that these two solutions are linearly independent and constitute a basis of the vector space of all solutions. In particular, there is a unique solution, up to a multiplicative constant, that tends to 0 at $+\infty$. Since $y \mapsto h_n(y\sqrt{2n+1})$ tends to 0 at $+\infty$, we may use the asymptotic form in [Olv97, page 399, Theorem 3.1 and page 403, Ex 4.2 and 4.3] and claim that there exists a constant C_n satisfying:

$$h_n(y\sqrt{2n+1}) = C_n\Big(\frac{-\zeta(y)}{1-y^2}\Big)^{\frac{1}{4}}\Big(\operatorname{Ai}\big((2n+1)^{\frac{2}{3}}\zeta(y)\big) + \varepsilon(n,y)\Big) \qquad \forall y \geq 0 \tag{41}$$

in which the function ζ and the remainder ε are now given. It should be noted that the factor of the Airy function is essentially constant for $n \gg 1$ (see (47)) and $y \to 1$ (see (42)):

$$C_n\Big(\frac{-\zeta(y)}{1-y^2}\Big)^{\frac{1}{4}} \underset{n\to+\infty}{\sim} \frac{\sqrt{2}}{(2n+1)^{\frac{1}{12}}} \times \frac{1}{2^{\frac{1}{6}}}.$$

The function ζ is a continuously differentiable solution of $\zeta(y)\zeta'(y)^2 = y^2 - 1$, yielding the classical formulas:

$$\begin{array}{lllll} 0 \leq y \leq 1 & \Rightarrow & \zeta(y) = -\Omega(y)^{\frac{2}{3}} \quad \text{with} \quad \Omega(y) & = & \frac{3}{4}\big(\arccos(y) - y\sqrt{1-y^2}\big) \\ & & & = & \frac{3}{2}\int_y^1 \sqrt{1-\omega^2}\,\mathrm{d}\omega. \\ 1 \leq y & \Rightarrow & \zeta(y) = \Phi(y)^{\frac{2}{3}} \quad \text{with} \quad \Phi(y) & = & \frac{3}{4}\big(y\sqrt{y^2-1} - \ln(y+\sqrt{y^2-1})\big) \\ & & & = & \frac{3}{2}\int_1^y \sqrt{\omega^2-1}\,\mathrm{d}\omega. \end{array} \tag{42}$$

Here is a remark useful for the end of the proof:

$$2\Omega(y) - (1-y^2)^{\frac{3}{2}} = \int_y^1 3(1-\omega)\sqrt{1-\omega^2}\,\mathrm{d}\omega \geq 0, \qquad \forall y \in [0,1]$$

which indeed reads

$$(-\zeta(y))^{\frac{3}{2}} \geq \frac{(1-y^2)^{\frac{3}{2}}}{2}, \qquad \forall y \in [0,1]. \tag{43}$$

Explicit bounds of the remainder ε for $y \in [0,1)$. Once numerical computations have been performed in the result presented in [Olv97, Theorem 3.1], the following explicit bounds are reported in [Olv97, page 403, Ex 4.3] for $y \in [0,1]$:

$$|\varepsilon(n,y)| \leq 0.97\left(e^{\frac{0.28}{2n+1}} - 1\right)\frac{M\big((2n+1)^{\frac{2}{3}}\zeta(y)\big)}{E\big((2n+1)^{\frac{2}{3}}\zeta(y)\big)}.$$

Using (36) and (37) from the previous section, the following inequality holds uniformly in the region $y \in [0,1)$ for $n \gg 1$:

$$|\varepsilon(n,y)| \leq \frac{0.28}{2n+1} \times \frac{1}{\sqrt{\pi}(2n+1)^{\frac{1}{6}}(-\zeta(y))^{\frac{1}{4}}} \leq \frac{0.16}{(2n+1)^{\frac{7}{6}}(-\zeta(y))^{\frac{1}{4}}}. \tag{44}$$

Negligible contribution of the remainder ε as $y \to +\infty$. We use [Olv97, page 397, line (2.15) and page 403, Ex 4.2] and (38) to obtain

$$|\varepsilon(n,y)| \leq \frac{\mathrm{Ai}((2n+1)^{\frac{2}{3}}\zeta(y))}{1.04}\Big[\exp\Big(\frac{1.05}{2n+1}\int_y^{+\infty} |H'(\omega)|\mathrm{d}\omega\Big) - 1\Big] \tag{45}$$

for a suitable explicit function $H : (-1,+\infty) \to [0,+\infty)$ given by

$$H(y) = \frac{5}{24|\zeta(y)|^{\frac{3}{2}}} + \frac{y^3-6y}{12|y^2-1|^{\frac{3}{2}}}.$$

In particular, the derivative of H on $(1,+\infty)$ can be expressed in terms of $\Phi(y)$ (see (42)) and satisfies, as $y \to +\infty$:

$$H'(y) = \frac{-5\Phi'(y)}{24\Phi(y)^2} + \frac{3y^2+2}{4(y^2-1)^{5/2}} = \mathcal{O}\Big(\frac{1}{y^3}\Big).$$

Hence, for a fixed n, (45) shows

$$|\varepsilon(n,y)| = o\Big(\mathrm{Ai}((2n+1)^{\frac{2}{3}}\zeta(y))\Big), \qquad y \to +\infty. \tag{46}$$

Computation and asymptotics of C_n in (41). We fix n and study the asymptotics, as $y \to +\infty$, of the two sides of (41). For the left-hand side of (41), we refer to (2) and recall that H_n is a polynomial with leading term 2^nX^n. For the right-hand side of (41), we refer to (46), (27) and use $\zeta(y)^{\frac{3}{2}} = \Phi(y) = \frac{3}{4}y^2 - \frac{3}{4}\ln(2y) - \frac{3}{8} + o(1)$. We then write

$$\begin{aligned}\frac{2^{\frac{n}{2}}(2n+1)^{\frac{n}{2}}}{\pi^{\frac{1}{4}}\sqrt{n!}}y^n e^{-\frac{(2n+1)}{2}y^2} &\underset{y\to+\infty}{\sim} \frac{C_n}{2\sqrt{\pi}(2n+1)^{\frac{1}{6}}\sqrt{y}}e^{-\frac{2}{3}(2n+1)\zeta(y)^{\frac{3}{2}}} \\ &\underset{y\to+\infty}{\sim} \frac{C_n}{2\sqrt{\pi}(2n+1)^{\frac{1}{6}}\sqrt{y}}e^{\frac{2n+1}{4}}(2y)^{\frac{2n+1}{2}}e^{-\frac{(2n+1)}{2}y^2}.\end{aligned}$$

The previous comparison leads to the following computation:

$$C_n = \pi^{\frac{1}{4}}\frac{(2n+1)^{\frac{n}{2}+\frac{1}{6}}e^{-\frac{2n+1}{4}}}{2^{\frac{n-1}{2}}\sqrt{n!}}.$$

In what follows, we require a two-term asymptotic expansion of the sequence (C_n) for $n \to +\infty$

$$C_n = \frac{\sqrt{2}}{(2n+1)^{\frac{1}{12}}} + \frac{\sqrt{2}}{24(2n+1)^{\frac{13}{12}}} + o\Big(\frac{1}{(2n+1)^{\frac{13}{12}}}\Big) \tag{47}$$

and also an approximation of $\frac{C_n}{\sqrt{\pi}(2n+1)^{\frac{1}{6}}}$. A numerical computation yields $\frac{\sqrt{2}}{24\sqrt{\pi}} < 0.1$ (at this point, we prefer a much larger bound to get a simple final constant 0.5 in (11)). Consequently, for $n \gg 1$, the following inequality holds:

$$\left|\frac{C_n}{\sqrt{\pi}(2n+1)^{\frac{1}{6}}} - \frac{\sqrt{2}}{\sqrt{\pi}(2n+1)^{\frac{1}{4}}}\right| \leq \frac{0.1}{(2n+1)^{\frac{5}{4}}}. \tag{48}$$

Conclusion. We now have all the ingredients to prove (40). We first exploit (48):

$$\left|h_n(y\sqrt{2n+1}) - \frac{\sqrt{2}}{\sqrt{\pi}(2n+1)^{\frac14}}\frac{\cos(g_n(y))}{(1-y^2)^{\frac14}}\right| \leq \left|h_n(y\sqrt{2n+1}) - \frac{C_n}{\sqrt{\pi}(2n+1)^{\frac16}}\frac{\cos(g_n(y))}{(1-y^2)^{\frac14}}\right| + \frac{0.1}{(2n+1)^{\frac54}(1-y^2)^{\frac14}}.$$

By the triangle inequality, the last upper bound can be majorized by

$$\left|h_n(y\sqrt{2n+1}) - C_n\Big(\frac{-\zeta(y)}{1-y^2}\Big)^{\frac14}\operatorname{Ai}((2n+1)^{\frac23}\zeta(y))\right| + \left|C_n\Big(\frac{-\zeta(y)}{1-y^2}\Big)^{\frac14}\operatorname{Ai}((2n+1)^{\frac23}\zeta(y)) - \frac{C_n}{\sqrt{\pi}(2n+1)^{\frac16}}\frac{\cos(g_n(y))}{(1-y^2)^{\frac14}}\right| + \frac{0.1}{(2n+1)^{\frac54}(1-y^2)^{\frac14}}.$$

We note that $g_n(y)$ equals $\frac23 X^{\frac32} - \frac{\pi}{4}$ with $X = -(2n+1)^{\frac23}\zeta(y)$ (see (10) and (42)). We may then use the Airy-function approximation (41) and (28) to bound the last term by

$$C_n\Big(\frac{-\zeta(y)}{1-y^2}\Big)^{\frac14}\big(|\varepsilon(n,y)| + |\widetilde{\varepsilon}(-(2n+1)^{\frac23}\zeta(y))|\big) + \frac{0.1}{(2n+1)^{\frac54}(1-y^2)^{\frac14}}.$$

Then (44) and the bound of the remainder in (28) give the following upper bound:

$$\frac{C_n\times 0.16}{(2n+1)^{\frac76}(1-y^2)^{\frac14}} + \frac{C_n\times 5}{(2n+1)^{\frac{14}{12}}\times 48\sqrt{\pi}(1-y^2)^{\frac14}(-\zeta(y))^{\frac64}} + \frac{0.1}{(2n+1)^{\frac54}(1-y^2)^{\frac14}}.$$

We are now in a position to conclude. We write

$$K = \frac{5\sqrt{2}}{24\sqrt{\pi}} + 2\delta \quad \text{with } \delta > 0$$

and we use (47) and (43) so that we obtain the following upper bound for $n \gg 1$:

$$\frac{\sqrt{2}\times 0.16+\delta}{(2n+1)^{\frac{15}{12}}(1-y^2)^{\frac14}} + \frac{\frac{\sqrt{2}\times 5}{48\sqrt{\pi}}\times 2+\delta}{(2n+1)^{\frac{15}{12}}(1-y^2)^{\frac74}} + \frac{0.1}{(2n+1)^{\frac54}(1-y^2)^{\frac14}} \tag{49}$$

which is less than

$$\frac{1}{(2n+1)^{\frac54}(1-y^2)^{\frac74}}\Big((0.33+\delta)(1-y^2)^{\frac64} + \frac{5\sqrt{2}}{24\sqrt{\pi}} + \delta\Big). \tag{50}$$

The term $(0.33+\delta)(1-y^2)^{\frac64}$ is the one we referred to as residual before the statement of Proposition 8. Such an upper bound finally proves Proposition 8 because

- the last factor is less than $K = \frac{5\sqrt{2}}{24\sqrt{\pi}} + 2\delta$ for $y \in [y_K, 1)$ for a suitable y_K near 1^-,
- one has $0.33 + \frac{5\sqrt{2}}{24\sqrt{\pi}} < 0.5$ and $(1-y^2)^{\frac64} \leq 1$. We deduce that, for $\delta > 0$ small enough, the factor in (50) is less than 0.5 uniformly for $y \in [0,1)$.

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