

# ALMOST GLOBAL EXISTENCE FOR HAMILTONIAN PDES ON COMPACT MANIFOLDS

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**ABSTRACT.** We prove an abstract result of almost global existence of small solutions to semi-linear Hamiltonian partial differential equations satisfying very weak non resonance conditions and natural multilinear estimates. Thanks to works by Delort–Szeftel, these assumptions turn out to typically hold for Hamiltonian PDEs on any smooth compact boundaryless Riemannian manifold. As a main application, we prove the almost global existence of small solutions to nonlinear Klein–Gordon equations on such manifolds: for almost all mass, any arbitrarily large  $r$  and sufficiently large  $s$ , solutions with initial data of sufficiently small size  $\varepsilon \ll 1$  in the Sobolev space  $H^s \times H^{s-1}$  exist and remain in  $H^s \times H^{s-1}$  for polynomial times  $|t| \leq \varepsilon^{-r}$ . This is the first result of almost global existence without specific assumptions on the compact manifold. We also apply this abstract result to nonlinear Schrödinger equations close to ground states and nonlinear Klein–Gordon equations on  $\mathbb{R}^d$  with positive quadratic potentials.

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## 1. INTRODUCTION

We consider the dynamics of small and smooth solutions to spatially confined nonlinear dispersive PDEs. We focus on models which, after diagonalizing the linear part can be rewritten as infinite systems of coupled harmonic oscillators of the form

$$(1) \quad i\partial_t u_j = \omega_j u_j + g_j(u), \quad j \in \mathbb{N}, \quad t \in \mathbb{R},$$

where  $\mathbb{N} = \{j \in \mathbb{Z} \mid j \geq 1\}$ ,  $u_j(t) \in \mathbb{C}$ ,  $\omega_j \in \mathbb{R}$  and  $g$  is at least of order  $\mathfrak{o} \geq 2$  at  $u = 0$  (at least formally). Since we are interested in PDEs, we assume the nonlinearity  $g$  to fulfill some good properties like tame estimates. Smoothness and size of solutions are measured in Sobolev norms defined by

$$h^s := \left\{ u \in \mathbb{C}^{\mathbb{N}} \mid \|u\|_{h^s}^2 := \sum_{j \in \mathbb{N}} j^{2s} |u_j|^2 < \infty \right\}, \quad s \in \mathbb{R}.$$

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Local well posedness of (1) usually implies that if  $s$  is large enough and  $\varepsilon := \|u(0)\|_{h^s}$  is small enough, then the solution exists for times of order  $\varepsilon^{-\sigma+1}$  (and remains of size  $\varepsilon$  in  $h^s$ ). For example, this property can be easily proven when  $g$  is smooth from  $h^s$  to  $h^s$  and satisfies  $\|g(u)\|_{h^s} = \mathcal{O}(\|u\|_{h^s}^\sigma)$ . This time scale is usually called linear because the solution remains close to the solution of the linear system up to this "local" time. For longer times, nonlinear effects may become predominant and the solution could blow up: its existence is no more ensured. This raises the question of the lifespan of small and smooth solutions.

Some nonlinear dispersive equations are globally well posed which completely solves the question (see e.g. [BGT04]). Nevertheless these results require the dimension (of the spacial domain) and/or the degree of the nonlinearity to be small enough. Here, we want to extend such results. To this end, we use a normal form approach to remove the nonlinear terms that could generate a growth of the  $h^s$  norms. To avoid resonances we need two extra assumptions on the system.

First, a geometric assumption on the system. Here, we focus on the Hamiltonian case, i.e. we assume the existence of a real valued function  $G \in C^\infty(h^s; \mathbb{R})$  for  $s$  is large enough such that for all  $j \in \mathbb{N}$

$$g_j = \partial_{\bar{u}_j} G.$$

Then, denoting by

$$G(u) = \sum_{q \geq 3} \sum_{j \in \mathbb{N}^q} \sum_{\sigma \in \{-1, 1\}^q} G_j^\sigma u_{j_1}^{\sigma_1} \cdots u_{j_q}^{\sigma_q}$$

the Taylor expansion of  $G$  in  $u = 0$  where  $G_j^\sigma \in \mathbb{C}$  and  $u_{j_i}^{-1} := \bar{u}_{j_i}$ , the point is to remove monomials  $u_{j_1}^{\sigma_1} \cdots u_{j_q}^{\sigma_q}$  which do not commute with a well chosen  $\tilde{h}^s$  norm equivalent to the  $h^s$  norm. This operation requires non resonance conditions on the frequencies. The two classical ones are

$$(NR_1) \quad |\sigma_1 \omega_{j_1} + \cdots + \sigma_q \omega_{j_q}| \geq C_q |\mathbf{j}_1^*|^{-a_q}$$

$$(NR_3) \quad |\sigma_1 \omega_{j_1} + \cdots + \sigma_q \omega_{j_q}| \geq C_q |\mathbf{j}_3^*|^{-a_q}$$

where  $C_q, a_q > 0$  are constants depending only on  $q$  and  $\mathbf{j}_1^*, \dots, \mathbf{j}_q^*$  denote the non-increasing arrangement of  $j_1, \dots, j_q$ .

For systems satisfying the non resonance condition (NR<sub>3</sub>), many results of *almost global existence and stability* have been proven. Mainly results of the following kind

**Meta-theorem 1.1** (Almost global existence and stability). *Assume (NR<sub>3</sub>). For all  $r \geq 1$  if  $u(0) \in h^s$  with  $s$  large enough and  $\|u(0)\|_{h^s} = \varepsilon$  small enough, then the solution  $u(t)$  exists in  $h^s$  for all time  $|t| \leq \varepsilon^{-r}$  and satisfies  $\|u(t)\|_{h^s} \lesssim_s \varepsilon$ .*

The first results in that direction was obtained by Bambusi in [Bam03] and generalized in Bambusi-Grébert [BG06]. Then this technics was applied in a many different contexts, see for instance [BDGS07, GIP09, FGL13, YZ14, Del15, BD17, FI21, BMM24, BG25]. We notice that in all the cited results the authors used external parameters to ensure that (NR<sub>3</sub>) holds true for almost all values of these external parameters (for instance the mass  $m$  in the case of the Klein–Gordon equation (KG)). Recently, but twenty years after a pioneering work by Bourgain [Bou00], similar results have been obtained without external parameters [BFG20a, BG21, BC24]. The bound  $\|u(t)\|_{h^s} \lesssim_s \varepsilon$  ensures that the zero solution is stable in  $h^s$  for times of order  $\varepsilon^{-r}$ , that is why we call it *almost global existence and stability*. Note that this question of stability is interesting in itself, even for globally well-posed equations (like in [Bou00, Bam03, GIP09, FGL13, YZ14, BFG20a, BG21]), but that it is strictly more specific than the one about the lifespan of the solutions.

When we only have condition  $(\text{NR}_1)$ , in general we can just extend a little the local time of existence and stability  $\varepsilon^{-\sigma+1}$ , obtaining typically a time of order  $\varepsilon^{-A(\sigma-1)}$  for some fixed  $A > 1$ , but not arbitrary large (see in particular [Del09, DI17, Brun23, FGI23]). What we claim in this paper is that, if we do not mind about the stability of the zero solution in  $h^s$  and just focus on the time of existence of the solutions, condition  $(\text{NR}_1)$ , complemented with multilinear estimates, is sufficient to obtain, for semi-linear equations, a result of the kind

**Meta-theorem 1.2** (Almost global existence). *Assume  $(\text{NR}_1)$ . For all  $r \geq 1$ , if  $u(0) \in h^s$  with  $s$  large enough and  $\|u(0)\|_{h^s} = \varepsilon$  small enough, then the solution  $u(t)$  exists in  $h^s$  for all time  $|t| \leq \varepsilon^{-r}$ .*

The precise abstract result is stated in Theorem 2.1 and the concrete applications are stated in Theorem 3.1 for Klein-Gordon equations on compact manifolds, Theorem 3.4 for nonlinear Schrödinger equations on compact manifolds close to ground states and Theorem 3.6 for nonlinear Klein-Gordon equations on  $\mathbb{R}^d$  with quadratic potentials. The important point is that condition  $(\text{NR}_1)$  is much weaker than  $(\text{NR}_3)$ . In particular, in the case of the Klein-Gordon equations, whereas  $(\text{NR}_3)$  has been proved to hold true for almost all value of the mass for some particular manifolds (Zoll manifolds) while  $(\text{NR}_1)$  holds true on any smooth compact boundaryless Riemannian manifolds ([DS04]).

The fact that  $(\text{NR}_1)$  is sufficient to ensure almost global existence was already proved in [BFG20b] for nonlinear Klein-Gordon equations on  $d$ -dimensional tori. Here we take advantage of this remark in a much more general situation.

Finally, we notice that, recently, using Bourgain's cluster decomposition, the almost global existence and stability of some  $(\text{NR}_1)$ -non-resonant Hamiltonian PDEs (not including Klein-Gordon) on compact Riemannian manifolds with globally integrable geodesic flow have been proved [BFM24, BFLM24] (see also [BL22]). The point is that thanks to the existence of a Bourgain's cluster decomposition these systems almost satisfy the  $(\text{NR}_3)$  non resonance condition (i.e. up to some quite trivial terms).

**Basic notations.** The Japanese bracket is defined by  $\langle a \rangle := \sqrt{1 + a^2}$  for  $a \in \mathbb{R}$ . We shall use the notation  $A \lesssim B$  to denote  $A \leq CB$  where  $C$  is a positive constant depending on parameters fixed once for all. We will emphasize by writing  $\lesssim_\alpha$  when the constant  $C$  depends on some other parameter  $\alpha$  and  $A \sim_\alpha B$  if  $A \lesssim_\alpha B \lesssim_\alpha A$ .

We rewrite as usual  $L^2 = \ell^2 := h^0$ . For all  $q \geq 3$  and  $\mathbf{k} \in \mathbb{C}^q$ ,  $\mathbf{k}_1^*, \dots, \mathbf{k}_q^*$  denotes the non-increasing arrangement of  $\mathbf{k}_1, \dots, \mathbf{k}_q$ . For all  $j \in \mathbb{N}$ , the operators  $\partial_{\bar{u}_j}, \partial_{u_j}$  are defined by  $2\partial_{\bar{u}_j} := \partial_{\Re u_j} + i\partial_{\Im u_j}$  and  $2\partial_{u_j} := \partial_{\Re u_j} - i\partial_{\Im u_j}$ . For all  $j \in \mathbb{N}$ ,  $\mathbf{1}_{\{j\}} \in \mathbb{C}^{\mathbb{N}}$  denotes the sequence such that  $(\mathbf{1}_{\{j\}})_i = 1$  if  $i = j$  and  $(\mathbf{1}_{\{j\}})_i = 0$  otherwise.

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## 2. ABSTRACT THEOREM

We give an abstract theorem to prove the *almost global existence* of small and smooth solutions to (1). First we present its setting and then we state the result. As we will see in the next section, these assumptions are natural for Hamiltonian PDEs on smooth compact boundaryless

Riemannian manifolds and more generally for Hamiltonian PDEs with a linear part which only has imaginary pure point spectrum.

Let  $\omega \in (\mathbb{R}_+^*)^{\mathbb{N}}$  be non-decreasing,  $s_0 \geq 0$ ,  $\mathcal{U} \subset h^{s_0}$  be an open neighborhood of the origin,  $G \in C^\infty(\mathcal{U}; \mathbb{R})$  be a function of order larger than or equal to 3 at the origin and set, for all  $j \in \mathbb{N}$ ,  $g_j := \partial_{u_j} G$  (so that the system is Hamiltonian).

Considering from now these objects as fixed, assume that

- *the nonlinearity is smooth, preserves the  $h^s$  regularity and is tame*:  $g$  is a  $C^\infty$  function from  $h^s \cap \mathcal{U}$  into  $h^s$  for all  $s \geq s_0$  and satisfying the following tame estimate

$$(2) \quad \forall u \in h^s \cap \mathcal{U}, \quad \|g(u)\|_{h^s} \lesssim_s \|u\|_{h^s}$$

- *Weyl law*: there exists  $\beta > 0$  such that provided that  $\lambda$  is large enough

$$(3) \quad \#\{j \in \mathbb{N} \mid \omega_j \leq \lambda\} \sim \lambda^\beta$$

- *Clustering*: there exist two positive numbers  $\alpha, \Upsilon > 0$  and a decomposition in disjoint subsets<sup>1</sup>

$$\mathbb{N} = \bigcup_{k \in \mathbb{N}} \mathcal{C}_k$$

such that

$$(4) \quad \sup_{k \in \mathbb{N}} \sup_{j \in \mathcal{C}_k} |\omega_j^{1/\alpha} - \Upsilon k| < \infty$$

and

- *the system is non resonant*: for all  $q \geq 1$ , there exists  $a_q > 0$  such that for  $\mathbf{j} \in \mathbb{N}^q$ ,  $\boldsymbol{\sigma} \in \{-1, 1\}^q$

$$(5) \quad \text{if } \exists k \in \mathbb{N}, \quad \sum_{i \text{ s.t. } j_i \in \mathcal{C}_k} \sigma_i \neq 0 \quad \text{then} \quad |\sigma_1 \omega_{j_1} + \cdots + \sigma_q \omega_{j_q}| \gtrsim_q |\mathbf{j}_1^*|^{-a_q},$$

- *the nonlinearity satisfies the following multilinear estimate*: there exists  $\nu \geq 0$  such that for all  $q \geq 3$ , all  $n \geq 0$ , all  $\mathbf{k} \in \mathbb{N}^q$  and all  $(u^{(\ell)})_{1 \leq \ell \leq q} \in \prod_{1 \leq \ell \leq q} E_{\mathbf{k}_\ell}$ ,

$$(6) \quad |d^q G(0)(u^{(1)}, \dots, u^{(q)})| \lesssim_{n,q} \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2^*}{\mathbf{k}_1^*} \right)^n \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_\ell^* \right)^\nu \prod_{\ell=1}^q \|u^{(\ell)}\|_{\ell^2}$$

where for all  $k \in \mathbb{N}$ ,  $E_k := \text{Span}_{\mathbb{C}}\{\mathbf{1}_{\{j\}} \mid j \in \mathcal{C}_k\} \subset \ell^2 := h^0$  and

$$(7) \quad \Gamma_{\mathbf{k}} := \sum_{\boldsymbol{\varsigma} \in \{-1, 1\}^q} \langle \boldsymbol{\varsigma}_1 \mathbf{k}_1 + \cdots + \boldsymbol{\varsigma}_q \mathbf{k}_q \rangle^{-3}.$$

**Theorem 2.1.** *For all  $r \geq 1$ ,  $s \gtrsim_r 1$ ,  $u^{(0)} \in h^s$  such that  $\varepsilon := \|u^{(0)}\|_{h^s} \lesssim_{r,s} 1$ , there exists a unique solution*

$$u \in C^0((-\varepsilon^{-r}, \varepsilon^{-r}); h^s) \cap C^1((-\varepsilon^{-r}, \varepsilon^{-r}); h^{s-\frac{1}{\beta}})$$

to (1) with initial datum  $u(0) = u^{(0)}$ . Furthermore, as long as  $|t| \leq \varepsilon^{-r}$ , one has  $\|u(t)\|_{h^{s_0}} \lesssim \varepsilon$ .

Some comments about the result:

- The bound  $s \gtrsim_r 1$  could be specified: the constant only depend on  $r, \alpha, \beta, \nu, a, s_0$ .
- We can choose  $s_0$  as large as we want but this has an incidence on the size of  $s$ .

<sup>1</sup>some of them may be empty.

- For  $s > s_0$ , the only control we have on the  $h^s$  norm of the solution comes from the tame estimate (2) and is of the form  $\|u(t)\|_{h^s} \leq \exp(C_s t)\varepsilon$  where  $C_s > 0$  is a constant depending only on  $s$ . In particular the  $h^s$  norm of  $u$  may become very large. In other words, our result does not exclude the existence of forward energy cascades.

Some comments about the assumptions:

- In order to prove the multilinear estimate (6) typically one uses that the operator  $\text{diag}(\omega_j^{1/\alpha})$  is pseudodifferential of order 1. That is why we introduced the exponent  $\alpha$ .
- By the Weyl law and the bound (4), one has

$$\# \bigcup_{k \leq \lambda} \mathcal{C}_k \sim \lambda^{\alpha\beta}.$$

- The non resonance condition (5) allows to remove monomials which do not commute with the super-actions

$$(8) \quad J_k(u) := \sum_{j \in \mathcal{C}_k} |u_j|^2.$$

This is the non resonance condition (NR<sub>1</sub>) of the introduction, in the sense that these monomials are exactly those do not commute with the  $H^s$  norms defined in Definition 5.5 below.

- The multilinear estimates (6) are new but they are implied by the multilinear estimates of the type of those proved by Delort–Szeftel in [DS06], namely

$$(9) \quad |d^q G(0)(u^{(1)}, \dots, u^{(q)})| \lesssim_{n,q} \frac{(\mathbf{k}_3^*)^{\nu+n} (\mathbf{k}_4^*)^\nu \dots (\mathbf{k}_q^*)^\nu}{(\mathbf{k}_1^* - \mathbf{k}_2^* + \mathbf{k}_3^*)^n} \prod_{\ell=1}^q \|u^{(\ell)}\|_{\ell^2}.$$

See Lemma A.7 for the fact that (9) implies (6).

- If the nonlinearity  $g$  is polynomial then the tame estimate (2) is a consequence of the multilinear estimates (6) (see Lemma 5.16 below for a proof).
- The following equivalence holds true

$$\Gamma_{\mathbf{k}} \sim_q \left(1 + \min_{\varsigma \in \{-1,1\}^q} |\varsigma_1 \mathbf{k}_1 + \dots + \varsigma_q \mathbf{k}_q|^3\right)^{-1}.$$

- the term  $\Upsilon_{\mathbf{k}}$  in (4) could be replaced by  $b_{\mathbf{k}}$ , where  $b \in (\mathbb{R}_+^*)^{\mathbb{N}}$  would be an increasing sequence satisfying  $b_{\mathbf{k}+1} - b_{\mathbf{k}} \sim 1$  (i.e. uniformly in  $\mathbf{k}$ ).

Finally, let us comment some technical novelties of this paper. An important part of the work consists of combining the techniques developed in [DS04, DS06, BFG20b]. Nevertheless, it is not direct. In particular, we point out two crucial points.

- The method introduced in [BFG20b] allows to prove the existence for times of order  $\varepsilon^{-cr_b/s_b}$  where  $c > 0$  is a universal constant,  $r_b$  is the number of Birkhoff normal form steps we perform and  $s_b$  is the minimal regularity for which we have tame estimates at the end of the Birkhoff normal form process (see e.g. equation (47) below). Basing our construction on Delort–Szeftel multilinear estimates (9),  $s_b$  would grow linearly with respect to  $r$  (see the exponent  $\nu$  in [DS06, Theorem 2.14]) and we could not conclude the almost global existence of the solutions. The point is that these estimates contain too much information and so are too costly to propagate in the normal form process. A natural alternative would have been the tame-modulus estimates of [BG06] but they are too weak: they do not allow to gain derivatives when computing brackets with  $h^s$  norms (which is a crucial property for our proof, see Proposition 6.8). That is why we proposed the new multilinear estimates (6) which are in between and overcome these obstructions.

- In [BFG20b], the solutions were estimated in mixed norms:  $h^{s_0}$  for high modes and  $h^s$  with  $s \gg r$  for low modes. Nevertheless, this disjunction does not allow for satisfactory treatment of remainder terms and generates technicalities. Here we simplified the approach by removing the mixed norms: the solution is just estimated in  $h^{s_0}$  norm.

### 3. APPLICATIONS

Now we present applications of this abstract result to two emblematic Hamiltonian PDEs on an arbitrary boundaryless smooth Riemannian manifold: the nonlinear Klein–Gordon equations and the nonlinear Schrödinger equations close to ground states. As in [BFM24, BFLM24], the result could be applied to other classical semi-linear equations like the beam equation or the nonlinear Schrödinger equations close to the origin with a spectral multiplier. We chose the nonlinear Klein–Gordon equations because they are the most emblematic ones and the nonlinear Schrödinger equations to emphasize that we do not need the nonlinearity to be smoothing. We also provide an application to nonlinear Klein–Gordon equations on  $\mathbb{R}^d$  with positive definite quadratic potentials to emphasize that our abstract theorem can even be applied beyond the framework of Hamiltonian PDEs on compact manifolds. Proofs are given in Section 4 below.

**3.1. Nonlinear Klein–Gordon equations on compact manifolds.** We consider nonlinear Klein–Gordon equations of the form

$$(KG) \quad \partial_t^2 \Psi(t, x) = (\Delta - m)\Psi(t, x) - V(x)\Psi(t, x) + f(x, \Psi(t, x)), \quad t \in \mathbb{R}_+^*, x \in \mathcal{M}$$

where  $\Psi(t, x) \in \mathbb{R}$ ,  $\mathcal{M}$  is a smooth compact boundaryless Riemannian manifold of dimension  $d \geq 1$ ,  $V \in C^\infty(\mathcal{M}; \mathbb{R}_+)$ ,  $m > 0$  is a parameter called *mass* and  $f \in C^\infty(\mathcal{M} \times \mathbb{R}; \mathbb{R})$  satisfies  $f(\cdot, 0) = \partial_\Psi f(\cdot, 0) = 0$  (to ensure that it is of order at least 2 with respect to  $\Psi$ ).

**Theorem 3.1.** *Fix  $s_1 \geq 0$ . For almost all  $m > 0$ , all  $r \geq 1$ , all  $s \gtrsim_{r,m} s_1$ , and any couple of real-valued functions  $(\Psi_0, \Phi_0) \in H^{s+1}(\mathcal{M}; \mathbb{R}) \times H^s(\mathcal{M}; \mathbb{R})$  such that  $\varepsilon := \|\Psi_0\|_{H^{s+1}} + \|\Phi_0\|_{H^s} \lesssim_{r,s,m} 1$ , there exists a unique solution*

$$\Psi \in C^0((-\varepsilon^{-r}, \varepsilon^{-r}); H^{s+1}(\mathcal{M}; \mathbb{R})) \cap C^1((-\varepsilon^{-r}, \varepsilon^{-r}); H^s(\mathcal{M}; \mathbb{R}))$$

to (KG) with initial datum  $\Psi(0) = \Psi_0$ ,  $\partial_t \Psi(0) = \Phi_0$ . Furthermore, as long as  $|t| \leq \varepsilon^{-r}$ , one has  $\|\Psi(t)\|_{H^{s+1}} + \|\partial_t \Psi(t)\|_{H^{s_1}} \lesssim \varepsilon$ .

**Remark 3.2.** *We recall that the Sobolev spaces  $H^s(\mathcal{M}; \mathbb{K})$ ,  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  and  $s \geq 0$ , are defined, as usual, by*

$$(10) \quad H^s(\mathcal{M}; \mathbb{K}) := \left\{ u \in L^2(\mathcal{M}; \mathbb{K}) \mid \|u\|_{H^s} := \|(1 - \Delta)^{s/2} u\|_{L^2} < \infty \right\}.$$

To insist on the fact that our aim is to obtain almost global existence without trying to control the solution in high Sobolev norms we state the following Corollary

**Corollary 3.3.** *For almost all  $m > 0$ , any couple of real-valued functions  $\Psi_0, \Phi_0 \in C^\infty(\mathcal{M}; \mathbb{R})$  and any  $\varepsilon \in (0, 1)$ , there exists a unique solution  $\Psi^{(\varepsilon)} \in C^\infty((-T_\varepsilon, T_\varepsilon) \times \mathcal{M}; \mathbb{R})$  to (KG) with initial datum  $\Psi^{(\varepsilon)}(0) = \varepsilon \Psi_0$ ,  $\partial_t \Psi^{(\varepsilon)}(0) = \varepsilon \Phi_0$  and*

$$\forall r \geq 3, \quad \lim_{\varepsilon \rightarrow 0} \frac{T_\varepsilon}{\varepsilon^{-r}} = +\infty.$$

**3.2. Nonlinear Schrödinger equations on compact manifolds.** We consider nonlinear Schrödinger equations of the form

$$(NLS) \quad i\partial_t z(t, x) = -\Delta z(t, x) + f(|z(t, x)|^2)z(t, x) \quad t \in \mathbb{R}, x \in \mathcal{M}$$

where  $z(t, x) \in \mathbb{C}$ ,  $\mathcal{M}$  is a smooth compact boundaryless connected Riemannian manifold of dimension  $d \geq 2$  and  $f \in C^\infty(\mathbb{R}; \mathbb{R})$  satisfies  $f(0) = 0$ .

It is immediate to verify that  $z_*(t) := \sqrt{p_0}e^{-i\nu t}$  is a solution of (NLS) if and only if  $\nu = f(p_0)$ . To ensure linear stability of this solution, we denote by  $\lambda_2^2 > 0$  the second smallest eigenvalue (with multiplicity) of  $-\Delta$  on  $\mathcal{M}$  and only consider values of  $p_0 > 0$  such that  $\lambda_2^2 + 2p_0 f'(p_0) > 0$ .

**Theorem 3.4.** *Fix  $s_1 \geq 0$ . There exists a zero measure set  $\mathcal{N} \subset \mathbb{R}$  such that for all  $p_0 > 0$  satisfying  $2p_0 f'(p_0) \in (-\lambda_2^2, +\infty) \setminus \mathcal{N}$ , all  $r \geq 1$ , all  $s \gtrsim_{r, p_0} s_1$ , and any functions  $z_0 \in H^s(\mathcal{M})$  such that*

$$\|z_0\|_{L^2}^2 = p_0, \quad \inf_{\theta \in \mathbb{T}} \|z_0 - \sqrt{p_0}e^{-i\theta}\|_{H^s} =: \varepsilon \lesssim_{r, s, p_0} 1$$

there exists a unique solution

$$z \in C^0((-\varepsilon^{-r}, \varepsilon^{-r}); H^s(\mathcal{M})) \cap C^1((-\varepsilon^{-r}, \varepsilon^{-r}); H^{s-2}(\mathcal{M}))$$

to (NLS) with initial datum  $z(0) = z_0$ . Furthermore, as long as  $|t| \leq \varepsilon^{-r}$ , one has

$$\inf_{\theta \in \mathbb{T}} \|z(t) - \sqrt{p_0}e^{-i\theta}\|_{H^{s_1}} \lesssim \varepsilon$$

**3.3. Nonlinear Klein–Gordon equations on  $\mathbb{R}^d$  with positive definite quadratic potential.** Here, we consider the following Klein–Gordon equation:

$$(11) \quad \partial_t^2 \Psi(t, x) = (\Delta - m)\Psi(t, x) - Q(x)\Psi(t, x) + f(\Psi(t, x)), \quad t \in \mathbb{R}, x \in \mathbb{R}^d$$

in which  $m > 0$ ,  $f \in C^\infty(\mathbb{R}; \mathbb{R})$  satisfies  $f(0) = f'(0) = 0$  and  $Q : \mathbb{R}^d \rightarrow \mathbb{R}$  is a positive definite quadratic form:

$$(12) \quad Q(x) = \sum_{i=1}^d \sum_{j=1}^d q_{ij} x_i x_j, \quad \text{with } q_{ij} = q_{ji} \quad \text{and} \quad Q(x) > 0 \quad \forall x \in \mathbb{R}^d \setminus \{0\}.$$

**Remark 3.5.** *Denoting by  $\varrho_d \geq \dots \geq \varrho_1 > 0$  the eigenvalues of the quadratic form  $Q$  one has that the spectrum of the quantum operator  $-\Delta + Q(x)$  is exactly*

$$\sum_{i=1}^d \sqrt{\varrho_i} (2\mathbb{N} + 1).$$

*In particular one has that if the numbers  $\sqrt{\varrho_i}$  are independent over the rational then their differences are dense on the real axis. The same is true for the eigenvalues of  $\sqrt{-\Delta + Q}$  and, generally speaking also those of  $\sqrt{-\Delta + Q + m}$  which are the frequencies in our problem.*

As we shall see in subsection 4.3, the adequate Sobolev space is as follows:

$$(13) \quad \mathcal{H}^s(\mathbb{R}^d) := \left\{ u \in H^s(\mathbb{R}^d) \mid \langle x \rangle^s u \in L^2(\mathbb{R}^d) \right\}.$$

We have exactly the same statement as Theorem 3.1:

**Theorem 3.6.** *Fix  $s_1 \geq 0$ . For almost all  $m > 0$ , all  $r \geq 1$ , all  $s \gtrsim_{r, m} s_1$ , and any couple of real-valued functions  $(\Psi_0, \Phi_0) \in \mathcal{H}^{s+1}(\mathbb{R}^d) \times \mathcal{H}^s(\mathbb{R}^d)$  such that  $\varepsilon := \|\Psi_0\|_{\mathcal{H}^{s+1}(\mathbb{R}^d)} + \|\Phi_0\|_{\mathcal{H}^s(\mathbb{R}^d)} \lesssim_{r, s, m} 1$ , there exists a unique solution*

$$\Psi \in C^0((-\varepsilon^{-r}, \varepsilon^{-r}); \mathcal{H}^{s+1}(\mathbb{R}^d)) \cap C^1((-\varepsilon^{-r}, \varepsilon^{-r}); \mathcal{H}^s(\mathbb{R}^d))$$

to (11) with initial datum  $\Psi(0) = \Psi_0$ ,  $\partial_t \Psi(0) = \Phi_0$ . Furthermore, as long as  $|t| \leq \varepsilon^{-r}$ , one has  $\|\Psi(t)\|_{\mathcal{H}^{s_1+1}(\mathbb{R}^d)} + \|\partial_t \Psi(t)\|_{\mathcal{H}^{s_1}(\mathbb{R}^d)} \lesssim \varepsilon$ .

#### 4. PROOFS OF THE APPLICATIONS

**4.1. Klein–Gordon on compact Riemannian manifold.** Let  $\mathcal{M}$  be a smooth compact boundaryless Riemannian manifold of dimension  $d \geq 1$ ,  $V \in C^\infty(\mathcal{M}; \mathbb{R}_+)$  and  $s_1 > d/2$ . In order to prove Theorem 3.1, we have first to explain how (KG) rewrites in the framework of Theorem 2.1 and then why it satisfies its assumptions.

*Step 1: Equivalence of the formalisms.* First we recall that the spectrum of the operator  $-\Delta + V$  acting on  $L^2(\mathcal{M}; \mathbb{R})$  is pure point. We denote by  $(\lambda_j^2)_{j \geq 1}$  the nondecreasing sequence of its eigenvalues (with multiplicities) and  $(e_j)_{j \in \mathbb{N}}$  an associated real Hilbertian basis. The eigenvalues satisfy the Weyl law [Hor68]

$$(14) \quad \forall y \gg 1, \quad \#\{j \in \mathbb{N} \mid \lambda_j \leq y\} \sim y^d.$$

We identify any function  $u \in L^2(\mathcal{M}; \mathbb{C})$  with its associated sequence of coefficients  $(u_j)_{j \in \mathbb{N}} \in \ell^2$  in this basis, i.e. such that

$$(u_j)_{j \in \mathbb{N}} \equiv u =: \sum_{j \in \mathbb{N}} u_j e_j.$$

Then, we prove in the following lemma that the standard Sobolev space  $H^s(\mathcal{M}; \mathbb{K})$  (defined by (10)) we used to state Theorem 3.1 are equivalent to the discrete Sobolev spaces  $h^s$  we used to state our abstract theorem.

**Lemma 4.1.** *For all  $s \geq 0$  and all  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , we have that*

$$H^s(\mathcal{M}; \mathbb{K}) = h^{s/d} \cap \mathbb{K}^{\mathbb{N}} \quad \text{and} \quad \|\cdot\|_{H^s} \sim_{s,V} \|\cdot\|_{h^{s/d}}.$$

*Proof.* First, we note that since  $V$  is bounded and nonnegative, for all  $u \in C^\infty(\mathcal{M}; \mathbb{C})$

$$\|u\|_{H^s}^2 = \|(\text{Id} - \Delta)^{s/2} u\|_{L^2}^2 \sim_{s,V} \|(\text{Id} - \Delta + V)^{s/2} u\|_{L^2}^2 = \sum_{j \in \mathbb{N}} \langle \lambda_j \rangle^{2s} |u_j|^2.$$

Since by the Weyl law, we have  $\langle \lambda_j \rangle \sim j^{1/d}$  (it suffices to apply (14) with  $y = \lambda_j$  and  $y = \lambda_j^-$ ), we deduce that  $\|u\|_{H^s} \sim_{s,V} \|u\|_{h^{s/d}}$ . The rest of the proof follows directly by density.  $\square$

Therefore, by setting, for all  $m > 0$ ,

$$u = \Lambda_m^{1/2} \Psi + i \Lambda_m^{-1/2} \partial_t \Psi,$$

where  $\Lambda_m$  is the spectral multiplier defined by

$$\Lambda_m = \sqrt{-\Delta + V + m},$$

we deduce that  $(\Psi, \partial_t \Psi)$  is solution of (KG) in  $H^{s+1}(\mathcal{M}; \mathbb{R}) \times H^s(\mathcal{M}; \mathbb{R})$  if and only if  $u$  is solution of (1) in  $H^{s+1/2}(\mathcal{M}; \mathbb{C})$  with

$$(15) \quad \forall j \in \mathbb{N}, \quad \omega_j := \sqrt{\lambda_j^2 + m} \quad \text{and} \quad g(u) := \Lambda_m^{-1/2} f\left(x, \Lambda_m^{-1/2} \left(\frac{u + \bar{u}}{2}\right)\right).$$

Finally, we set naturally,  $s_0 := s_1/d$  so that  $H^{s_1}(\mathcal{M}; \mathbb{C}) = h^{s_0}$ .

*Step 2: Validity of the assumptions.* For simplicity, we choose  $\mathcal{U} = B_{H^{s_1}}(0, 1)$ . The fact that  $g$  is smooth from  $\overline{H^s \cap \mathcal{U}}$  to  $\overline{H^s}$  for all  $s \geq s_1$  is just a consequence of the algebra property of the Sobolev spaces (because we chose  $s_1 > d/2$ ). Due to (15), the tame estimate (2) will be a consequence of the following one

$$(16) \quad \forall s > d/2, \forall \Psi \in H^s(\mathcal{M}; \mathbb{R}) \quad \|f(\cdot, \Psi)\|_{H^s(\mathcal{M})} \lesssim_{s, \|\Psi\|_{L^\infty}} \|\Psi\|_{H^s(\mathcal{M})}$$



where  $f$  is given in (KG) and we thus refer to [BCD11, Thm 2.87 page 104] for a proof. Finally, for the Hamiltonian structure, it suffices to note that for all  $j \in \mathbb{N}$ ,  $g_j = \partial_{u_j} G$  where

$$G(u) := \int_{\mathcal{M}} F\left(x, \Lambda_m^{-1/2} \frac{u(x) + \overline{u(x)}}{2}\right) dx$$

where  $F \in C^\infty(\mathbb{R} \times \mathbb{R}; \mathbb{R})$  is the function satisfying  $F(\cdot, 0) = 0$  and  $\partial_\Psi F = f$ .

Recalling that  $(\lambda_j)_{j \in \mathbb{N}}$  are the eigenvalues of the operator  $-\Delta + V$ , the estimate (3) with  $\beta := d$  is a direct consequence of the Weyl law (14) for this operator.

Finally, it only remains to construct the clustering and to check the associated properties. The construction relies on the following lemma.

**Lemma 4.2.** *There exists an increasing sequence  $(c_k)_{k \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$  such that*

$$\sigma(\sqrt{-\Delta + V}) = \{\lambda_j \mid j \in \mathbb{N}\} \subset \bigcup_{k \in \mathbb{N}} [c_{2k-1}, c_{2k}]$$

and

$$(17) \quad \forall k \in \mathbb{N}, \quad k^{-d} \lesssim |c_{2k+1} - c_{2k}| \quad \text{and} \quad |c_{2k} - k| + |c_{2k} - c_{2k-1}| \lesssim 1.$$

*Proof.* Recalling that by the Weyl law there exists  $C \in \mathbb{N}$  such that for all  $k \geq 1$

$$\#\{j \in \mathbb{N} \mid \lambda_j \leq k\} \leq Ck^d$$

Now split the interval  $[k - \frac{1}{2}, k]$  into  $2Ck^d$  subintervals of the same length (that is  $\frac{1}{4Ck^d}$ ) and of the form  $[k - \frac{\ell-1}{4Ck^d}, k - \frac{\ell}{4Ck^d})$  with  $\ell$  being an integer. Due to the last inequality, there is at least one subinterval that contains no eigenvalue and we may define it to be at a number  $\ell = \ell_k$ . Thus, it suffices to set  $c_1 = 0$  and for all  $k \in \mathbb{N}$

$$c_{2k} := k - \frac{\ell_k - 1}{4Ck^d} \quad \text{and} \quad c_{2k+1} := k - \frac{\ell_k}{4Ck^d}.$$

In particular, we get the inequalities  $\frac{1}{2} \leq c_{2k+2} - c_{2k+1} \leq \frac{3}{2}$  as illustrated in the figure below:



□

Then we set

$$\mathcal{C}_k = \{j \in \mathbb{N} \mid c_{2k-1} \leq \lambda_j \leq c_{2k}\}.$$

and, recalling that  $\omega_j = \sqrt{\lambda_j^2 + m}$ , we note that thanks to the second estimate of (17), the bound (4) on the frequencies and the clusters holds with  $\alpha = \Upsilon = 1$ . The non-resonance estimate (5), was proved in [DS04] (see also [DI17]). Here we do not repeat the proof, we just emphasize that it is based on the fact that the frequencies are analytic functions of  $m$  and of the eigenvalues  $\lambda_j$  and furthermore the intervals actually containing frequencies are separated among them by a quantity that goes to zero not faster than  $k^{-n}$  with some  $n$  (given by (17) here).

The proof of the multilinear estimate (6) is done in the appendix. The proof consists of first proving an estimate of the kind of (9) and then deducing the estimate (6). We emphasize that the estimate (9) was essentially proved in [DS06]. Here we give a slightly simplified version of its proof which does not make use of the Helffer–Sjöstrand formula.

Therefore all the hypothesis of Theorem 2.1 are satisfied and, when we apply it, we obtain Theorem 3.1.

**4.2. Schrödinger.** Let  $\mathcal{M}$  be a smooth compact connected boundaryless Riemannian manifold of dimension  $d \geq 1$  and  $s_1 > d/2$ . In order to prove Theorem 3.4, we have first to explain how, thanks to the Gauge symmetry, (NLS) rewrites in the framework of Theorem 2.1 and then why it satisfies its assumptions. These explanations are similar to those given in [FGL13, BFLM24]. To avoid normalization constants, we assume without loss of generality that

$$\text{Vol}(\mathcal{M}) := \int_{\mathcal{M}} 1 \, dx = 1.$$

*Step 1: Formalism.* As previously, we denote by  $(\lambda_j^2)_{j \in \mathbb{N}}$  the non-decreasing sequence of eigenvalues of the Laplace–Beltrami operator  $-\Delta$  acting on  $L^2(\mathcal{M}; \mathbb{R})$  and by  $(e_j)_{j \in \mathbb{N}}$  an associated real Hilbertian basis such that  $e_1 = 1$ . We denote by  $L_0^2(\mathcal{M}; \mathbb{C})$  the space of the zero integral functions in  $L^2$ , i.e.

$$L_0^2(\mathcal{M}; \mathbb{C}) := \left\{ v \in L^2(\mathcal{M}; \mathbb{C}) \mid \int_{\mathcal{M}} v(x) \, dx = 0 \right\}.$$

We identify any function  $u \in L_0^2(\mathcal{M}; \mathbb{C})$  with its sequence of coefficients  $(u_j)_{j \in \mathbb{N}} \in \ell^2$ , i.e.

$$u = \sum_{j \in \mathbb{N}} u_j e_{j+1}.$$

It follows that by Lemma 4.1, we have

$$\forall s \geq 0, \quad H_0^s(\mathcal{M}; \mathbb{C}) := H^s(\mathcal{M}; \mathbb{C}) \cap L_0^2(\mathcal{M}; \mathbb{C}) \equiv h^{s/d}$$

and that the associated norms are equivalent.

Now, we summarize in the following proposition how the construction of Faou–Glauckler–Lubich in [FGL13] (which is also recalled in [BFLM24]) reduces the analysis of (NLS) close to a plane wave to the analysis of an equation of the form (1).

**Proposition 4.3** (Faou–Glauckler–Lubich [FGL13]). *For all  $p_0 > 0$  satisfying  $2p_0 f'(p_0) > \lambda_2^2$ , there exist a function  $K \in C^\infty(\Omega; \mathbb{R})$  defined on an open neighborhood  $\Omega$  of the origin in  $\mathbb{C} \times \mathbb{R}$  and a  $\mathbb{R}$ -linear isomorphism<sup>2</sup>  $S : \ell^2 \rightarrow \ell^2$  which is diagonal in the sense that*

$$(18) \quad \forall i \geq 2, \quad S C e_i = C e_i$$

such that setting, for all  $j \geq 1$ ,

- $\omega_j := \sqrt{\lambda_{j+1}^4 + 2p_0 f'(p_0) \lambda_{j+1}^2}$
- $g_j := \partial_{\bar{u}_j} G$  where  $G$  denotes the function defined on a neighborhood of the origin in  $H_0^{s_1}$  by

$$(19) \quad G(u) = \int_{\mathcal{M}} K(Su(x), \|Su\|_{L^2}^2) \, dx,$$

we have the following property. For any  $T > 0$  and any solution  $u \in C^0([0, T]; H_0^{s_1})$  to (1), there exists a continuous function  $\theta \in C^0([0, T]; \mathbb{R})$  such that

$$(20) \quad z := e^{-i\theta} \left( \sqrt{p_0 - \|Su\|_{L^2}^2} + Su \right) \in C^0([0, T]; H^{s_1}) \quad \text{is a solution to (NLS).}$$

<sup>2</sup>which is also actually a symplectomorphism.

**Remark 4.4.** *The proof is fully constructive. In particular there are explicit formulas for  $K, S$  and  $\theta$ . However, they are not useful for us. To explain why  $S$  enjoys these good properties let us just mention that it satisfies*

$$\forall i \geq 2, \forall a, b \in \mathbb{R}, \quad S(a + ib)e_i = (\alpha_i a + i\alpha_i^{-1}b)e_i \quad \text{where } \alpha_i \sim 1.$$

*The proof in [FGL13] is done in  $\mathbb{T}^d$  with a cubic nonlinearity but, as explained in [BFLM24], there is no difficulty to extend it in our setting. The key point is to note that the map  $(p, \theta, u) \mapsto z$  defined by (20) is symplectic (in some classical sense not recalled here).*

**Remark 4.5.** *Since  $S$  is diagonal, it is also an isomorphism from  $H_0^s$  to  $H_0^s$  for all  $s \in \mathbb{R}$ .*

**Remark 4.6.** *Let us note that  $G$  is a well defined smooth function on a bounded neighborhood of the origin in  $H_0^{s_1}$  and so that  $g$  is well defined. Indeed, it is a consequence of the fact that  $S$  is continuous on  $H^{s_1}$  and of the algebra property of  $H^{s_1}$  (because  $s_1 > d/2$ ).*

Then it suffices to note that if  $u$  is small enough in  $L^2$  and  $z$  is given by (20) then for all  $s \geq 0$  we have

$$\|u\|_{H^s} \sim_{s, p_0} \inf_{\varphi \in \mathbb{T}} \|e^{i\varphi} \sqrt{p_0} - z\|_{H^s}$$

to deduce that if (1) is almost globally well posed (in the sense that Theorem 2.1 applies) then Theorem 3.4 holds.

*Step 2: Validity of the assumptions.* Most of the discussions are the same as the ones we presented for Klein–Gordon, so we just explain briefly the constructions. The open set  $\mathcal{U}$  can be chosen as a centered open ball in  $H_0^{s_1}$  of sufficiently small radius. The fact that  $g$  is smooth from  $H_0^s \cap \mathcal{U}$  to  $H_0^s$  for all  $s \geq s_1$  is just a consequence of the algebra property of the Sobolev spaces and of the continuity of  $S$  in  $H_0^s$ . As previously, the tame estimate (2) relies on para-differential calculus techniques and we refer to [BCD11, Thm 2.87 page 104] for a proof. Recalling that  $(\lambda_j)_{j \in \mathbb{N}}$  are the eigenvalues of the operator  $-\Delta$ , the estimate (3) with  $\beta := d/2$  is a direct consequence of the Weyl law (14) for this operator.

Now we focus on the clustering property. Let  $(c_k)_{k \geq 1}$  be the sequence given by Lemma 4.2 in the case  $V = 0$ . As previously we define the cluster by

$$\mathcal{C}_k = \{j \in \mathbb{N} \mid c_{2k-1} \leq \lambda_{j+1} \leq c_{2k}\}.$$

We note that by construction of the sequence  $(c_k)_k$  the cluster satisfies the bound (4) with  $\alpha := 2$  and  $\Upsilon = 1$ . The nonresonance condition (5) was proved (following [DS06]) in [BFLM24], see Section 7.2 and in particular Lemma 7.5 (using  $2p_0 f'(p_0)$  as parameter).

We come to the multilinear estimate (6). We have to prove that if one considers the Taylor expansion of a functions of the form (19), each Taylor polynomial fulfills the estimate (6). The spectral multiplier  $S$  being diagonal (in the sense of (18)) and bounded it can be ignored in this discussion. The proof is almost the same as for Klein–Gordon. The only novelty is the extra dependence of  $G$  with respect to the  $L^2$  norm. We prove in the following lemma that this dependence is not an obstruction.

**Lemma 4.7.** *Let  $P : \ell^2 \rightarrow \mathbb{R}$  be a locally bounded  $\mathbb{R}$ –homogeneous polynomial of degree  $q \geq 3$  fulfilling the estimate (6). Then  $Q := P \cdot \|\cdot\|_{L^2}^2$  fulfills the estimate (6).*

*Proof.* The function  $Q$  being a homogeneous polynomial of degree  $q + 2$  it suffices to consider  $d^{q+2}Q(0)$  which is equal to

$$d^{q+2}Q(0)(v^{(1)}, \dots, v^{(q+2)}) = \sum_{\tau \in \mathfrak{S}_{q+2}} d^q P(0)(v^{(\tau(1))}, \dots, v^{(\tau(q))}) \int_{\mathcal{M}} v^{(\tau(q+1))} \bar{v}^{(\tau(q+2))} dx$$

with  $v^{(1)}, \dots, v^{(q+2)} \in \ell^2$  and  $\mathfrak{S}_{q+2}$  the group of the permutations of the first  $q+2$  integers. We analyze only the identical permutation, the other being equal.

Denote  $(\mathbf{h}_1, \dots, \mathbf{h}_q) := (\mathbf{k}_1, \dots, \mathbf{k}_q)$ , then, if  $v^{(j)} \in E_{\mathbf{k}_j}$  for all  $j \in \{1, \dots, q+2\}$  one has

$$\left| d^q P(0)(v^{(1)}, \dots, v^{(q)}) \int_{\mathcal{M}} v^{(q+1)} \bar{v}^{(q+2)} dx \right| \lesssim \mathbb{1}_{\mathbf{k}_{q+1}=\mathbf{k}_{q+2}} \Gamma_{\mathbf{h}} \left( \frac{\mathbf{h}_2^*}{\mathbf{h}_1^*} \right)^n \prod_{l=3}^q (\mathbf{h}_l^*)^\nu \prod_{\ell=1}^{q+2} \|v^{(\ell)}\|_{\ell^2},$$

so it enough to show that

$$\mathbb{1}_{\mathbf{k}_{q+1}=\mathbf{k}_{q+2}} \Gamma_{\mathbf{h}} \left( \frac{\mathbf{h}_2^*}{\mathbf{h}_1^*} \right)^n \prod_{l=3}^q (\mathbf{h}_l^*)^\nu \lesssim \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2^*}{\mathbf{k}_1^*} \right)^n \prod_{l=3}^{q+2} (\mathbf{k}_l^*)^\nu.$$

In turn, since  $\Gamma_{\mathbf{k}} \geq \Gamma_{\mathbf{h}}$  this is implied by

$$(21) \quad \mathbb{1}_{\mathbf{k}_{q+1}=\mathbf{k}_{q+2}} \left( \frac{\mathbf{h}_2^*}{\mathbf{h}_1^*} \right)^n \prod_{l=3}^q (\mathbf{h}_l^*)^\nu \lesssim \left( \frac{\mathbf{h}_2^*}{\mathbf{h}_1^*} \right)^n \prod_{l=3}^q (\mathbf{h}_l^*)^\nu,$$

which is the one we now prove. To this end we denote (just for this prove) by  $K$  the l.h.s. of (21). We distinguish three cases.

- Case 1:  $\mathbf{k}_{q+1} \geq \mathbf{h}_1^*$ . In this case we have (since  $\mathbf{k}_{q+1} = \mathbf{k}_{q+2}$ )

$$K \leq \prod_{l=3}^q (\mathbf{h}_l^*)^\nu \leq \left( \frac{\mathbf{k}_{q+2}}{\mathbf{k}_{q+1}} \right)^n \prod_{l=1}^q (\mathbf{k}_l^*)^\nu,$$

which is the wanted estimate.

- Case 2:  $\mathbf{h}_1^* > \mathbf{k}_{q+1} \geq \mathbf{h}_2^*$ . Here we have

$$K \leq \left( \frac{\mathbf{k}_{q+1}}{\mathbf{h}_1^*} \right)^n \prod_{l=3}^q (\mathbf{h}_l^*)^\nu < \left( \frac{\mathbf{k}_{q+1}}{\mathbf{k}_1^*} \right)^n \prod_{l=3}^q (\mathbf{h}_l^*)^\nu \mathbf{k}_{q+2}^\nu$$

which is the wanted estimate.

- Case 3:  $\mathbf{h}_2^* > \mathbf{k}_{q+1}$ . In this case the estimate trivially holds.

□

**4.3. Klein-Gordon equation on  $\mathbb{R}^d$  with positive definite quadratic potential.** We now briefly explain the proof of Theorem 3.6 by emphasizing the differences with the compact case. Here, the Sobolev space naturally associated to the potential  $Q$  for any  $s \geq 0$  is defined as follows:

$$(22) \quad \mathcal{H}_Q^s(\mathbb{R}^d) := \text{Dom}((-\Delta + Q)^{\frac{s}{2}}) = \{u \in L^2(\mathbb{R}^d), (-\Delta + Q)^{\frac{s}{2}} u \in L^2(\mathbb{R}^d)\}.$$

It is well-known that the norms

$$(23) \quad \left\| (-\Delta + Q)^{s/2} u \right\|_{L^2} \quad \text{and} \quad \left\| (1 - \Delta)^{s/2} u \right\|_{L^2} + \left\| Q^{s/2} u \right\|_{L^2}$$

are equivalent (see e.g. [YZ04]) and

$$\mathcal{H}_Q^s(\mathbb{R}^d) = \{u \in H^s(\mathbb{R}^d) \mid Q^{\frac{s}{2}} u \in L^2(\mathbb{R}^d)\}.$$

Since  $Q$  is 2-homogeneous, we clearly find  $\mathcal{H}_Q^s(\mathbb{R}^d) = \mathcal{H}^s(\mathbb{R}^d)$  defined in (13).

As above, we denote by  $(\lambda_j^2)_{j \in \mathbb{N}}$  the nondecreasing sequence (with multiplicities) of the eigenvalues of  $-\Delta + Q$  and by  $(-\Delta + Q)e_j = \lambda_j^2 e_j$  the corresponding relation. The Weyl law here reads (see Theorem XIII.81 of [RS80]):

$$\forall y \gtrsim 1, \quad \#\{j \in \mathbb{N} \mid \lambda_j \leq y\} \sim \int_{Q(x) \leq y^2} (y^2 - Q(x))^{\frac{d}{2}} dx \sim y^{2d}$$

and in particular  $\lambda_j \sim j^{\frac{1}{2d}}$ . Then we may repeat the same argumentation as in Subsection 4.1 upon making a few modifications. For instance, once we have identified a  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  valued function  $u = \sum_{j \in \mathbb{N}} u_j e_j$  with the sequence of coefficients  $(u_j)_{j \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ , Lemma 4.1 now reads

$$\mathcal{H}^s(\mathbb{R}^d; \mathbb{K}) = h^{\frac{s}{2d}} \cap \mathbb{K}^{\mathbb{N}}.$$

Let us now turn to the tame estimate (2) which, as above, relies on an analogue of (16). Thanks to (23), it is sufficient to prove the following inequality for any smooth function  $f$  vanishing at the origin and any  $u \in \mathcal{H}^s(\mathbb{R}^d)$  for  $s > \frac{d}{2}$ :

$$\|f(u)\|_{H^s(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} \langle x \rangle^{2s} |f(u(x))|^2 dx \lesssim_{s,f,\|u\|_{L^\infty}} \|u\|_{H^s(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} \langle x \rangle^{2s} |u(x)|^2 dx$$

in which we recall that the constant may depend on  $s, f$  and  $\|u\|_{L^\infty}$ . The bound on  $\|f(u)\|_{H^s(\mathbb{R}^d)}$  is already done in [BCD11, Thm 2.87 page 104] as used above. In order to bound the second term, we use the embedding  $\mathcal{H}^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$  and we set

$$C := \sup_{|t| \leq \|u\|_{L^\infty(\mathbb{R}^d)}} |f'(t)|,$$

then we immediately get

$$\int_{\mathbb{R}^d} \langle x \rangle^{2s} |f(u(x))|^2 dx \leq C^2 \int_{\mathbb{R}^d} \langle x \rangle^{2s} |u(x)|^2 dx.$$

The proof of the multilinear estimate (6) is done in the appendix. It mainly relies on multilinear estimates proved in [Brun23]. The rest of the proof of Theorem 3.6 is similar to the one of Theorem 3.1 done in Subsection 4.1.

## 5. FUNCTIONAL SETTING

From now and until the end of this paper, we fix a non-decreasing sequence of frequencies  $\omega \in (\mathbb{R}_+^*)^{\mathbb{N}}$  satisfying the Weyl law (3) and a cluster decomposition

$$\mathbb{N} = \bigsqcup_{k \in \mathbb{N}} \mathcal{C}_k$$

satisfying the bound (4). We recall that the spaces  $E_k$ ,  $k \in \mathbb{N}$  are defined by

$$E_k := \text{Span}_{\mathbb{C}}\{\mathbf{1}_{\{j\}} \mid j \in \mathcal{C}_k\}.$$

In this section, first we introduce some basic notations and definitions. Then, we introduce a class of polynomials and prove the associated multilinear estimates. Finally, we prove some estimates on the Hamiltonian flows generated by these polynomials.

### 5.1. Formalism.

**Definition 5.1** ( $\ell^2$  scalar product). *We endow  $\ell^2$  of the real scalar product*

$$(u, v)_{\ell^2} := \Re \sum_{k \in \mathbb{N}} u_k \overline{v_k}$$

*and of the symplectic form  $(i \cdot, \cdot)_{\ell^2}$ .*

**Definition 5.2** (Projections  $\Pi_k$ ). *For all  $k \in \mathbb{N}$  and all  $u \in \mathbb{C}^{\mathbb{N}}$ , we define  $\Pi_k u \in \mathbb{C}^{\mathbb{N}}$  by*

$$\forall j \in \mathbb{N}, \quad (\Pi_k u)_j = \mathbf{1}_{j \in \mathcal{C}_k} u_j.$$

*Moreover, we set*

$$\Pi_k^{-1} u := \overline{\Pi_k u},$$

and for all  $N \in \mathbb{N}$ ,

$$\Pi_{\leq N} = \sum_{k \leq N} \Pi_k \quad \text{and} \quad \Pi_{> N} = \text{Id} - \Pi_{\leq N}.$$

**Remark 5.3.**  $\Pi_k$  is nothing but the orthogonal projection on  $E_k$  in  $\ell^2$ .

**Definition 5.4** (Super-actions  $J_k$ ). For all  $k \in \mathbb{N}$  and  $u \in \mathbb{C}^{\mathbb{N}}$ , we set

$$J_k(u) := \|\Pi_k u\|_{\ell^2}^2 = \sum_{j \in \mathcal{C}_k} |u_j|^2.$$

**Definition 5.5** ( $H^s$  spaces). For all  $s \in \mathbb{R}$ , we set

$$H^s := \left\{ u \in \mathbb{C}^{\mathbb{N}} \mid \|u\|_{H^s}^2 := \sum_{k \in \mathbb{N}} k^{2s} J_k(u) < \infty \right\}.$$

For all  $\varepsilon$ , we denote by  $B_{H^s}(0, \varepsilon)$  the open ball of  $H^s$  of center 0 and radius  $\varepsilon$ .

Then we note in the following lemma that thanks to the Weyl law (see (3)), the Sobolev spaces  $H^s$  are equivalent to the discrete Sobolev spaces  $h^s$  we used to state our abstract result. Therefore, from now, we only work with  $H^s$  norms.

**Lemma 5.6.** For all  $s \geq 0$ , we have

$$H^s = h^{s/\alpha\beta} \quad \text{and} \quad \|\cdot\|_{h^{s/\alpha\beta}} \sim_s \|\cdot\|_{H^s}.$$

*Proof.* First, we note that thanks to the bound (2), if  $j \in \mathcal{C}_k$  we have  $\omega_j \sim k^\alpha$ . Therefore, we have, for all  $u \in \mathbb{C}^{\mathbb{N}}$

$$\|u\|_{H^s}^2 = \sum_{k \in \mathbb{N}} \sum_{j \in \mathcal{C}_k} k^{2s} |u_j|^2 \sim_s \sum_{j \in \mathbb{N}} \omega_j^{2s/\alpha} |u_j|^2.$$

Finally, to conclude the proof, it suffices to note that, thanks to the Weyl law (3), we have  $\langle \omega_j \rangle \sim j^{1/\beta}$  (it suffices to consider  $\lambda = \omega_j$  and  $\lambda = \omega_{\bar{j}}$  in (3) and to use that  $\omega$  is non decreasing).  $\square$

**Definition 5.7** (Gradient  $\nabla$ ). Being given  $s \geq 0$ ,  $\mathcal{V}$  an open subset of  $H^s$  and  $P \in C^1(\mathcal{V}; \mathbb{R})$ ,  $\nabla P : \mathcal{U} \rightarrow H^{-s}$  denotes the unique function satisfying

$$\forall u \in \mathcal{V}, \forall v \in H^s, \quad (\nabla P(u), v)_{\ell^2} = \text{d}P(u)(v).$$

**Remark 5.8.** As usual, for all  $j \in \mathbb{N}$ , one has  $(\nabla P(u))_j = 2\partial_{u_j} P(u)$ .

**Definition 5.9** (Symplectic map). Being given  $s \geq 0$  and  $\mathcal{V}$  an open subset of  $H^s$ , a map  $\tau \in C^1(\mathcal{V}; h^s)$  is symplectic if

$$\forall u \in \mathcal{V}, \forall v, w \in H^s, \quad (\text{id}\tau(u)(v), \text{d}\tau(u)(w))_{\ell^2} = (iv, w)_{\ell^2}.$$

**Definition 5.10** (Poisson bracket). Let  $s \geq 0$ ,  $\mathcal{V}$  an open subset of  $H^s$ , and two functions  $P, Q \in C^1(\mathcal{V}; \mathbb{R})$  such that  $\nabla P$  or  $\nabla Q$  is  $H^s$  valued. The Poisson bracket of  $P, Q$  is defined by

$$\{P, Q\} := (i\nabla P, \nabla Q)_{\ell^2}.$$

**Remark 5.11.** As usual, one has

$$\{P, Q\} = 2i \sum_{j \in \mathbb{N}} \partial_{u_j} P \partial_{u_j} Q - \partial_{u_j} P \partial_{u_j} Q.$$

**Definition 5.12** (Multilinear forms  $\mathcal{L}_{\mathbf{k}}$ ). Given  $q \geq 3$  and  $\mathbf{k} \in \mathbb{N}^q$ ,  $\mathcal{L}_{\mathbf{k}}$  denote the space of the  $\mathbb{C}$  multilinear forms on  $E_{\mathbf{k}_1} \times \cdots \times E_{\mathbf{k}_q}$ . Moreover, we endow  $\mathcal{L}_{\mathbf{k}}$  of its canonical norm

$$\forall M \in \mathcal{L}_{\mathbf{k}}, \quad \|M\|_{\mathcal{L}_{\mathbf{k}}} := \sup_{\substack{u^{(1)} \in E_{\mathbf{k}_1} \\ \|u^{(1)}\|_{\ell^2} \leq 1}} \cdots \sup_{\substack{u^{(q)} \in E_{\mathbf{k}_q} \\ \|u^{(q)}\|_{\ell^2} \leq 1}} |M(u^{(1)}, \dots, u^{(q)})|.$$

## 5.2. Homogeneous polynomials.

**Definition 5.13** (Spaces of homogeneous polynomials  $\mathcal{H}_q^{\nu,n}$ ). *Given  $q \geq 3$ ,  $\nu, n \geq 0$ ,  $\mathcal{H}_q^{\nu,n}$  denotes the space of the formal real valued homogeneous polynomial  $P$  of degree  $q$  on  $\mathbb{C}^{\mathbb{N}}$  of the form*

$$(24) \quad P(u) = \sum_{\mathbf{k} \in \mathbb{N}^q} \sum_{\sigma \in \{-1,1\}^q} P_{\mathbf{k}}^{\sigma}(\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_q}^{\sigma_q} u)$$

where  $P_{\mathbf{k}}^{\sigma} \in \mathcal{L}_{\mathbf{k}}$  satisfies the symmetry condition

$$P_{\mathbf{k}}^{\sigma}(u^{(1)}, \dots, u^{(q)}) = P_{\varphi \mathbf{k}}^{\varphi \sigma}(u^{(\varphi_1)}, \dots, u^{(\varphi_q)})$$

for all permutation  $\varphi$  of  $\{1, \dots, q\}$  and all  $(u^{(\ell)})_{1 \leq \ell \leq q} \in \prod_{1 \leq \ell \leq q} E_{\mathbf{k}_{\ell}}$ , the reality condition  $P_{\mathbf{k}}^{\sigma} = \overline{P_{\mathbf{k}}^{-\sigma}(\bar{\cdot}, \dots, \bar{\cdot})}$  and the bound

$$\|P\|_{\mathcal{H}_q^{\nu,n}} := \sup_{\mathbf{k} \in \mathbb{N}^q} \sup_{\sigma \in \{-1,1\}^q} \left[ \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2^*}{\mathbf{k}_1^*} \right)^n \prod_{3 \leq \ell \leq q} (\mathbf{k}_{\ell}^*)^{\nu} \right]^{-1} \|P_{\mathbf{k}}^{\sigma}\|_{\mathcal{L}_{\mathbf{k}}} < \infty$$

with  $\Gamma_{\mathbf{k}}$  defined by (7).

First, we note that formal polynomials are polynomials functions on some Sobolev spaces.

**Lemma 5.14.** *Given  $q \geq 3$ ,  $\nu, n \geq 0$ , all formal polynomial  $P \in \mathcal{H}_q^{\nu,n}$  defines a  $C^{\infty}$  real valued function on  $H^{1+\nu}$ .*

*Proof.* It suffices to note that

$$\|P_{\mathbf{k}}^{\sigma}\|_{\mathcal{L}_{\mathbf{k}}} \leq \|P\|_{\mathcal{H}_q^{\nu,n}} (\mathbf{k}_3^*)^{\nu} \cdots (\mathbf{k}_q^*)^{\nu}$$

to get that for all  $u \in H^{1+\nu}$

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{N}^q} \sum_{\sigma \in \{-1,1\}^q} |P_{\mathbf{k}}^{\sigma}(\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_q}^{\sigma_q} u)| &\lesssim_q \|P\|_{\mathcal{H}_q^{\nu,n}} \sum_{\mathbf{k} \in \mathbb{N}^q} (\mathbf{k}_3^*)^{\nu} \cdots (\mathbf{k}_q^*)^{\nu} \prod_{i=1}^q \|\Pi_{\mathbf{k}_i} u\|_{\ell^2} \\ &\lesssim_q \|P\|_{\mathcal{H}_q^{\nu,n}} \left( \sum_{k \in \mathbb{N}} k^{\nu} \|\Pi_k u\|_{\ell^2} \right)^q \lesssim_q \|P\|_{\mathcal{H}_q^{\nu,n}} \|u\|_{H^{1+\nu}}^q. \end{aligned}$$

Thus the series defining  $P$  converge. The fact that  $P$  is real valued is a direct consequence of the reality condition on its coefficients. Finally, since  $P$  is a locally bounded homogeneous polynomial, it is smooth (see e.g. [BS71]).  $\square$

**Remark 5.15.** *Thanks to the symmetry condition, the polynomial functions characterize the formal polynomial: from now, we identify the formal polynomials with the associated function.*

Now, we prove vector field estimates.

**Lemma 5.16.** *Given  $q \geq 3$ ,  $\nu, n \geq 0$  and  $P \in \mathcal{H}_q^{\nu,n}$ , for all  $s \in [0, n]$ ,  $\nabla P$  is a  $q-1$  homogeneous polynomial from  $H^s$  to  $H^s$  satisfying*

$$\forall u \in H^s \cap H^{1+\nu}, \quad \|\nabla P(u)\|_{H^s} \lesssim_q \|P\|_{\mathcal{H}_q^{\nu,n}} \|u\|_{H^s} \|u\|_{H^{1+\nu}}^{q-2}.$$

*Proof.* Without loss of generality, we assume that  $\|P\|_{\mathcal{H}_q^{\nu,n}} = 1$ . First, we note that thanks to the symmetry condition

$$(25) \quad \forall u, v \in H^{1+\nu}, \quad dP(u)(v) = q \sum_{\mathbf{k} \in \mathbb{N}^q} \sum_{\sigma \in \{-1,1\}^q} P_{\mathbf{k}}^{\sigma}(\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_{q-1}}^{\sigma_{q-1}} u, \Pi_{\mathbf{k}_q}^{\sigma_q} v).$$

Then we recall that by duality and density

$$(26) \quad \|\nabla P(u)\|_{H^s} = \sup_{\substack{v \in H^{1+\nu} \\ \|v\|_{H^{-s}}=1}} |(\nabla P(u), v)_{\ell^2}| = \sup_{\substack{v \in H^{1+\nu} \\ \|v\|_{H^{-s}}=1}} |dP(u)(v)|.$$

So, we fix  $u \in H^s \cap H^{1+\nu}$ ,  $v \in H^{1+\nu}$  such that  $\|v\|_{H^{-s}} = 1$ , we set  $w := \sum_k k^{-2s} \Pi_k v$  and for all  $p \in \mathbb{N}$  we set  $p' = 1 + \mathbb{1}_{p=1}$ . We apply the triangular inequality in (25) to get that (since  $s \leq n$ )

$$\begin{aligned} |dP(u)(v)| &\lesssim_q \sum_{\mathbf{k} \in \mathbb{N}^q} \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2^*}{\mathbf{k}_1^*} \right)^n \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_\ell^* \right)^\nu \mathbf{k}_q^{2s} \|\Pi_{\mathbf{k}_q} w\|_{\ell^2} \prod_{i=1}^{q-1} \|\Pi_{\mathbf{k}_i} u\|_{\ell^2} \\ &\lesssim_q \sum_{1 \leq p \leq q} \sum_{\mathbf{k}_1 \geq \dots \geq \mathbf{k}_q} \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2}{\mathbf{k}_1} \right)^n \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_\ell \right)^\nu \mathbf{k}_p^{2s} \|\Pi_{\mathbf{k}_p} w\|_{\ell^2} \prod_{i \neq p} \|\Pi_{\mathbf{k}_i} u\|_{\ell^2} \\ &\lesssim_q \sum_{1 \leq p \leq q} \sum_{\mathbf{k}_1 \geq \dots \geq \mathbf{k}_q} \Gamma_{\mathbf{k}} \left( \prod_{\ell \neq p, p'} \mathbf{k}_\ell \right)^\nu \mathbf{k}_p^s \mathbf{k}_{p'}^s \|\Pi_{\mathbf{k}_p} w\|_{\ell^2} \prod_{i \neq p} \|\Pi_{\mathbf{k}_i} u\|_{\ell^2} \\ &\lesssim_q \sum_{\varsigma \in \{-1, 1\}^q} \sum_{\mathbf{k} \in \mathbb{N}^q} \langle \varsigma_1 \mathbf{k}_1 + \dots + \varsigma_q \mathbf{k}_q \rangle^{-3} \left( \prod_{\ell \geq 3} \mathbf{k}_\ell \right)^\nu \mathbf{k}_1^s \mathbf{k}_2^s \|\Pi_{\mathbf{k}_1} w\|_{\ell^2} \prod_{i \geq 2} \|\Pi_{\mathbf{k}_i} u\|_{\ell^2} \\ &\lesssim_q \|w\|_{H^s} \|u\|_{H^s} \|u\|_{H^{1+\nu}}^{q-2}. \end{aligned}$$

where the last estimate is just the Young convolution inequality. Since  $\|w\|_{H^s} = 1$ , we get the expected estimate by (26).  $\square$

**Remark 5.17.** *Since  $\nabla P$  is locally bounded homogeneous polynomial from  $H^s$  to  $H^s$  (see e.g. [BS71] for details about polynomials), it is  $C^\infty$  and satisfies*

$$(27) \quad \forall u, v \in H^s, \quad \|d\nabla P(u)(v)\|_{H^s} \lesssim_{n,q} \|P\|_{\mathcal{H}_q^{\nu,n}} \|u\|_{H^s}^{q-2} \|v\|_{H^s}.$$

Finally, we prove that these spaces of polynomials are stable by Poisson bracket.

**Proposition 5.18.** *Let  $\nu \geq 0$ ,  $n \geq \nu + 1$ ,  $q, q' \geq 3$ . For all  $P \in \mathcal{H}_q^{n,\nu}$  and  $Q \in \mathcal{H}_{q'}^{n,\nu}$ , their Poisson bracket  $\{P, Q\}$  belongs to  $\mathcal{H}_{q+q'-2}^{n,\nu}$  and satisfies*

$$\|\{P, Q\}\|_{\mathcal{H}_{q+q'-2}^{n,\nu}} \lesssim qq' \|P\|_{\mathcal{H}_q^{n,\nu}} \|Q\|_{\mathcal{H}_{q'}^{n,\nu}}.$$

*Proof.* We divide the proof in 3 steps. Actually, the proof is done in the first step up to the technical estimates (30) which are proved in the two last steps.

$\triangleright$  *Step 1 : Core of the proof.* First, we note that, (since  $n \geq \nu + 1$ )  $\nabla P$  is smooth from  $H^{1+\nu}$  into  $H^{1+\nu}$  (see Lemma 5.16) and that  $Q$  is smooth on  $H^{1+\nu}$  (see Lemma 5.14). Therefore  $\{P, Q\}$  is a well defined function on  $H^{1+\nu}$ .

Then, we note that for all  $j \in \mathbb{N}$  and  $\sigma \in \{1, 1\}$ , we have

$$\partial_{u_j^\sigma} P(u) = q \sum_{\mathbf{k} \in \mathbb{N}^{q-1}} \sum_{\sigma \in \{-1, 1\}^{q-1}} P_{\mathbf{k}, \underline{j}}^{\sigma, \sigma} (\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_{q-1}}^{\sigma_{q-1}} u, \mathbf{1}_{\{j\}})$$

where  $\underline{j} \in \mathbb{N}$  denotes the index such that  $j \in \mathcal{C}_{\underline{j}}$ . It follows that for all  $u \in H^{1+\nu}$ , we have

$$(28) \quad \{P, Q\}(u) = 2iqq' \sum_{\mathbf{k}'' \in \mathbb{N}^{q''}} \sum_{\sigma'' \in \{-1, 1\}^{q''}} R_{\mathbf{k}'', \underline{j}}^{\sigma'', \sigma''} (\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_{q''}}^{\sigma_{q''}} u)$$



where  $q'' = q + q' - 2$  and decomposing  $\mathbf{k}'' = (\mathbf{k}, \mathbf{k}') \in \mathbb{N}^{q-1} \times \mathbb{N}^{q'-1}$ ,  $\boldsymbol{\sigma}'' = (\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in \{-1, 1\}^{q-1} \times \{-1, 1\}^{q'-1}$ ,

$$R_{\mathbf{k}''}^{\boldsymbol{\sigma}''} := \sum_{\ell \in \mathbb{N}} \sum_{\sigma \in \{-1, 1\}} \sigma \sum_{j \in \mathcal{C}_\ell} P_{\mathbf{k}, \ell}^{\sigma, -\sigma}(\cdot, \mathbf{1}_{\{j\}}) \otimes Q_{\mathbf{k}', \ell}^{\boldsymbol{\sigma}', \sigma}(\cdot, \mathbf{1}_{\{j\}}).$$

Then, it suffices to note that by duality

$$\left( \sum_{j \in \mathcal{C}_\ell} |P_{\mathbf{k}, \ell}^{\sigma, -\sigma}(\cdot, \mathbf{1}_{\{j\}})|^2 \right)^{1/2} = \sup_{\substack{v \in E_\ell \\ \|v\|_{\ell^2} \leq 1}} |P_{\mathbf{k}, \ell}^{\sigma, -\sigma}(\cdot, v)|$$

to get by Cauchy–Schwarz that

$$(29) \quad \|R_{\mathbf{k}''}^{\boldsymbol{\sigma}''}\|_{\mathcal{L}_{\mathbf{k}''}} \leq 2 \sum_{\ell \in \mathbb{N}} \|P\|_{\mathcal{L}_{\mathbf{k}, \ell}} \|Q\|_{\mathcal{L}_{\mathbf{k}', \ell}} \leq 2 \sum_{\ell \in \mathbb{N}} \Gamma_{\mathbf{k}, \ell} \Gamma_{\mathbf{k}', \ell} A_{\mathbf{k}, \ell} A_{\mathbf{k}', \ell}$$

where we have assumed by homogeneity that  $\|P\|_{\mathcal{H}_q^{n, \nu}} = \|Q\|_{\mathcal{H}_{q'}^{n, \nu}} = 1$  and used the notation

$$\forall p \geq 2, \forall \mathbf{h} \in \mathbb{N}^p, \quad A_{\mathbf{h}} = \left( \frac{\mathbf{h}_2^*}{\mathbf{h}_1^*} \right)^n \prod_{i=3}^p (\mathbf{h}_i^*)^\nu.$$

In the two next steps, we are going to prove that

$$(30) \quad \sum_{\ell \in \mathbb{N}} \Gamma_{\mathbf{k}, \ell} \Gamma_{\mathbf{k}', \ell} \lesssim \Gamma_{\mathbf{k}, \mathbf{k}'} \quad \text{and} \quad \sup_{\ell \in \mathbb{N}} A_{\mathbf{k}, \ell} A_{\mathbf{k}', \ell} \leq A_{\mathbf{k}, \mathbf{k}'}$$

These estimates imply that

$$\|R_{\mathbf{k}''}^{\boldsymbol{\sigma}''}\|_{\mathcal{L}_{\mathbf{k}''}} \lesssim \Gamma_{\mathbf{k}, \mathbf{k}'} A_{\mathbf{k}, \mathbf{k}'}$$

This estimates on  $R_{\mathbf{k}''}^{\boldsymbol{\sigma}''}$  is almost the expected one. The only missing property is the symmetry condition on the coefficients of  $\{P, Q\}$ . To get it, it suffices to average (28) by the action of the group of the permutations of  $\{1, \dots, q''\}$  and to note that it does not affect the multilinear estimates we proved on the coefficients  $R_{\mathbf{k}''}^{\boldsymbol{\sigma}''}$ .

▷ Step 2 : Estimate on  $\Gamma$ . The first estimate in (30) is a consequence of the more general fact: for all  $b > 1$ , all  $x, y \in \mathbb{R}$

$$\sum_{\ell \in \mathbb{N}} \frac{1}{\langle \ell - x \rangle^b} \frac{1}{\langle \ell - y \rangle^b} \lesssim_b \frac{1}{\langle x - y \rangle^b}$$

which in turn can be proved as follow: as  $\langle x - y \rangle \leq \langle \ell - x \rangle + \langle \ell - y \rangle$  we deduce that for all  $\ell \in \mathbb{N}$ , either  $\langle x - y \rangle \leq 2\langle \ell - x \rangle$  or  $\langle x - y \rangle \leq 2\langle \ell - y \rangle$ . Thus

$$\sum_{\ell \in \mathbb{N}} \frac{1}{\langle \ell - x \rangle^b} \frac{1}{\langle \ell - y \rangle^b} \leq \frac{2^b}{\langle x - y \rangle^b} \left( \sum_{\ell \in \mathbb{N}} \frac{1}{\langle \ell - x \rangle^b} + \sum_{\ell \in \mathbb{N}} \frac{1}{\langle \ell - y \rangle^b} \right) \lesssim_b \frac{1}{\langle x - y \rangle^b}.$$

▷ Step 3 : Estimate on  $A$ . We want to prove that,  $\forall p, p' \geq 2, \forall \mathbf{k} \in \mathbb{N}^p, \forall \mathbf{k}' \in \mathbb{N}^{p'}, \forall \ell \in \mathbb{N}$ , we have

$$A_{\mathbf{k}, \ell} A_{\mathbf{k}', \ell} \leq A_{\mathbf{k}, \mathbf{k}'}$$

Without loss of generality we can assume that  $\mathbf{k}$  and  $\mathbf{k}'$  are ordered and we denote by  $\mathbf{k}''$  the ordered version of  $(\mathbf{k}, \mathbf{k}')$  and we set  $p'' = p + p'$ . Then we rewrite  $A_{\mathbf{k}}$  as

$$A_{\mathbf{k}} = a_{\mathbf{k}}^\nu b_{\mathbf{k}}^{n-\nu}$$

where

$$a_{\mathbf{k}} = \frac{\prod_{i=1}^p \mathbf{k}_i}{\mathbf{k}_1^2}, \quad b_{\mathbf{k}} = \frac{\mathbf{k}_2}{\mathbf{k}_1}.$$

As  $n \geq \nu$  it remains to prove

$$(31) \quad a_{\mathbf{k},\ell} a_{\mathbf{k}',\ell} \leq a_{\mathbf{k}''} \quad \text{and} \quad b_{\mathbf{k},\ell} b_{\mathbf{k}',\ell} \leq b_{\mathbf{k}''}.$$

To begin with we have

$$a_{\mathbf{k},\ell} a_{\mathbf{k}',\ell} = \frac{\ell^2 \prod_{i=1}^{p''} \mathbf{k}_i''}{\max(\mathbf{k}_1, \ell)^2 \max(\mathbf{k}'_1, \ell)^2} \leq \frac{\ell^2 \prod_{i=1}^{p''} \mathbf{k}_i''}{(\mathbf{k}'_1)'^2 \ell^2} = a_{\mathbf{k}''}.$$

For the second estimate in (31) we can assume without loss of generality that  $\mathbf{k}_1 = \mathbf{k}'_1$  (and thus  $\mathbf{k}'_1 \leq \mathbf{k}''_1$ ). Then we argue according to the place of  $\ell$  with respect to  $\mathbf{k}'_1$  and  $\mathbf{k}''_1$ :

- If  $\ell \leq \mathbf{k}''_1$  then  $b_{\mathbf{k},\ell} = \frac{\max(\mathbf{k}_2, \ell)}{\mathbf{k}'_1} \leq \frac{\mathbf{k}''_2}{\mathbf{k}'_1} = b_{\mathbf{k}''}$ . Thus  $b_{\mathbf{k},\ell} b_{\mathbf{k}',\ell} \leq b_{\mathbf{k}''}$  since  $b_{\mathbf{k}',\ell} \leq 1$ .
- If  $\mathbf{k}''_1 \leq \ell \leq \mathbf{k}'_1$  then  $b_{\mathbf{k},\ell} = \frac{\ell}{\mathbf{k}'_1}$  and  $b_{\mathbf{k}',\ell} = \frac{\mathbf{k}'_1}{\ell} \leq \frac{\mathbf{k}''_1}{\ell}$  and thus  $b_{\mathbf{k},\ell} b_{\mathbf{k}',\ell} \leq b_{\mathbf{k}''}$ .
- If  $\mathbf{k}'_1 \leq \ell$  then

$$b_{\mathbf{k},\ell} b_{\mathbf{k}',\ell} = \frac{\mathbf{k}_1}{\ell} \frac{\mathbf{k}'_1}{\ell} \leq \frac{\mathbf{k}'_1}{\ell} \leq \frac{\mathbf{k}''_1}{\ell} = b_{\mathbf{k}''}.$$

□

### 5.3. Inhomogeneous polynomials and flows.

**Definition 5.19** (Space of inhomogeneous polynomials  $\mathcal{S}_{\leq r}^{\nu,n}$ ). *For all  $n, \nu \geq 0$  and  $r \geq 3$ , we define a space of polynomials of degree smaller or equal to  $r$  by*

$$\mathcal{S}_{\leq r}^{\nu,n} := \bigoplus_{3 \leq q \leq r} \mathcal{H}_q^{\nu,n}.$$

For all  $P \in \mathcal{S}_{\leq r}^{\nu,n}$ , we denote by

$$P =: P^{(3)} + \dots + P^{(r)} \quad \text{with} \quad P^{(q)} \in \mathcal{H}_q^{\nu,n}, \quad \forall q \in \{3, \dots, r\}$$

the associated decomposition and we set, for all  $\gamma > 0$ ,

$$\|P\|_{\mathcal{S}_{\leq r}^{\nu,n}} := \sup_{3 \leq q \leq r} \gamma^{q-3} \|P^{(q)}\|_{\mathcal{H}_q^{\nu,n}}.$$

As a consequence of Lemma 5.14 and Lemma 5.16, we have the following tame estimate.

**Lemma 5.20.** *Let  $n, \nu \geq 0$  and  $r \geq 3$ . All polynomial  $P \in \mathcal{S}_{\leq r}^{\nu,n}$  defines a smooth real valued function on  $H^{1+\nu}$ . Moreover, for all  $s \in [1 + \nu, n]$ ,  $\nabla P$  is a smooth function from  $H^s$  into  $H^s$  and it satisfies, for all  $\gamma \in (0, 1]$ , all  $u \in H^s$  of size  $\|u\|_{H^{1+\nu}} \leq \gamma$ ,*

$$\|\nabla P(u)\|_{H^s} \lesssim_{r,n} \|P\|_{\mathcal{S}_{\leq r}^{\nu,n}} \|u\|_{H^s} \|u\|_{H^{1+\nu}} (1 + \gamma^{-1} \|u\|_{H^{1+\nu}})^{q-3}.$$

Now, we are going to consider Hamiltonian systems of the form

$$(32) \quad i\partial_t v = \nabla \chi(v)$$

where  $\chi \in \mathcal{S}_{\leq r}^{\nu,n}$  some  $r, n, \nu \geq 0$ . Thanks to the tame estimate of Lemma 5.20 the local Cauchy theory of this equation is given by the Cauchy–Lipschitz Theorem. In particular, we have the following classical proposition about its flow.

**Proposition 5.21** (Flows  $\Phi_\chi^t$ ). *Let  $n \geq \nu + 1 \geq 0$ ,  $r \geq 3$  and  $\chi \in \mathcal{S}_{\leq r}^{\nu,n}$ . There exists a unique open set  $\mathcal{V}_\chi \subset \mathbb{R} \times H^{1+\nu}$  and a unique map  $\Phi_\chi \equiv (t, u) \mapsto \Phi_\chi^t(u) \in C^\infty(\mathcal{V}_\chi; H^{1+\nu})$  such that*

$$\forall (t, u) \in \mathcal{V}_\chi, \quad i\partial_t \Phi_\chi^t(u) = \nabla \chi(\Phi_\chi^t(u)) \quad \text{and} \quad \Phi_\chi^0(u) = u$$

and satisfying the following properties

- maximality:  $\mathcal{V}_\chi$  is of the form

$$\mathcal{V}_\chi = \bigcup_{u \in H^{1+\nu}} (-T_u^-, T_u^+) \times \{u\}, \quad \text{with } T_u^-, T_u^+ \in (0, +\infty)$$

and for all  $u \in H^{1+\nu}$  and  $\sigma \in \{-1, 1\}$ ,

$$\text{either } T_u^\sigma = +\infty \text{ or } \lim_{t \rightarrow T_u^\sigma} \|\Phi_\chi^t(u)\|_{H^{1+\nu}} = +\infty.$$

- symplecticity: For all  $t \in \mathbb{R}$ ,  $\Phi_\chi^t$  is symplectic on  $\{u \in H^{1+\nu} \mid (t, u) \in \mathcal{V}_\chi\}$ .
- preservation of regularity: for all  $s \in [1 + \nu, n]$ ,  $\Phi_\chi \in \mathcal{C}^\infty(\mathcal{V}_\chi \cap (\mathbb{R} \times H^s); H^s)$ .

Finally, we prove some useful estimates on small solutions.

**Lemma 5.22.** *Let  $n, \nu \geq 0$ ,  $s \in [\nu + 1, n]$ ,  $r \geq 3$  and  $\chi \in \mathcal{S}_{\leq r}^{\nu, n}$ . There exists*

$$(33) \quad \varepsilon_{n, \gamma, \chi} \sim_{n, r} \min(\|\chi\|_{\mathcal{S}_{\leq r}^{\nu, n}}^{-1}, \gamma)$$

such that, for  $t \in [-1, 1]$ ,  $\Phi_\chi^t$  is well defined on  $B_{H^{1+\nu}}(0, 2\varepsilon_{n, \gamma, \chi})$ , i.e.

$$(34) \quad [-1, 1] \times B_{H^{1+\nu}}(0, 2\varepsilon_{n, \gamma, \chi}) \subset \mathcal{V}_\chi,$$

and it is close to the identity, i.e. for all  $u \in H^s \cap B_{H^{1+\nu}}(0, 2\varepsilon_{n, \gamma, \chi})$  and  $t \in [-1, 1]$  we have

$$(35) \quad \|\Phi_\chi^t(u) - u\|_{H^s} \leq \frac{\|u\|_{H^{1+\nu}}}{\varepsilon_{n, \gamma, \chi}} \|u\|_{H^s},$$

$$(36) \quad \|\mathrm{d}\Phi_\chi^t(u) - \mathrm{Id}\|_{H^s \rightarrow H^s} \leq \frac{\|u\|_{H^s}}{\varepsilon_{n, \gamma, \chi}} \left\langle \frac{\|u\|_{H^s}}{2\varepsilon_{n, \gamma, \chi}} \right\rangle^{r-3}.$$

*Proof.* Without loss of generality, we focus only on positive times. We set

$$\varepsilon_{n, \gamma, \chi} := c_{n, r} \min(\|\chi\|_{\mathcal{S}_{\leq r}^{\nu, n}}^{-1}, \gamma)$$

where  $c_{n, r}$  is a constant depending only on  $r$  and  $n$  that we will choose small enough.

We fix  $u \in B_{H^{1+\nu}}(0, 2\varepsilon_{n, \gamma, \chi})$  and we set

$$T^* = \sup \{t \leq \min(1, T_u^+) \mid \forall \tau \leq t, \|\Phi_\chi^\tau(u)\|_{H^{1+\nu}} \leq 2\|u\|_{H^{1+\nu}}\}.$$

Let  $t < T^*$  and  $s \in [\nu + 1, n]$  such that  $u \in H^s$ . First, we note that, provided that  $c_{n, r} \leq \frac{1}{4}$  then  $\|\Phi_\chi^t(u)\|_{H^{1+\nu}} < \gamma$ . Thus, by Lemma 5.20, we have that

$$\|\Phi_\chi^t(u) - u\|_{H^s} \lesssim_{r, n} |t| \|\chi\|_{\mathcal{S}_{\leq r}^{\nu, n}} \|u\|_{H^s} \|u\|_{H^{1+\nu}}.$$

Therefore, provided that  $c_{n, r}$  is small enough, we have that

$$\|\Phi_\chi^t(u) - u\|_{H^s} \leq \frac{1}{2} \frac{\|u\|_{H^{1+\nu}}}{\varepsilon_{n, \gamma, \chi}} \|u\|_{H^s}.$$

This estimate implies that  $T^* = 1$  (i.e. (34)) and that (35) holds.

It only remains to prove (36). So let  $u \in B_{H^{1+\nu}}(0, 2\varepsilon_{n, \gamma, \chi})$ ,  $s \in [\nu + 1, n]$  and  $t \leq 1$ . Deriving (32), it comes

$$\forall w \in H^{1+\nu}, \quad \mathrm{id}_t \mathrm{d}\Phi_\chi^t(u)(w) = \mathrm{d}\nabla\chi(\Phi_\chi^t(u))(\mathrm{d}\Phi_\chi^t(u)(w))$$

and so

$$(37) \quad \|\mathrm{d}\Phi_\chi^t(u) - \mathrm{Id}\|_{H^s \rightarrow H^s} \leq \int_0^t \|\mathrm{d}\nabla\chi(\Phi_\chi^\tau(u))\|_{H^s \rightarrow H^s} \|\mathrm{d}\Phi_\chi^\tau(u)\|_{H^s \rightarrow H^s} \mathrm{d}\tau.$$

Using the bound (27) on  $d\nabla\chi$  it comes

$$\|d\nabla\chi(\Phi_\chi^t(u))\|_{H^s \rightarrow H^s} \lesssim_{n,r} \|\chi\|_{\mathcal{H}_\gamma^{\nu,n}} \sum_{3 \leq q \leq r} \gamma^{-q+3} \|\Phi_\chi^t(u)\|_{H^s}^{q-2}.$$

Therefore, provided that  $c_{n,r}$  is small enough, we have

$$\|d\nabla\chi(\Phi_\chi^t(u))\|_{H^s \rightarrow H^s} \leq \frac{1}{8} \frac{\|u\|_{H^s}}{\varepsilon_{n,\gamma,\chi}} \left\langle \frac{\|u\|_{H^s}}{2\varepsilon_{n,\gamma,\chi}} \right\rangle^{r-3}.$$

Hence, it suffices to apply the Grönwall estimate to (37) to get (36).  $\square$

## 6. NORMAL FORM

In this section the setting is the same as in the previous one. We frequencies  $\omega$  and the cluster decomposition  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  are considered as given. The aim of this section is to prove a Birkhoff normal form theorem to remove by symplectic changes of variables terms which "do not commute enough" with

$$Z_2(u) := \frac{1}{2} \sum_{j \in \mathbb{N}} \omega_j |u_j|^2, \quad u \in H^{\frac{\alpha}{2}}.$$

This Hamiltonian correspond to the linear part of the abstract Hamiltonian PDE (1), i.e.

$$\nabla Z_2(u) = \Omega u =: (\omega_j u_j)_{j \in \mathbb{N}}.$$

In order to quantify how much terms are resonant it is useful to introduce the following operator.

**Definition 6.1** (Operator  $\mathcal{L}_{\mathbf{k},\sigma}$ ). *For all  $q \geq 3$ ,  $\mathbf{k} \in \mathbb{N}^q$ ,  $\sigma \in \{-1, 1\}^q$ , we define the endomorphism  $\mathcal{L}_{\mathbf{k},\sigma} : \mathcal{L}_{\mathbf{k}} \rightarrow \mathcal{L}_{\mathbf{k}}$  by the relation*

$$\mathcal{L}_{\mathbf{k},\sigma}(M)(u^{(1)}, \dots, u^{(q)}) := \sum_{1 \leq i \leq q} \sigma_i M(u^{(1)}, \dots, u^{(i-1)}, \Omega u^{(i)}, u^{(i+1)}, \dots, u^{(q)})$$

for all  $M \in \mathcal{L}_{\mathbf{k}}$  and all  $(u^{(1)}, \dots, u^{(q)}) \in E_{\mathbf{k}_1} \times \dots \times E_{\mathbf{k}_q}$ .

As stated in the following lemma, it appears naturally when considering Poisson brackets with  $Z_2$ . Indeed, by a straightforward calculation, we have:

**Lemma 6.2.** *Let  $q \geq 3$ ,  $n, \nu \geq 0$  and  $P \in \mathcal{H}_q^{n,\nu}$ . Then, for all  $u \in \mathbb{C}^{\mathbb{N}}$  with finite support, we have*

$$\{Z_2, P\}(u) = \sum_{\mathbf{k} \in \mathbb{N}^q} \sum_{\sigma \in \{-1, 1\}^q} (i\mathcal{L}_{\mathbf{k},\sigma} P_{\mathbf{k}}^\sigma)(\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_q}^{\sigma_q} u).$$

Now, we define the polynomials which "almost commute" with  $Z_2$ .

**Definition 6.3** ( $\gamma$ -resonance). *Let  $\gamma > 0$ ,  $q \geq 3$  and  $n, \nu \geq 0$ . A couple  $(\mathbf{k}, \sigma) \in \mathbb{N}^q \times \{-1, 1\}^q$  is non  $\gamma$ -resonant if*

$$(38) \quad \mathcal{L}_{\mathbf{k},\sigma} \text{ is invertible and } \forall M \in \mathcal{L}_{\mathbf{k}}, \quad \|\mathcal{L}_{\mathbf{k},\sigma}^{-1} M\|_{\mathcal{L}_{\mathbf{k}}} \leq \gamma^{-1} \|M\|_{\mathcal{L}_{\mathbf{k}}}.$$

A homogeneous polynomial  $P \in \mathcal{H}_q^{\nu,n}$  is  $\gamma$ -resonant if for all  $(\mathbf{k}, \sigma) \in \mathbb{N}^q \times \{-1, 1\}^q$  either  $P_{\mathbf{k}}^\sigma = 0$  or  $(\mathbf{k}, \sigma)$  is  $\gamma$ -resonant. An inhomogeneous polynomial is  $\gamma$ -resonant if it is the sum of  $\gamma$ -resonant homogeneous polynomials.

Note that (38) is stronger than standard small divisor estimates. Nevertheless, it is a natural multi-dimensional extension. We provide examples of  $\gamma$ -non resonant terms in subsection 6.2.

**6.1. A Birkhoff normal form theorem.** Now, we prove a Birkhoff normal form theorem which allows to remove the non- $\gamma$ -resonant terms (see e.g. [BG25, BC24] for similar formulations).

**Theorem 6.4.** *Let  $r \geq 3$ ,  $\nu \geq \alpha$ . For all  $n \geq \nu + 1$ , all  $P \in \mathcal{S}_{\leq r}^{\nu, n}$  of norm  $B := \|P\|_{\mathcal{S}_1^{\nu, n}}$  and all  $\gamma \in (0, 1)$ , there exists  $\chi \in \mathcal{S}_{\leq r}^{\nu, n}$  such that  $\|\chi\|_{\mathcal{S}_\gamma^{\nu, n}} \lesssim_{n, \nu, r, B} \gamma^{-1}$  and*

$$(Z_2 + P) \circ \Phi_\chi^{-1} = Z_2 + Q + R \quad \text{on} \quad B_{H^{1+\nu}}(0, \varepsilon_{n, \gamma, \chi})$$

where  $\varepsilon_{n, \gamma, \chi} \gtrsim_{n, r, B, \nu} \gamma$  is given by Lemma 5.22<sup>3</sup> and

- $Q \in \mathcal{S}_{\leq r}^{\nu, n}$  is a  $\gamma$ -resonant polynomial of norm  $\|Q\|_{\mathcal{S}_\gamma^{\nu, n}} \lesssim_{n, \nu, r, B} 1$ ,
- $R$  is a  $C^\infty$  real valued function on  $B_{H^{1+\nu}}(0, \varepsilon_{n, \gamma, \chi})$  satisfying, for all  $s \in [1 + \nu, n]$  and all  $u \in B_{H^s}(0, \varepsilon_{n, \gamma, \chi})$

$$(39) \quad \|\nabla R(u)\|_{H^s} \lesssim_{n, r, \nu, B} \gamma^{-r+2} \|u\|_{H^s}^r.$$

*Proof.* Let  $n \geq \nu + 1$ . Then, let  $\gamma \in (0, 1)$ ,  $P \in \mathcal{S}_{\leq r}^{\nu, n}$  and set  $B := \|P\|_{\mathcal{S}_1^{\nu, n}}$ . We divide the proof in 3 steps.

▷ *Step 1: Expansion.* Let  $\chi \in \mathcal{S}_{\leq r}^{\nu, n}$ . It will be fixed at the second step but for the moment we do not impose any constraint on it.

First, we note that if  $F \in C^\infty(H^{1+\nu}; \mathbb{R})$  then

$$\partial_t F \circ \Phi_\chi^{-t}(u) = \{\chi, F\} \circ \Phi_\chi^{-t}(u) \quad \text{for} \quad (t, u) \in \mathcal{V}_\chi.$$

Therefore denoting  $\text{ad}_\chi := \{\chi, \cdot\}$  and performing the Taylor expansion of  $(Z_2 + P) \circ \Phi_\chi^{-1}$ , it comes, on  $B_{H^{1+\nu}}(0, \varepsilon_{n, \gamma, \chi})$ <sup>4</sup>

$$(Z_2 + P) \circ \Phi_\chi^{-1} = \sum_{\ell=0}^{r-2} \frac{\text{ad}_\chi^\ell}{\ell!} (Z_2 + P) + \int_0^1 \frac{(1-t)^{r-2}}{(r-2)!} \text{ad}_\chi^{r-1} (Z_2 + P) \circ \Phi_\chi^{-t} dt.$$

Then, we expand all the polynomial and group terms depending on their homogeneity. More precisely, we set

$$K^{(d)} = \sum_{\ell=1}^{r-2} \frac{K^{(\ell, d)}}{\ell!} \quad \text{with} \quad K^{(\ell, d)} := \sum_{\mathbf{q}_1 + \dots + \mathbf{q}_{\ell+1} = d + 2\ell} \text{ad}_{\chi^{(\mathbf{q}_1)}} \cdots \text{ad}_{\chi^{(\mathbf{q}_\ell)}} P^{(\mathbf{q}_{\ell+1})},$$

$$H^{(d)} = \sum_{\ell=2}^{r-2} \frac{H^{(\ell, d)}}{(\ell+1)!} \quad \text{with} \quad H^{(\ell, d)} := \sum_{\mathbf{q}_1 + \dots + \mathbf{q}_{\ell+1} = d + 2\ell} \text{ad}_{\chi^{(\mathbf{q}_1)}} \cdots \text{ad}_{\chi^{(\mathbf{q}_\ell)}} \{\chi^{(\mathbf{q}_{\ell+1})}, Z_2\},$$

and it comes

$$(Z_2 + P) \circ \Phi_\chi^{-1} = Z_2 + Q + R \quad \text{on} \quad B_{H^{1+\nu}}(0, \varepsilon_{n, \gamma, \chi})$$

where

$$(40) \quad Q := P + \{\chi, Z_2\} + \sum_{3 \leq d \leq r} K^{(d)} + H^{(d)},$$

$$R := \underbrace{\sum_{r+1 \leq d \leq r^2} K^{(d)} + H^{(d)}}_{=: R^{(\text{pol})}} + \underbrace{\sum_{r+1 \leq d \leq r^2} \int_0^1 \frac{(1-t)^r}{r!} (K^{(r+1, d)} + H^{(r, d)}) \circ \Phi_\chi^{-t} dt}_{=: R^{(\text{Tay})}}.$$

<sup>3</sup>to ensure that  $\Phi_\chi^{-1}$  is well defined

<sup>4</sup>because by Lemma 5.22, one has that  $[-1, 1] \times B_{H^{1+\nu}}(0, \varepsilon_{n, \gamma, \chi}) \subset \mathcal{V}_\chi$ .

▷ *Step 2: Resolution of the cohomological equations.* Now, we have to design  $\chi$  so that  $Q$ , defined by (40), belongs to  $\mathcal{S}_{\leq r}^{\nu, n}$  and is  $\gamma$ -resonant. Let us note that decomposing  $Q$  as a sum of homogeneous polynomials, we have

$$Q = \sum_{d=3}^r Q^{(d)} \quad \text{where} \quad Q^{(d)} := P^{(d)} + K^{(d)} + H^{(d)} - \{Z_2, \chi^{(d)}\}.$$

We proceed by induction. We aim at proving that, for all  $d \in \{3, \dots, r\}$  there exists  $\chi^{(d)} \in \mathcal{H}_d^{\nu, n}$  so that  $Q^{(d)}, H^{(d)}, K^{(d)} \in \mathcal{H}_d^{\nu, n}$ ,  $Q^{(d)}$  is  $\gamma$  resonant and the following bounds holds

$$(41) \quad \gamma \|\chi^{(d)}\|_{\mathcal{H}_d^{\nu, n}} + \|H^{(d)}\|_{\mathcal{H}_d^{\nu, n}} + \|K^{(d)}\|_{\mathcal{H}_d^{\nu, n}} + \|Q^{(d)}\|_{\mathcal{H}_d^{\nu, n}} \lesssim_{n, B} \gamma^{-d+3}.$$

To prove it, we consider  $d \in \{3, \dots, r\}$  and assume that  $\chi^{(q)}$  have been designed for all  $q < d$ . The key point is to note that  $K^{(d)}$  and  $H^{(d)}$  only depend on  $\chi$  through the homogeneous terms  $\chi^{(q)}$  with  $q < d$  (which makes the system triangular in some sense).

First, we control  $K^{(d)}$ . Indeed, applying Proposition 5.18 to control the Poisson brackets, using the induction hypothesis (41) and the bound  $\|P\|_{\mathcal{S}_\gamma^{\nu, n}} \leq \|P\|_{\mathcal{S}_1^{\nu, n}} = B$ , it comes that for all  $\ell \leq d$ ,  $K^{(\ell, d)} \in \mathcal{H}_d^{\nu, n}$  and

$$(42) \quad \begin{aligned} \|K^{(\ell, d)}\|_{\mathcal{H}_d^{\nu, n}} &\lesssim_{n, B} \sum_{\mathbf{q}_1 + \dots + \mathbf{q}_{\ell+1} = d+2\ell} \|\chi^{(\mathbf{q}_1)}\|_{\mathcal{H}_{\mathbf{q}_1}^{\nu, n}} \dots \|\chi^{(\mathbf{q}_\ell)}\|_{\mathcal{H}_{\mathbf{q}_\ell}^{\nu, n}} \|P^{(\mathbf{q}_{\ell+1})}\|_{\mathcal{H}_{\mathbf{q}_{\ell+1}}^{\nu, n}} \\ &\lesssim_{n, B} \sum_{\mathbf{q}_1 + \dots + \mathbf{q}_{\ell+1} = d+2\ell} \gamma^{-\mathbf{q}_1+2} \dots \gamma^{-\mathbf{q}_\ell+2} \gamma^{-\mathbf{q}_{\ell+1}+3} \sim_{n, B} \gamma^{-d+3}. \end{aligned}$$

Therefore, we have  $\|K^{(d)}\|_{\mathcal{H}_d^{\nu, n}} \lesssim_{n, \nu, d, B} \gamma^{-d+3}$ .

Now we focus on  $H^{(d)}$ . We note that, by construction, for all  $q < d$

$$\{Z_2, \chi^{(q)}\} = P^{(q)} + K^{(q)} + H^{(q)} - Q^{(q)}.$$

Therefore, by induction hypothesis, we get that for all  $q < d$ ,  $\{Z_2, \chi^{(q)}\} \in \mathcal{H}_q^{\nu, n}$  and that it satisfies the bound

$$\|\{Z_2, \chi^{(q)}\}\|_{\mathcal{H}_q^{\nu, n}} \lesssim_{n, B} \gamma^{-q+3}.$$

As a consequence, proceeding exactly as we did for  $K^{(\ell, d)}$ , we deduce that

$$\|H^{(d)}\|_{\mathcal{H}_d^{\nu, n}} \lesssim_{n, B} \gamma^{-d+3}.$$

Now, we design  $\chi^{(d)}$ . For all  $\mathbf{k} \in \mathbb{N}^d$  and  $\boldsymbol{\sigma} \in \{-1, 1\}^d$ , we set

$$(\chi^{(d)})_{\mathbf{k}}^{\boldsymbol{\sigma}} := (i\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}})^{-1}(P^{(d)} + K^{(d)} + H^{(d)})_{\mathbf{k}}^{\boldsymbol{\sigma}} \quad \text{if } (\mathbf{k}, \boldsymbol{\sigma}) \text{ is non-}\gamma\text{-resonant}$$

and  $(\chi^{(d)})_{\mathbf{k}}^{\boldsymbol{\sigma}} = 0$  else. By definition of  $\gamma$ -resonance, using the bounds we proved on  $K^{(d)}$  and  $H^{(d)}$ , it implies that  $\chi^{(d)} \in \mathcal{H}_d^{\nu, n}$  and that it satisfies the bound

$$\|\chi^{(d)}\|_{\mathcal{H}_d^{\nu, n}} \lesssim_{n, B} \gamma^{-d+2}.$$

Finally, using the formula of Lemma 6.2, it implies that for all  $\mathbf{k} \in \mathbb{N}^d$  and  $\boldsymbol{\sigma} \in \{-1, 1\}^d$ ,

$$(Q^{(d)})_{\mathbf{k}}^{\boldsymbol{\sigma}} = (P^{(d)} + K^{(d)} + H^{(d)})_{\mathbf{k}}^{\boldsymbol{\sigma}} \quad \text{if } (\mathbf{k}, \boldsymbol{\sigma}) \text{ is } \gamma\text{-resonant}$$

and  $(Q^{(d)})_{\mathbf{k}}^{\boldsymbol{\sigma}} = 0$  else. Thus, we get that  $Q^{(d)} \in \mathcal{H}_d^{\nu, n}$  is  $\gamma$ -resonant and satisfies the bound  $\|Q^{(d)}\|_{\mathcal{H}_d^{\nu, n}} \lesssim_{n, B} \gamma^{-d+3}$ . This concludes the proof of the induction.

▷ *Step 3: Bound on the remainder terms.* Finally, we have to control the remainder term  $R = \overline{R^{(\text{pol})}} + R^{(\text{Tay})}$ .

The key point here is to note that proceeding exactly as we did at the previous step (see (42)), we have that for all  $d \leq r^2$  and  $\ell \leq r - 1$ ,  $K^{(\ell,d)} \in \mathcal{H}_d^{\nu,n}$ ,  $H^{(\ell,d)} \in \mathcal{H}_d^{\nu,n}$  satisfy the bound

$$(43) \quad \|K^{(\ell,d)}\|_{\mathcal{H}_d^{\nu,n}} + \|H^{(\ell,d)}\|_{\mathcal{H}_d^{\nu,n}} \lesssim_{n,B} \gamma^{-d+3}.$$

Now, we fix  $u \in B_{H^s}(0, \varepsilon_{n,\gamma,\chi})$  and  $s \in [1 + \nu, n]$ . We note that by construction of  $\chi$ , we have  $\|\chi\|_{\mathcal{S}_\gamma^{\nu,n}} \lesssim_{n,\nu,r,B} \gamma^{-1}$  and so  $\varepsilon_{n,\gamma,\chi} \sim_{n,r,B,\nu} \gamma$  (see (33) and (41)). Therefore, applying Lemma 5.16 and using (43), we directly have that

$$\|\nabla R^{(\text{pol})}(u)\|_{H^s} \lesssim_{n,\nu,r,B} \gamma^{-r+2} \|u\|_{H^s}^r.$$

To estimate  $R^{(\text{Tay})}$ , we first note that, for all  $r+1 \leq d \leq r^2$  and  $t \in [0, 1]$ , since  $\Phi_\chi^{-t}$  is symplectic and  $\Phi_\chi^{-t} \circ \Phi_\chi^t = \text{id}$ , we have

$$\nabla[(K^{(r+1,d)} + H^{(r,d)}) \circ \Phi_\chi^{-t}](u) = -\text{id} \Phi_\chi^t(\Phi_\chi^{-t}(u)) \left( i[\nabla(K^{(r+1,d)} + H^{(r,d)})] \circ \Phi_\chi^{-t}(u) \right).$$

Then, we use that by Lemma 5.22, we have

$$\|\Phi_\chi^{-t}(u)\|_{H^s} \leq 2\|u\|_{H^s} \lesssim_{n,B} \varepsilon_{n,\gamma,\chi}$$

to deduce by Lemma 5.16, the estimate (43) and the estimate on  $d\Phi_\chi^t$  given by Lemma 5.22 that

$$\|\nabla[(K^{(r+1,d)} + H^{(r,d)}) \circ \Phi_\chi^{-t}](u)\|_{H^s} \lesssim_{n,B} \gamma^{-d+3} \|u\|_{H^s}^{d-1}.$$

Finally, using that  $d \geq r+1$  and  $\|u\|_{H^s} \lesssim_{n,B} \gamma$ , we deduce that

$$\|\nabla R^{(\text{Tay})}\|_{H^s} \lesssim_{n,B} \gamma^{-r+2} \|u\|_{H^s}^r.$$

□

**6.2. About  $\gamma$ -resonant terms.** To deduce dynamical corollary of the Birkhoff normal form theorem we have just proved, we have to know more about  $\gamma$ -resonant terms (defined in Definition 6.3).

**Lemma 6.5.** *Let  $q \geq 3$ ,  $\mathbf{k} \in \mathbb{N}^q$  and  $\boldsymbol{\sigma} \in \{-1, 1\}^q$  be such that*

$$\mathbf{k}_1 \geq \dots \geq \mathbf{k}_q \quad \text{and} \quad |\boldsymbol{\sigma}_1 \mathbf{k}_1^\alpha + \boldsymbol{\sigma}_2 \mathbf{k}_2^\alpha| \gtrsim_q \max(\mathbf{k}_3^\alpha, \mathbf{k}_1^{\alpha-1}).$$

*Then,  $(\mathbf{k}, \boldsymbol{\sigma})$  is non-1-resonant.*

*Proof.* Without loss of generality we assume that  $\boldsymbol{\sigma}_1 = 1$ . First, we note that thanks to (2) and the mean value inequality

$$\forall k \in \mathbb{N}, \forall u \in E_k \cap B_{L^2}(0, 1), \quad \|\Omega u_k - (\Upsilon k)^\alpha u_k\|_{\ell^2} \leq \sup_{j \in \mathcal{C}_k} |\omega_j - (\Upsilon k)^\alpha| \lesssim k^{\alpha-1}.$$

Now we consider  $M \in \mathcal{L}_\mathbf{k}$ . It follows that

$$\|\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}}(M) - \Upsilon^\alpha (\mathbf{k}_1^\alpha + \boldsymbol{\sigma}_2 \mathbf{k}_2^\alpha) M\|_{\mathcal{L}_\mathbf{k}} \lesssim \|M\|_{\mathcal{L}_\mathbf{k}} (\mathbf{k}_1^{\alpha-1} + \mathbf{k}_2^{\alpha-1} + \mathbf{k}_3^\alpha + \dots + \mathbf{k}_q^\alpha).$$

Thus, provided that  $\mathbf{k}$  satisfies an assumption of the kind  $|\boldsymbol{\sigma}_1 \mathbf{k}_1^\alpha + \boldsymbol{\sigma}_2 \mathbf{k}_2^\alpha| \gtrsim_q \max(\mathbf{k}_3^\alpha, \mathbf{k}_1^{\alpha-1})$  we have

$$\|\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}}(M)\|_{\mathcal{L}_\mathbf{k}} \geq \|M\|_{\mathcal{L}_\mathbf{k}}.$$

Therefore, the rank-nullity theorem implies that  $(\mathbf{k}, \boldsymbol{\sigma})$  is non-1-resonant.

□

A refined version of the following lemma can be found in [DI17, Proposition 10].

**Lemma 6.6.** *Assume the non resonance condition (5) on the frequencies  $\omega$ . Then for all  $q \geq 3$ ,  $\mathbf{k}_1 \geq \dots \geq \mathbf{k}_q$  and  $\boldsymbol{\sigma} \in \{-1, 1\}^q$  such that*

$$\sum_{1 \leq \ell \leq q} \sigma_\ell \mathbb{1}_{\{\mathbf{k}_\ell\}} \neq 0,$$

there exists  $\gamma \gtrsim_q \mathbf{k}_1^{-a_q - q\alpha\beta}$  such that  $(\mathbf{k}, \boldsymbol{\sigma})$  is non- $\gamma$ -resonant.

*Proof.* Let  $M \in \mathcal{L}_{\mathbf{k}}$  and  $u^{(1)} \in E_{\mathbf{k}_1}, \dots, u^{(q)} \in E_{\mathbf{k}_q}$  of norm  $\|u^{(1)}\|_{\ell^2} \leq 1, \dots, \|u^{(q)}\|_{\ell^2} \leq 1$ . Expanding  $\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}}(M)$  it comes

$$\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}}(M)(u^{(1)}, \dots, u^{(q)}) = \sum_{j \in \mathcal{C}_{\mathbf{k}_1} \times \dots \times \mathcal{C}_{\mathbf{k}_q}} \left( \sum_{\ell=1}^q \sigma_\ell \omega_{j_\ell} \right) M_j \prod_{\ell=1}^q u_{j_\ell}^{(\ell)} \quad \text{where } M_j := M(\mathbb{1}_{\{j_1\}}, \dots, \mathbb{1}_{\{j_q\}}).$$

Therefore,  $\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}}$  is clearly invertible of invert given by

$$\forall j \in \mathcal{C}_{\mathbf{k}_1} \times \dots \times \mathcal{C}_{\mathbf{k}_q}, \quad (\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}}^{-1}(M))_j = \left( \sum_{\ell=1}^q \sigma_\ell \omega_{j_\ell} \right)^{-1} M_j.$$

As a consequence, using the small divisor estimate (5), the Cauchy–Schwarz inequality and the assumptions (3) and (4) on the clusters, it comes

$$\begin{aligned} |\mathcal{L}_{\mathbf{k}, \boldsymbol{\sigma}}^{-1}(M)(u^{(1)}, \dots, u^{(q)})| &\lesssim_q \mathbf{k}_1^{a_q} \left( \sum_{j \in \mathcal{C}_{\mathbf{k}_1} \times \dots \times \mathcal{C}_{\mathbf{k}_q}} |M_j|^2 \right)^{1/2} \prod_{\ell=1}^q \|u^{(\ell)}\|_{\ell^2} \\ &\lesssim_q \mathbf{k}_1^{a_q} \|M\|_{\mathcal{L}_{\mathbf{k}}} \prod_{\ell=1}^q (\#\mathcal{C}_\ell)^{1/2} \lesssim_q \mathbf{k}_1^{a_q + q\frac{\alpha\beta}{2}} \|M\|_{\mathcal{L}_{\mathbf{k}}}. \end{aligned}$$

□

As a direct consequence of these lemmas we deduce the following lemma.

**Lemma 6.7.** *Assume the non resonance condition (5) on the frequencies  $\omega$ . For all  $q \geq 3$  and all  $N \gtrsim_q 1$ , setting*

$$\gamma = N^{-2a_q - q\alpha\beta - 2},$$

all  $\gamma$ -resonant couple  $(\mathbf{k}, \boldsymbol{\sigma}) \in \mathbb{N}^q \times \{-1, 1\}^q$  is of one of the following type

- Type I :  $\mathbf{k}_3^* > N$ ,
- Type II :  $\mathbf{k}_1^* \leq N$  and  $\sigma_1 \mathbb{1}_{\{\mathbf{k}_1\}} + \dots + \sigma_q \mathbb{1}_{\{\mathbf{k}_q\}} = 0$ ,
- Type III :  $\mathbf{k}_2^* > N$ ,  $\mathbf{k}_3^* \leq N$  and  $\sigma_1^* = -\sigma_2^*$ .

where  $\boldsymbol{\sigma}^*$  denotes an arrangement of  $\boldsymbol{\sigma}$  corresponding to  $\mathbf{k}^{*5}$ .

*Proof.* Let  $(\mathbf{k}, \boldsymbol{\sigma}) \in \mathbb{N}^q \times \{-1, 1\}^q$  be a  $\gamma$ -resonant couple such that  $\mathbf{k}_1 \geq \dots \geq \mathbf{k}_q$  and  $\mathbf{k}_3 \leq N$ . We aim at proving that it is either of type II or of type III. We distinguish 3 basic cases.

▷ Case 1 :  $\mathbf{k}_1 \leq N$ . Provided that  $N$  is large enough, we have by Lemma 6.6 that  $(\mathbf{k}, \boldsymbol{\sigma})$  is of type II.

▷ Case 2 :  $N < \mathbf{k}_1 \leq N^2$ . Provided that  $N$  is large enough, we have by Lemma 6.6 that

$$\sigma_1 \mathbb{1}_{\{\mathbf{k}_1\}} + \dots + \sigma_q \mathbb{1}_{\{\mathbf{k}_q\}} = 0.$$

Since  $\mathbf{k}_3 < N$  it implies that  $\sigma_1 = -\sigma_2$  and  $\mathbf{k}_1 = \mathbf{k}_2$ . Therefore  $(\mathbf{k}, \boldsymbol{\sigma})$  is of type III.

<sup>5</sup>i.e.  $\boldsymbol{\sigma}^*$  is an element of  $\{-1, 1\}^q$  such that there exists  $\varphi \in \mathfrak{S}_q$  satisfying  $\sigma_j^* = \sigma_{\varphi_j}$  and  $\mathbf{k}_j^* = j_{\varphi_j}$  for all  $j \in \{1, \dots, q\}$ .



▷ *Case 3 :  $N^2 < \mathbf{k}_1$ .* Without loss of generality, we assume that  $\sigma_1 = 1$ . Lemma 6.5 implies that, provided that  $N$  is large enough,

$$\sigma_2 \mathbf{k}_2^\alpha \geq \mathbf{k}_1^\alpha - C_q(\mathbf{k}_3^\alpha + \mathbf{k}_1^{\alpha-1}) \geq N^{2\alpha} - C_q(N^\alpha + N^{2(\alpha-1)}) > N^\alpha$$

where  $C_q > 0$  is the implicit constant in Lemma 6.5. Therefore  $\sigma_2 = -1$  and  $\mathbf{k}_2 > N$ , i.e.  $(\mathbf{k}, \sigma)$  is of type III.  $\square$

Finally, in the following proposition, we deduce a result about the dynamics generated by the  $\gamma$ -resonant terms.

**Proposition 6.8.** *Assume the non resonance condition (5) on the frequencies  $\omega$ . For all  $q \geq 3$ , all  $N \gtrsim_q 1$ , all  $n \geq 2 + \nu$  and all  $s \in [\nu + 2, n]$ , setting*

$$\gamma = N^{-2a_q - q\alpha\beta - 2},$$

*we have that for all  $\gamma$ -resonant homogeneous polynomial  $Q \in \mathcal{H}_q^{n,\nu}$  of norm  $\|Q\|_{\mathcal{H}_q^{n,\nu}} = 1$  and all  $u \in H^s$*

$$\begin{aligned} |\{\|\Pi_{\leq N} \cdot \|\ell^2, Q\}(u)| &\lesssim_q \|\Pi_{>N} u\|_{H^s}^2 \|u\|_{H^{1+\nu}}^{q-2}, \\ |\{\|\Pi_{>N} \cdot \|\ell^2, Q\}(u)| &\lesssim_q \|\Pi_{>N} u\|_{\ell^2}^2 \|\Pi_{>N} u\|_{H^{1+\nu}} \|u\|_{H^{1+\nu}}^{q-3}, \\ |\{\|\Pi_{>N} \cdot \|\ell^2, Q\}(u)| &\lesssim_q \|\Pi_{>N} u\|_{H^s}^{2-\frac{1}{s}} \|\Pi_{>N} u\|_{\ell^2}^{\frac{1}{s}} \|u\|_{H^{2+\nu}}^{q-2} + \|\Pi_{>N} u\|_{H^s}^2 \|\Pi_{>N} u\|_{H^{1+\nu}} \|u\|_{H^{2+\nu}}^{q-3}. \end{aligned}$$

*Proof.* By Lemma 6.7, it suffices to consider all the types separately.

▷ *Case 1 :  $Q$  is supported on indices of type II.* In that case  $Q$  commutes with all the super actions  $J_k$  for  $k \in \mathbb{N}$ . A fortiori it commutes with  $\|\Pi_{\leq N} \cdot \|\ell^2, \|\Pi_{>N} \cdot \|\ell^2$  and  $\|\Pi_{>N} \cdot \|\ell^2$  (i.e. the considered Poisson bracket are identically equal to 0).

▷ *Case 2 :  $Q$  is supported on indices of type I.* A minor extension of the proof of Lemma 5.16 (i.e. paying attention to the support of the functions) provides the estimate

$$\|\nabla Q(u)\|_{H^{s_*}} \lesssim_q \|\Pi_{>N} u\|_{H^{s_*}} \|\Pi_{>N} u\|_{H^{1+\nu}} \|u\|_{H^{1+\nu}}^{q-3}, \quad \text{for } s_* \in [0, n].$$

Thus all the Poisson bracket estimates follow by Cauchy-Schwarz inequality.

▷ *Case 3 :  $Q$  is supported on indices of type III.* First we note that  $\{\|\Pi_{>N} \cdot \|\ell^2, Q\} = 0$ . Then, using the mean value inequality and the Young convolution inequality, we have

$$\begin{aligned} |\{\|\Pi_{>N} \cdot \|\ell^2, Q\}(u)| &= \left| \sum_{\mathbf{k} \in \mathbb{N}^q} \sum_{\sigma \in \{-1, 1\}^q} (\sigma_1^*(\mathbf{k}_1^*)^{2s} + \sigma_2^*(\mathbf{k}_2^*)^{2s}) Q_{\mathbf{k}}^\sigma(\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_q}^{\sigma_q} u) \right| \\ &\lesssim_q \sum_{\mathbf{k}_1 \geq \mathbf{k}_2 > N \geq \mathbf{k}_3 \dots \geq \mathbf{k}_q} \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2}{\mathbf{k}_1} \right)^n \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_\ell \right)^\nu (\mathbf{k}_1^{2s} - \mathbf{k}_2^{2s}) \prod_{\ell=1}^q \|\Pi_{\mathbf{k}_\ell}^{\sigma_\ell} u\|_{\ell^2} \\ &\lesssim_q \sum_{\substack{\mathbf{k}_1 \geq \mathbf{k}_2 > N \geq \mathbf{k}_3 \dots \geq \mathbf{k}_q \\ \varsigma \in \{-1, 1\}^q}} \frac{(\mathbf{k}_1 - \mathbf{k}_2) \mathbf{k}_1^{2s-1}}{\langle \varsigma_1 \mathbf{k}_1 + \dots + \varsigma_q \mathbf{k}_q \rangle^3} \left( \frac{\mathbf{k}_2}{\mathbf{k}_1} \right)^n \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_\ell \right)^\nu \prod_{\ell=1}^q \|\Pi_{\mathbf{k}_\ell}^{\sigma_\ell} u\|_{\ell^2} \\ &\lesssim_q \sum_{\substack{\mathbf{k}_1 \geq \mathbf{k}_2 > N \geq \mathbf{k}_3 \dots \geq \mathbf{k}_q \\ \varsigma \in \{-1, 1\}^q}} \frac{\mathbf{k}_3 \mathbf{k}_1^{s-\frac{1}{2}} \mathbf{k}_2^{s-\frac{1}{2}}}{\langle \varsigma_1 \mathbf{k}_1 + \dots + \varsigma_q \mathbf{k}_q \rangle^2} \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_\ell \right)^\nu \prod_{\ell=1}^q \|\Pi_{\mathbf{k}_\ell}^{\sigma_\ell} u\|_{\ell^2} \\ &\lesssim_q \|\Pi_{>N} u\|_{H^{s-1/2}}^2 \|u\|_{H^{2+\nu}}^{q-2} \\ &\lesssim_q \|\Pi_{>N} u\|_{H^s}^{2-\frac{1}{s}} \|\Pi_{>N} u\|_{\ell^2}^{\frac{1}{s}} \|u\|_{H^{2+\nu}}^{q-2} \end{aligned}$$

where the last estimate comes from the Hölder inequality. Finally, applying the Young convolution inequality as previously, we have

$$\begin{aligned}
|\{\|\Pi_{\leq N} \cdot\|_{H^s}^2, Q\}(u)| &= \left| \sum_{\mathbf{k} \in \mathbb{N}^q} \sum_{\sigma \in \{-1, 1\}^q} (\sigma_3^*(\mathbf{k}_3^*)^{2s} + \cdots + \sigma_q^*(\mathbf{k}_q^*)^{2s}) Q_{\mathbf{k}}^{\sigma}(\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_q}^{\sigma_q} u) \right| \\
&\lesssim_q \sum_{\mathbf{k}_1 \geq \mathbf{k}_2 > N \geq \mathbf{k}_3 \cdots \geq \mathbf{k}_q} \Gamma_{\mathbf{k}} \mathbf{k}_3^{2s} \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_{\ell} \right)^{\nu} \prod_{\ell=1}^q \|\Pi_{\mathbf{k}_{\ell}}^{\sigma_{\ell}} u\|_{\ell^2} \\
&\lesssim_q \sum_{\mathbf{k}_1 \geq \mathbf{k}_2 > N \geq \mathbf{k}_3 \cdots \geq \mathbf{k}_q} \Gamma_{\mathbf{k}} \mathbf{k}_1^s \mathbf{k}_2^s \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_{\ell} \right)^{\nu} \prod_{\ell=1}^q \|\Pi_{\mathbf{k}_{\ell}}^{\sigma_{\ell}} u\|_{\ell^2} \\
&\lesssim_q \|\Pi_{> N} u\|_{H^s}^2 \|u\|_{H^{1+\nu}}^{q-2}.
\end{aligned}$$

□

## 7. DYNAMICAL CONSEQUENCES : PROOF OF THEOREM 2.1

From now, we are in the setting of Theorem 2.1. In particular, we assume all the related hypothesis. The only difference is that we work with  $H^s$  spaces instead of  $h^s$  spaces, we recall that since for all  $s \geq 0$

$$\|\cdot\|_{h^{s/\alpha\beta}} \sim_s \|\cdot\|_{H^s}$$

this is equivalent. We just set

$$\mathfrak{s}_0 := \frac{s_0}{\alpha\beta}$$

so that the index  $s_0$  of the theorem becomes  $\mathfrak{s}_0$  in this setting. Without loss of generality, we assume that

$$\mathfrak{s}_0 \geq \nu + 2,$$

that the sequence  $(a_q)_q$  (related to the assumptions (5)) is increasing and that  $\mathcal{U}$  is a ball, i.e. there exists  $\rho_0 > 0$  such that

$$\mathcal{U} = B_{H^{\mathfrak{s}_0}}(0, \rho_0).$$

▷ Step 1: Parameters. Let  $r \geq 17$ ,  $\mathfrak{s} \geq 1$  be large enough so that

$$\mathfrak{s} \geq \mathfrak{s}_0 + \alpha \quad \text{and} \quad \mathfrak{s} \geq \mathfrak{s}_0 + 9r^2(2a_r + r\alpha\beta + 2).$$

Let  $\varepsilon \in (0, 1)$ . All along the proof we will add smallness assumptions on  $\varepsilon$  with respect to  $r$  and  $\mathfrak{s}$ . We set

$$N := \varepsilon^{-\frac{8r}{\mathfrak{s}-\mathfrak{s}_0}} \quad \text{and} \quad \gamma := N^{-(2a_r + r\alpha\beta + 2)}.$$

We note that, we have the useful estimate

$$(44) \quad \gamma^{-r} = N^{r(2a_r + r\alpha\beta + 2)} \leq N^{\frac{\mathfrak{s}-\mathfrak{s}_0}{9r}} = \varepsilon^{-\frac{8}{9}}.$$

We define for  $u \in H^{\mathfrak{s}_0}$

$$P(u) := \sum_{q=3}^r P^{(q)}(u) \quad \text{where} \quad P^{(q)}(u) := \frac{1}{q!} d^q G(0)(u, \dots, u).$$

Since  $g$  is a smooth function from  $B_{H^{\mathfrak{s}_0}}(0, \rho_0)$  into  $H^{\mathfrak{s}_0}$ , it implies that provided that  $5\varepsilon < \rho_0$

$$(45) \quad \forall u \in B_{H^{\mathfrak{s}_0}}(0, 5\varepsilon), \quad \|\nabla(G - P)(u)\|_{H^{\mathfrak{s}_0}} \lesssim_r \varepsilon^r.$$

Then, we set  $n \in \mathbb{N}$  as the integer satisfying

$$n + 1 \leq \mathfrak{s} < n + 2.$$

Finally, we have to prove that  $P \in \mathcal{S}_{\leq r}^{n,\nu}$  (i.e.  $P^{(q)} \in \mathcal{H}_q^{n,\nu}$  for all  $q \geq 3$ ). We define the coefficients  $(P^{(q)})_{\mathbf{k}}^{\sigma}$  by the polarization formula<sup>6</sup>

$$(P^{(q)})_{\mathbf{k}}^{\sigma}(\Pi_{\mathbf{k}_1} u^{(1)}, \dots, \Pi_{\mathbf{k}_q} u^{(q)}) = 2^{-q} \sum_{\boldsymbol{\eta} \in \{0,1\}^q} \left( \prod_{\ell=1}^q (-i\sigma_{\ell})^{\eta_{\ell}} \right) \frac{d^q G(0)}{q!} (i^{\eta_1} \Pi_{\mathbf{k}_1}^{\sigma_1} u^{(1)}, \dots, i^{\eta_q} \Pi_{\mathbf{k}_q}^{\sigma_q} u^{(q)})$$

for  $q \geq 3$ ,  $\mathbf{k} \in \mathbb{N}$ ,  $\boldsymbol{\sigma} \in \{-1, 1\}^q$  and  $u^{(1)}, \dots, u^{(q)} \in \mathbb{C}^{\mathbb{N}}$ . By a tedious calculation, it can be checked that  $(P^{(q)})_{\mathbf{k}}^{\sigma} \in \mathcal{L}_{\mathbf{k}}$  (i.e. that it is  $\mathbb{C}$ -multilinear) and that

$$(46) \quad P(u) = \sum_{q=3}^r \sum_{\mathbf{k} \in \mathbb{N}^q} \sum_{\boldsymbol{\sigma} \in \{-1, 1\}^q} (P^{(q)})_{\mathbf{k}}^{\sigma}(\Pi_{\mathbf{k}_1}^{\sigma_1} u, \dots, \Pi_{\mathbf{k}_q}^{\sigma_q} u)$$

when  $u \in \mathbb{C}^{\mathbb{N}}$  with finite support. Moreover, we note that the reality and the symmetry condition are obvious. Finally, we note that thanks to the polarization formula above and the bound (6) we assumed on  $d^q G(0)$ , we have that  $\|P\|_{\mathcal{S}_{\leq r}^{n,\nu}} \lesssim_{n,q} 1$  (and so that (46) holds in  $H^{1+\nu}$  by density).

▷ *Step 2: Setting of the bootstrap.* Without loss of generality, we only focus on positive times. Let  $u^{(0)} \in H^s$  such that  $\|u^{(0)}\|_{H^s} = \varepsilon$  and let

$$u \in C^0([0, T^{\max}); H^s) \cap C^1([0, T^{\max}); H^{s-\alpha})$$

be the maximal solution of (1) with initial datum  $u(0) = u^{(0)}$ . We note that thanks to the tame estimate (2), we have

$$\text{either } T^{\max} = +\infty \text{ or } \lim_{t \rightarrow T^{\max}} \|u(t)\|_{H^{s_0}} = \rho_0.$$

Then, we set

$$(47) \quad T_{\varepsilon} = \sup \left\{ t \leq \min(T^{\max}, \varepsilon^{-\frac{r}{9s_0}}) \mid \sup_{\tau \leq t} \|u(\tau)\|_{H^{s_0}} < 5\varepsilon \right\}.$$

Since  $r$  can be chosen arbitrarily large, in order to prove Theorem 2.1, it suffices<sup>7</sup> to prove that  $T_{\varepsilon} = \varepsilon^{-\frac{r}{9s_0}}$ . By a continuity argument, it means that it suffices to prove that for all  $t \leq T_{\varepsilon}$ , we have  $\|u(t)\|_{H^{s_0}} \leq 4\varepsilon$ .

▷ *Step 3: Normal form.* Now, we apply the Birkhoff normal form Theorem 6.4. We note that by (44), we have that

$$\varepsilon_{n,\gamma,\chi} \gtrsim \gamma \geq \varepsilon^{\frac{8}{9r}} \gg \varepsilon.$$

So, by Lemma 5.22, it makes sense to define, for all  $t \leq T_{\varepsilon}$ ,

$$v(t) := \Phi_{\chi}^1(u(t)).$$

Since  $u \in C^1((0, T_{\varepsilon}); H^{s_0})$  (because  $s - \alpha \geq s_0$ ) and  $\Phi_{\chi}^1$  is smooth on  $B_{H^{s_0}}(0, \varepsilon_{n,\gamma,\chi})$  (because  $s_0 \geq \nu + 1$ ), we have that

$$v \in C^1((0, T_{\varepsilon}); H^{s_0}).$$

Moreover, since  $\chi$  is symplectic, it satisfies

$$(48) \quad i\partial_t v = \nabla(Z_2 + Q + R)(v) - \underbrace{\text{id}\Phi_{\chi}^1(u)(i\nabla(G - P)(u))}_{=:f}.$$

Using the estimates of Lemma 5.22 on  $\Phi_{\chi}^1$ , the bootstrap assumption and the estimate (45) on  $\nabla(G - P)(u)$ , we have that for all  $t < T_{\varepsilon}$

$$(49) \quad \|v(0)\|_{H^s} \leq 2\varepsilon, \quad \|v(t)\|_{H^{s_0}} \leq 6\varepsilon, \quad \|f(t)\|_{H^{s_0}} \lesssim_r \varepsilon^r.$$

<sup>6</sup>coming from the Fourier inversion formula in the Abelian group  $(\mathbb{Z}/2\mathbb{Z})^q$ .

<sup>7</sup>note that here we use that  $s_0$  does not depend on  $r$ .

We note that in particular

$$(50) \quad \|\Pi_{>N}v(0)\|_{H^{s_0}} \leq \|v(0)\|_{H^s} N^{-(s-s_0)} \leq 2\varepsilon N^{-(s-s_0)} = 2\varepsilon^{8r+1}.$$

Moreover, using the bound on  $\|v(t)\|_{H^{s_0}}$ , the estimate (39) on  $\nabla R$  and the estimate (44) on  $\gamma^{-r}$ , we have that for all  $t < T_\varepsilon$

$$(51) \quad \|\nabla R(v(t))\|_{H^{s_0}} \leq \varepsilon^{r-1}.$$

Finally, using that  $\Phi_\chi^1$  is close to the identity as proven in Lemma 5.22, we point out that it suffices to prove that  $\|v(t)\|_{H^{s_0}} \leq 3\varepsilon$  for  $t < T_\varepsilon$  to deduce that  $\|u(t)\|_{H^{s_0}} \leq 4\varepsilon$  and so to conclude the proof.

▷ *Step 4: system of estimates.* Since  $v \in C^1((0, T_\varepsilon); H^{s_0})$  solve the equation (48) and  $Z_2$  commutes with  $\|\cdot\|_{H^{s_0}}^2$ , we have

$$\partial_t \|\Pi_{>N}v\|_{H^{s_0}}^2 = \{\|\Pi_{>N} \cdot\|_{H^{s_0}}^2, Q\} + 2(\mathrm{i}\Pi_{>N}v, \nabla R(v) + f)_{\ell^2}.$$

Then by using the Cauchy–Schwarz inequality and the estimates (49), (51) on the remainder terms (and that  $\|v(t)\|_{H^{s_0}} \lesssim \varepsilon$ ; see (49)), it comes

$$|(\mathrm{i}\Pi_{>N}v, \nabla R(v) + f)_{\ell^2}| \lesssim \varepsilon^r.$$

Since  $\varepsilon \leq \gamma$  (by (44)),  $Q$  is  $\gamma$ -resonant and satisfies  $\|Q\|_{\mathcal{S}_\gamma^{n,\nu}} \lesssim_{r,n} 1$  (by Theorem 6.4), we get by Proposition 6.8 that

$$|\{\|\Pi_{>N} \cdot\|_{H^{s_0}}^2, Q\}(v)| \lesssim_{r,n} \varepsilon \|\Pi_{>N}v\|_{H^{s_0}}^{2-\frac{1}{s_0}} \|\Pi_{>N}v\|_{\ell^2}^{\frac{1}{s_0}} + \|\Pi_{>N}v\|_{H^{s_0}}^3.$$

Thus we have proven that

$$(52) \quad \partial_t \|\Pi_{>N}v\|_{H^{s_0}}^2 \lesssim_{r,n} \varepsilon \|\Pi_{>N}v\|_{H^{s_0}}^{2-\frac{1}{s_0}} \|\Pi_{>N}v\|_{\ell^2}^{\frac{1}{s_0}} + \|\Pi_{>N}v\|_{H^{s_0}}^3 + \varepsilon^r.$$

Then, by using the same arguments we get that

$$(53) \quad \partial_t \|\Pi_{\leq N}v\|_{H^{s_0}}^2 \lesssim_{r,n} \varepsilon \|\Pi_{>N}v\|_{H^{s_0}}^2 + \varepsilon^r.$$

$$(54) \quad \partial_t \|\Pi_{>N}v\|_{\ell^2}^2 \lesssim_{r,n} \|\Pi_{>N}v\|_{\ell^2}^2 \|\Pi_{>N}v\|_{H^{s_0}} + \varepsilon^r.$$

▷ *Step 5: last bootstrap and conclusion.* In order to fully exploit the system of estimates ((52), (53), (54)) we do a second bootstrap argument. We set

$$T^* = \sup \{t \leq T_\varepsilon \mid \forall \tau \leq t, \quad \|\Pi_{\leq N}v(\tau)\|_{H^{s_0}}^2 \leq 3\varepsilon \quad \text{and} \quad \|\Pi_{>N}v(\tau)\|_{H^{s_0}}^2 \leq \varepsilon^{\frac{r}{8}}\}$$

and we note that by (49) and (50) we have  $T^* > 0$ . We note that now, to conclude the proof, it suffices to prove that if  $t < T^*$  then  $\|\Pi_{\leq N}v(\tau)\|_{H^{s_0}}^2 < 3\varepsilon$  and  $\|\Pi_{>N}v(\tau)\|_{H^{s_0}}^2 < \varepsilon^{\frac{r}{8}}$ .

So let  $t < T^*$  and we note that

$$t < T^* \leq T_\varepsilon \leq \varepsilon^{-\frac{r}{9s_0}} \leq \varepsilon^{-\frac{r}{18}}.$$

First, by using (53), we get that

$$\partial_t \|\Pi_{\leq N}v(t)\|_{H^{s_0}}^2 \lesssim_{r,n} \varepsilon^{1+\frac{r}{4}}$$

and so that (since  $r \geq 17$ )

$$\|\Pi_{\leq N}v(t)\|_{H^{s_0}}^2 \leq T_* \varepsilon^{\frac{r}{4}} \leq \varepsilon^{\frac{r}{8}} < \varepsilon^2.$$

Thus it suffices to focus on estimating  $\|\Pi_{>N}v(t)\|_{H^{s_0}}^2$ .

Then by applying the Gröwnwall inequality in (54), we get a constant  $C_{q,n} > 0$  depending only on  $q$  and  $n$  such that

$$\|\Pi_{>N}v(t)\|_{\ell^2}^2 \leq \|\Pi_{>N}v(0)\|_{\ell^2}^2 e^{C_{q,n}t\varepsilon^{\frac{r}{8}}} + C_{q,n} \int_0^t e^{C_{q,n}(t-\tau)\varepsilon^{\frac{r}{8}}} \varepsilon^r d\tau.$$

Now recalling that that by (50)  $\|\Pi_{>N}v(0)\|_{\ell^2}^2 \leq \varepsilon^r$ , we get that (provided that  $\varepsilon$  is small enough)

$$\|\Pi_{>N}v(t)\|_{\ell^2}^2 \leq 2\varepsilon^r + 2T^*\varepsilon^r \leq \varepsilon^{\frac{r}{2}}.$$

Then plugging this estimate in (52) and using the bootstrap assumption, we get that

$$\partial_t \|\Pi_{>N}v(t)\|_{H^{s_0}}^2 \lesssim_{r,n} \varepsilon^{\frac{r}{8}(2-\frac{1}{s_0})} \varepsilon^{\frac{r}{4s_0}} + \varepsilon^{\frac{3r}{8}}.$$

and so using the bound (50) on  $\|\Pi_{>N}v(0)\|_{H^{s_0}}^2$ , we get that

$$\|\Pi_{>N}v(t)\|_{H^{s_0}}^2 \lesssim_{r,n} \varepsilon^r + T^*\varepsilon^{\frac{r}{4}}(\varepsilon^{\frac{r}{8}} + \varepsilon^{\frac{r}{8s_0}}) \lesssim_{r,n} \varepsilon^r + \varepsilon^{-\frac{r}{9s_0}}\varepsilon^{\frac{r}{4}}(\varepsilon^{\frac{r}{8}} + \varepsilon^{\frac{r}{8s_0}}) \ll \varepsilon^{\frac{r}{4}}.$$

This last estimate ensures that, provided that  $\varepsilon$  is small enough,  $\|\Pi_{>N}v(t)\|_{H^{s_0}} < \varepsilon^{\frac{r}{8}}$  which concludes the proof of Theorem 2.1.

## APPENDIX A. MULTILINEAR ESTIMATES

Let  $\mathcal{M}$  be a Riemannian manifold which is either  $\mathbb{R}^d$  or a boundaryless compact manifold. Let us consider an unbounded operator of the form  $-\Delta + V$  on  $L^2(\mathcal{M})$  in which  $V$  is a positive smooth potential. We assume that  $-\Delta + V$  is essentially self-adjoint and has pure-point spectrum.

Let us write a few words about Sobolev spaces. There is no issue to define the Sobolev space  $W^{N,\infty}(\mathcal{M})$  (eventually using a finite atlas in the manifold framework) as the space of functions such that the  $N$  first derivatives are bounded. For any fixed number  $s_0 > \frac{d}{2}$ , due to the Sobolev embedding  $H^{s_0}(\mathcal{M}) \subset L^\infty(\mathcal{M})$ , we may deduce the following one

$$(55) \quad H^{N+s_0}(\mathcal{M}) \subset W^{N,\infty}(\mathcal{M}).$$

By positivity and self-adjointness, one may define the following operator by functional calculus

$$P := \sqrt{-\Delta + V}.$$

In all the examples of our paper, the following inequalities hold true

$$(56) \quad \|u\|_{H^s(\mathcal{M})} \lesssim \|P^s u\|_{L^2(\mathcal{M})}, \quad \forall s \in \mathbb{N}, \forall u \in \text{Dom}(P^s).$$

We shall set an abstract definition of admissible operators but before stating it, let us write two examples of admissible operators we have in mind :

- $-\Delta + V$  on a boundaryless Riemannian compact manifold (see [DS06]). For (56), this is actually and equivalence. For complements about checking Definition A.1, one may see a proof due to J.M. Delort presented in the thesis of F. Monzani [Mon24, Section 2.3].
- $-\Delta + V$  on  $\mathbb{R}^d$  and  $V$  being a positive-definite quadratic form (see (23) and [Brun23, Section 3]). Note that, in this case, in the rest of the paper, the quadratic potential is denoted by  $Q$  instead of  $V$ .

**Definition A.1.** *The linear operator  $-\Delta + V$  is admissible if the following assumptions is true : there exists an integer  $\nu > 0$  such that for any  $a \in C^\infty(\mathcal{M})$  with bounded derivatives, the operator  $A$  of multiplication by  $a$  satisfies for any  $n \in \mathbb{N}$*

$$(57) \quad \sup_{0 \leq \ell \leq n} \|(\text{Ad}_P^\ell(A))\|_{L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})} \leq C_n \|a\|_{W^{n+\nu,\infty}(\mathcal{M})}$$

in which  $\text{Ad}_P : L \mapsto [P, L]$  is the adjoint mapping.

Let us now explain the consequences for multilinear estimates. Let us denote  $(I_k)_{k \in \mathbb{N}}$  a sequence of bounded intervals of  $\mathbb{R}$  which satisfy (for some constant  $C > 0$ ):

$$(58) \quad \min(I_k) = Ck + O(1) \quad k \rightarrow +\infty$$

$$(59) \quad \max(I_k) - \min(I_k) = O(1).$$

We shall need to select once and for all an element of  $I_k$ , say  $\theta_k \in I_k$ . We also have the asymptotic

$$\theta_k = Ck + O(1)$$

and thus there exists a positive constant  $\tau > 0$  fulfilling

$$(60) \quad \mathbf{k}_1 - \mathbf{k}_2 \geq \tau \quad \implies \quad \theta_{\mathbf{k}_1} - \theta_{\mathbf{k}_2} \gtrsim \mathbf{k}_1 - \mathbf{k}_2.$$

The goal of this part is to prove the next result in which, for all  $k \in \mathbb{N}$ ,  $\Pi_{\mathcal{E}_k} : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$  is the orthonormal projector on  $\mathcal{E}_k$  the subspace spanned by the eigenvectors of  $P$  whose associated eigenvalue belong to  $I_k$ , i.e.

$$\Pi_{\mathcal{E}_k} = \sum_{\lambda \in I_k} \Pi_\lambda$$

where  $\Pi_\lambda$  denotes the spectral projector associated with the eigenvalue  $\lambda$  of  $P$ .

**Proposition A.2.** *There exists  $\nu \geq 0$  such that for all  $q \geq 3$ , for all  $n \geq 4$ , all  $\mathbf{k} \in \mathbb{N}^q$  and all  $u^{(1)}, \dots, u^{(q)} \in L^2(\mathcal{M})$ , the following two sets of inequalities hold true*

$$(61) \quad \left| \int_{\mathcal{M}} (\Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}) \cdots (\Pi_{\mathcal{E}_{\mathbf{k}_q}} u^{(q)}) dx \right| \lesssim_{n,q} \frac{(\mathbf{k}_3^*)^{\nu+n} (\mathbf{k}_4^*)^\nu \cdots (\mathbf{k}_q^*)^\nu}{(\mathbf{k}_1^* - \mathbf{k}_2^* + \mathbf{k}_3^*)^n} \prod_{\ell=1}^q \|u^{(\ell)}\|_{L^2}$$

and

$$(62) \quad \left| \int_{\mathcal{M}} (\Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}) \cdots (\Pi_{\mathcal{E}_{\mathbf{k}_q}} u^{(q)}) dx \right| \lesssim_{n,q} \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2^*}{\mathbf{k}_1^*} \right)^n \left( \prod_{3 \leq \ell \leq q} \mathbf{k}_\ell^* \right)^\nu \prod_{\ell=1}^q \|u^{(\ell)}\|_{L^2}$$

where  $dx$  is the Riemannian volume on  $\mathcal{M}$  and  $\Gamma_{\mathbf{k}}$  is defined in (7).

**Remark A.3.** • A proof of (61) in the compact setting is done in [DS06] and involves the Helffer-Sjöstrand formula (precisely Lemma 1.2.3 of [DS06]). We give here an elementary alternative proof by exploiting that  $-\Delta + V$  and hence its square root  $P$  have pure-point spectrum.

• In [Brun23], a Bernstein inequality is also proved in order to follow the proof of Delort and Szeftel. But it turns out that we may overcome such an inequality thanks to the Sobolev embedding (55).

Let us begin the proof of Proposition A.2. We first need the following result.

**Lemma A.4.** *Assume  $\mathbf{k}_1 - \mathbf{k}_2 \geq \tau$  as in (60). For any bounded self-adjoint operator  $A : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ , the following bound holds true*

$$(63) \quad |\langle A \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle| \lesssim_n \sup_{0 \leq \ell \leq n} \|(\text{Ad}_P^\ell(A))\|_{L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})} \frac{\|u^{(1)}\|_{L^2(\mathcal{M})} \|u^{(2)}\|_{L^2(\mathcal{M})}}{(\mathbf{k}_1 - \mathbf{k}_2)^n}$$

where  $\langle \cdot, \cdot \rangle$  denotes the  $L^2(\mathcal{M}; \mathbb{R})$  scalar product.

*Proof.* We shall prove (63) by induction on  $n$ . For  $n = 0$ , (63) is merely the Cauchy-Schwarz inequality. Let us assume that (63) is true for  $n$  and let us prove it for  $n + 1$ . For any eigenvalue  $\lambda$  of  $P = \sqrt{-\Delta + V}$ , let us denote by  $\Pi_\lambda$  the spectral projector associated with the eigenvalue  $\lambda$  of  $P$ . By definition, we may write

$$\Pi_{\mathcal{E}_{\mathbf{k}_1}} = \sum_{\lambda \in I_{\mathbf{k}_1}} \Pi_\lambda \quad \text{and} \quad P \Pi_\lambda = \lambda \Pi_\lambda$$

which may be used to reformulate

$$\begin{aligned}
\langle [A, P] \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle &= \langle AP \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle - \langle P A \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle \\
&= \langle P \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, A \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle - \langle A \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, P \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle \\
&= \sum_{\substack{\lambda \in I_{\mathbf{k}_1} \\ \mu \in I_{\mathbf{k}_2}}} (\lambda - \mu) \langle \Pi_{\lambda} u^{(1)}, A \Pi_{\mu} u^{(2)} \rangle.
\end{aligned}$$

We now formalize the idea that the eigenvalues in  $I_{\mathbf{k}_1}$  and  $I_{\mathbf{k}_2}$  can be compared to  $\theta_{\mathbf{k}_1}$  and  $\theta_{\mathbf{k}_2}$ . More precisely, the last right-hand side can be written  $S_0 + S_1 + S_2$  with

$$\begin{aligned}
S_0 &:= \sum_{\substack{\lambda \in I_{\mathbf{k}_1} \\ \mu \in I_{\mathbf{k}_2}}} (\theta_{\mathbf{k}_1} - \theta_{\mathbf{k}_2}) \langle \Pi_{\lambda} u^{(1)}, A \Pi_{\mu} u^{(2)} \rangle = (\theta_{\mathbf{k}_1} - \theta_{\mathbf{k}_2}) \langle \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, A \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle, \\
S_1 &:= \sum_{\substack{\lambda \in I_{\mathbf{k}_1} \\ \mu \in I_{\mathbf{k}_2}}} (\lambda - \theta_{\mathbf{k}_1}) \langle \Pi_{\lambda} u^{(1)}, A \Pi_{\mu} u^{(2)} \rangle = \underbrace{\left\langle \sum_{\lambda \in I_{\mathbf{k}_1}} (\lambda - \theta_{\mathbf{k}_1}) \Pi_{\lambda} u^{(1)}, A \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \right\rangle}_{:=v^{(1)}}, \\
S_2 &:= \sum_{\substack{\lambda \in I_{\mathbf{k}_1} \\ \mu \in I_{\mathbf{k}_2}}} (\theta_{\mathbf{k}_2} - \mu) \langle \Pi_{\lambda} u^{(1)}, A \Pi_{\mu} u^{(2)} \rangle = \left\langle A \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \underbrace{\sum_{\mu \in I_{\mathbf{k}_2}} (\theta_{\mathbf{k}_2} - \mu) \Pi_{\mu} u^{(2)}}_{:=v^{(2)}} \right\rangle.
\end{aligned}$$

In the previous formulas, we remark that  $S_1$  and  $S_2$  can be written as

$$\begin{aligned}
S_1 &= \langle v^{(1)}, A \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle \quad \text{with} \quad \Pi_{\mathcal{E}_{\mathbf{k}_1}} v^{(1)} = v^{(1)}, \\
S_2 &= \langle A \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, v^{(2)} \rangle \quad \text{with} \quad \Pi_{\mathcal{E}_{\mathbf{k}_2}} v^{(2)} = v^{(2)}.
\end{aligned}$$

Moreover, thanks to (59), the following bounds hold true

$$(64) \quad \|v^{(1)}\|_{L^2(\mathcal{M})} \lesssim \|u^{(1)}\|_{L^2(\mathcal{M})} \quad \text{and} \quad \|v^{(2)}\|_{L^2(\mathcal{M})} \lesssim \|u^{(2)}\|_{L^2(\mathcal{M})}.$$

Since  $S_0$  contains the inner product we want to control in (63), we bound as follows

$$\begin{aligned}
S_0 &= -S_1 - S_2 + \langle [A, P] \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle \\
|S_0| &\leq |\langle \Pi_{\mathcal{E}_{\mathbf{k}_1}} v^{(1)}, A \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle| + |\langle A \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \Pi_{\mathcal{E}_{\mathbf{k}_2}} v^{(2)} \rangle| + |\langle [A, P] \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle|.
\end{aligned}$$

Note that if the commutator operator  $[A, P]$  is not bounded then (63) is of no interest since the upper bound is infinite. So there is no loss of generality to assume that  $[A, P]$  is bounded. Then we invoke the induction assumption (63) at rank  $n$  to get the upper bound

$$\begin{aligned}
|S_0| &\lesssim_n \sup_{0 \leq \ell \leq n} \|(\text{Ad}_P^\ell(A))\|_{L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})} \frac{\|v^{(1)}\|_{L^2(\mathcal{M})} \|u^{(2)}\|_{L^2(\mathcal{M})} + \|u^{(1)}\|_{L^2(\mathcal{M})} \|v^{(2)}\|_{L^2(\mathcal{M})}}{(\mathbf{k}_1 - \mathbf{k}_2)^n} \\
&\quad + \sup_{0 \leq \ell \leq n} \|(\text{Ad}_P^\ell([A, P]))\|_{L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})} \frac{\|u^{(1)}\|_{L^2(\mathcal{M})} \|u^{(2)}\|_{L^2(\mathcal{M})}}{(\mathbf{k}_1 - \mathbf{k}_2)^n}.
\end{aligned}$$

We get the conclusion (namely (63) for  $n+1$ ) by looking at (64), (60) and the equality

$$|S_0| = |(\theta_{\mathbf{k}_1} - \theta_{\mathbf{k}_2})| |\langle \Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}, A \Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)} \rangle|.$$

□

**Lemma A.5.** *Set the function  $a$  given by the following product*

$$(65) \quad a = (\Pi_{\mathcal{E}_{\mathbf{k}_3}} u^{(3)}) \cdots (\Pi_{\mathcal{E}_{\mathbf{k}_q}} u^{(q)})$$

Then following bound holds true

$$(66) \quad \|a\|_{W^{n,\infty}(\mathcal{M})} \lesssim_{n,q} \mathbf{k}_3^{\nu+n} \mathbf{k}_4^\nu \dots \mathbf{k}_q^\nu \times \prod_{\ell=3}^q \|\Pi_{\mathcal{E}_{k_\ell}} u^{(\ell)}\|_{L^2}.$$

*Proof.* Note that, thanks to (55) and upon increasing  $\nu$ , it is sufficient to prove

$$(67) \quad \|a\|_{H^n(\mathcal{M})} \lesssim_{n,q} \mathbf{k}_3^{\nu+n} \mathbf{k}_4^\nu \dots \mathbf{k}_q^\nu \times \prod_{\ell=3}^q \|\Pi_{\mathcal{E}_{k_\ell}} u^{(\ell)}\|_{L^2}.$$

For any  $s > 0$  and  $(u, w) \in (H^s(\mathcal{M}) \cap L^\infty(\mathcal{M}))^2$ , let us recall the following bound of a product of two functions (see [AG07, Proposition 2.1.1]):

$$\|uw\|_{H^s(\mathcal{M})} \lesssim_s \|u\|_{H^s(\mathcal{M})} \|w\|_{L^\infty(\mathcal{M})} + \|u\|_{L^\infty(\mathcal{M})} \|w\|_{H^s(\mathcal{M})}.$$

Remembering the Sobolev embedding  $H^{s_0}(\mathcal{M}) \subset L^\infty(\mathcal{M})$  once we fix a real number  $s_0 > \frac{d}{2}$  (say  $s_0 = d$ ), we deduce

$$\|uw\|_{H^s(\mathcal{M})} \lesssim_s \|u\|_{H^s(\mathcal{M})} \|w\|_{H^d(\mathcal{M})} + \|u\|_{H^d(\mathcal{M})} \|w\|_{H^s(\mathcal{M})}.$$

By using (56), (58) and (59), we get

$$\|(\Pi_{\mathcal{E}_k} u)\|_{H^s(\mathcal{M})} \lesssim (1+k)^s \|u\|_{L^2(\mathcal{M})}.$$

We then understand that  $\|a\|_{H^{n+\nu}(\mathcal{M})}$  is bounded (up to a multiplicative constant depending on  $n$ ) by

$$\left( \sum_{\ell=3}^q (1+k_\ell)^{n+\nu} \prod_{j \neq \ell} (1+k_j)^d \right) \times \prod_{\ell=3}^q \|\Pi_{\mathcal{E}_{k_\ell}} u^{(\ell)}\|_{L^2}$$

and we immediately get (67).  $\square$

In view of Definition A.1, we have the following corollary

**Corollary A.6.** *The first set of inequalities (61) hold true.*

*Proof.* Upon permuting the terms, one may assume  $\mathbf{k}_1 \geq \dots \geq \mathbf{k}_q$ . We distinguish two cases.

$\triangleright$  Case 1:  $\mathbf{k}_1 - \mathbf{k}_2 \geq \max(\mathbf{k}_3, \tau)$ . In particular, we have  $\mathbf{k}_1 - \mathbf{k}_2 \geq \tau$ . Let  $A$  be the bounded operator by the product function  $a$  given in (65). We now use (63) and (57) to get

$$(68) \quad \left| \int_{\mathcal{M}} (\Pi_{\mathcal{E}_{k_1}} u^{(1)}) \dots (\Pi_{\mathcal{E}_{k_q}} u^{(q)}) dx \right| = |\langle A \Pi_{\mathcal{E}_{k_1}} u^{(1)}, \Pi_{\mathcal{E}_{k_2}} u^{(2)} \rangle| \|\Pi_{\mathcal{E}_{k_1}} u^{(1)}\|_{L^2(\mathcal{M})} \|\Pi_{\mathcal{E}_{k_2}} u^{(2)}\|_{L^2(\mathcal{M})} \\ \lesssim_n \frac{\|a\|_{W^{n+\nu,\infty}(\mathcal{M})}}{(\mathbf{k}_1 - \mathbf{k}_2)^n} \|\Pi_{\mathcal{E}_{k_1}} u^{(1)}\|_{L^2(\mathcal{M})} \|\Pi_{\mathcal{E}_{k_2}} u^{(2)}\|_{L^2(\mathcal{M})}.$$

Again upon modifying  $\nu$  and looking at (66), we then obtain

$$(69) \quad \left| \int_{\mathcal{M}} (\Pi_{\mathcal{E}_{k_1}} u^{(1)}) \dots (\Pi_{\mathcal{E}_{k_q}} u^{(q)}) dx \right| \lesssim_{n,q} \frac{\mathbf{k}_3^{\nu+n} \mathbf{k}_4^\nu \dots \mathbf{k}_q^\nu}{(\mathbf{k}_1 - \mathbf{k}_2)^n} \prod_{\ell=1}^q \|u^{(\ell)}\|_{L^2}.$$

Since our assumption implies  $\mathbf{k}_1 - \mathbf{k}_2 \geq \mathbf{k}_3$ , we may write

$$1 \leq \frac{\mathbf{k}_3 + \mathbf{k}_1 - \mathbf{k}_2}{\mathbf{k}_1 - \mathbf{k}_2} \leq 2.$$

Hence (69) is exactly the expected upper bound (61).



▷ *Case 2:*  $\max(\mathbf{k}_3, \tau) > \mathbf{k}_1 - \mathbf{k}_2$ . By using the notation (65), we may invoke the Cauchy-Schwarz inequality:

$$\left| \int_{\mathcal{M}} (\Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}) \cdots (\Pi_{\mathcal{E}_{\mathbf{k}_q}} u^{(q)}) dx \right| \leq \|(\Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)})\|_{L^2(\mathcal{M})} \|(\Pi_{\mathcal{E}_{\mathbf{k}_2}} u^{(2)})\|_{L^2(\mathcal{M})} \|a\|_{L^\infty(\mathcal{M})}.$$

The computations made in the proof of Lemma A.5 are again available (see (66) for  $n = 0$ ) and we obtain the following inequality

$$\left| \int_{\mathcal{M}} (\Pi_{\mathcal{E}_{\mathbf{k}_1}} u^{(1)}) \cdots (\Pi_{\mathcal{E}_{\mathbf{k}_q}} u^{(q)}) dx \right| \lesssim_{n,q} \mathbf{k}_3^\nu \mathbf{k}_4^\nu \cdots \mathbf{k}_q^\nu \prod_{\ell=1}^q \|u^{(\ell)}\|_{L^2}.$$

Under the assumption  $\max(\tau, \mathbf{k}_3) > \mathbf{k}_1 - \mathbf{k}_2$ , the last upper bound is equivalent to

$$\left( 1 + \frac{\mathbf{k}_1 - \mathbf{k}_2}{\mathbf{k}_3} \right)^{-n} \mathbf{k}_3^\nu \cdots \mathbf{k}_q^\nu \prod_{\ell=1}^q \|u^{(\ell)}\|_{L^2}$$

which is exactly the right-hand side of (61).  $\square$

The second set of inequalities (62) is finally a consequence of the following lemma (by keeping in mind that  $n$  may be chosen arbitrary large in (61)).

**Lemma A.7.** *Let us consider  $q \geq 3$ ,  $n \geq 4$ ,  $\nu \geq 0$ ,  $\mathbf{k}_1 \geq \cdots \geq \mathbf{k}_q$ , then the following inequalities hold*

$$\frac{\mathbf{k}_3^{\nu+n} \mathbf{k}_4^\nu \cdots \mathbf{k}_q^\nu}{(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3)^n} \lesssim_{q,n} \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_2}{\mathbf{k}_1} \right)^{n-3} \mathbf{k}_3^{\nu'} \cdots \mathbf{k}_q^{\nu'}$$

where  $\nu' = \nu + 3$ .

*Proof.* Without loss of generality, we assume that  $\nu = 0$ . First, we note that

$$\frac{\mathbf{k}_3}{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3} \lesssim_q \frac{\mathbf{k}_3}{\mathbf{k}_1 - \mathbf{k}_2 + q\mathbf{k}_3} \lesssim_q \frac{\mathbf{k}_3}{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 + \cdots + \mathbf{k}_q}.$$

Looking at the definition of  $\Gamma_{\mathbf{k}}$  in (7), it follows that

$$\begin{aligned} \frac{\mathbf{k}_3^n}{(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3)^n} &\lesssim_{q,n} \left( \frac{\mathbf{k}_3}{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3} \right)^3 \left( \frac{\mathbf{k}_3}{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3} \right)^{n-3} \\ &\lesssim_{q,n} \mathbf{k}_3^3 \Gamma_{\mathbf{k}} \left( \frac{\mathbf{k}_3}{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3} \right)^{n-3}. \end{aligned}$$

Since the function  $t \mapsto \frac{t}{\mathbf{k}_1 - \mathbf{k}_2 + t}$  is non-decreasing for  $t \in [0, \mathbf{k}_2]$ , its highest value is  $\frac{\mathbf{k}_2}{\mathbf{k}_1}$ .  $\square$

## REFERENCES

- [AG07] S. ALINHAC, P. GÉRARD, *Pseudo-differential operators and the Nash-Moser Theorem*, American Mathematical Society, volume 82, (2007)
- [BCD11] H. BAHOURI, J.-Y. CHEMIN, R. DANCHIN *Fourier Analysis and Nonlinear Partial Differential Equations*, *Grundlehren der mathematischen Wissenschaften*, Springer-Verlag Berlin Heidelberg, 2011
- [Bam03] D. BAMBUSI, *Birkhoff Normal Form for Some Nonlinear PDEs*, *Commun. Math. Phys.* (2003) 234: 253.
- [BDGS07] D. BAMBUSI, J.M. DELORT, B. GRÉBERT, J. SZEFTTEL, *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, *Comm. Pure Appl. Math.*, 60: 1665-1690 (2007).
- [BFLM24] D. BAMBUSI, R. FEOLA, B. LANGELLA, F. MONZANI, *Almost global existence for some Hamiltonian PDEs on manifolds with globally integrable geodesic flow*, [arXiv:2402.00521](https://arxiv.org/abs/2402.00521), to appear in *Nonlinearity* (2024)
- [BFM24] D. BAMBUSI, R. FEOLA, R. MONTALTO, *Almost global existence for some Hamiltonian PDEs with small Cauchy data on general tori*, *Commun. Math. Phys.*, 405, 15 (2024).

- [BG06] D. BAMBUSI, B. GRÉBERT, *Birkhoff normal form for partial differential equations with tame modulus*, *Duke Math. J.* 135 (2006), no. 3, 507–567.
- [BL22] D. BAMBUSI, B. LANGELLA, *Growth of Sobolev norms in quasi integrable quantum systems*, arXiv:2202.04505, to appear in *Annales scientifiques de l'École normale supérieure* (2022).
- [BC24] J. BERNIER, N. CAMPS, *Long time stability for cubic nonlinear Schrödinger equations on non-rectangular flat tori*, arXiv:2402.04122, (2024)
- [BFG20a] J. BERNIER, E. FAOU, B. GRÉBERT, *Rational normal forms and stability of small solutions to nonlinear Schrödinger equations*, *Annals of PDE* 6, no. 14 (2020)
- [BFG20b] J. BERNIER, E. FAOU, B. GRÉBERT, *Long time behavior of the solutions of NLW on the  $d$ -dimensional torus*, *Forum of Mathematics, Sigma* 8, E12, (2020).
- [BG21] J. BERNIER, B. GRÉBERT, *Long time dynamics for generalized Korteweg-de Vries and Benjamin-Ono equations*, *Arch. Rational Mech. Anal.* 241, 1139–1241 (2021)
- [BG25] J. BERNIER, B. GRÉBERT, *Almost global existence for some nonlinear Schrödinger equations on  $\mathbb{T}^d$  in low regularity*, *Annales de l'Institut Fourier* (2025)
- [BD17] M. BERTI, J.M. DELORT, *Almost global solutions of capillary-gravity water waves equations on the circle*, *UMI Lecture Notes, 2017* (awarded UMI book prize 2017).
- [BMM24] M. BERTI, A. MASPERO, F. MURGANTE, *Hamiltonian Birkhoff normal form for gravity-capillary water waves with constant vorticity: almost global existence*, *Ann. PDE* 10, 22 (2024).
- [BS71] J. BOCHNAK, J. SICIĄK, *Polynomials and multilinear mappings in topological vector-spaces*, *Studia Mathematica* 39.1 (1971): 59-76.
- [Bou00] J. BOURGAIN, *On diffusion in high-dimensional Hamiltonian systems and PDE*, *J. Anal. Math.* 80, 1–35 (2000)
- [Brun23] P. BRUN, *Partial Normal Form for the Semilinear Klein-Gordon Equation with Quadratic Potentials and Algebraic Non-resonant Masses*, *J Dyn Diff Equat* 35, 2641-2675 (2023).
- [BGT04] N. BURQ, P. GÉRARD, N. TZVETKOV, *Strichartz Inequalities and the Nonlinear Schrödinger Equation on Compact Manifolds*, *American Journal of Mathematics*, 126, no. 3 (2004): 569-605.
- [Del09] J.M. DELORT, *On long time existence for small solutions of semi-linear Klein-Gordon equations on the torus*, *J Anal Math*, 107, 161-194 (2009).
- [Del15] J.M. DELORT, *Quasi-linear perturbations of Hamiltonian Klein-Gordon equations on spheres*. *Memoirs of the American Mathematical Society*, 234 (2015), no. 1103, vi+80 pp.
- [DS04] J.M. DELORT, J. SZEFTTEL, *Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres*, *International Mathematics Research Notices*, vol. 2004, no. 37, pp. 1897-1966, 2004.
- [DS06] J.M. DELORT, J. SZEFTTEL, *Long-Time Existence for Semi-Linear Klein-Gordon Equations with Small Cauchy Data on Zoll Manifolds*, *American Journal of Mathematics*, vol. 128, no. 5, 2006, pp. 1187-1218.
- [DI17] J.M. DELORT, R. IMEKRAZ, *Long-time existence for the semilinear Klein-Gordon equation on a compact boundary-less Riemannian manifold*, *Communications in Partial Differential Equations*, 42(3), pages 388-416, 2017.
- [FGL13] E. FAOU, L. GAUCKLER, C. LUBICH, *Sobolev Stability of Plane Wave Solutions to the Cubic Nonlinear Schrödinger Equation on a Torus*. *Communications in Partial Differential Equations*, 38:1123-1140, 2013.
- [FGI23] R. FEOLA, B. GRÉBERT, F. IANDOLI, *Long time solutions for quasi-linear Hamiltonian perturbations of Schrödinger and Klein-Gordon equations on tori*, *Analysis & PDE*, 16 (2023),1133–1203.
- [FI21] R. FEOLA, F. IANDOLI, *Long time existence for fully nonlinear NLS with small Cauchy data on the circle*, *Annali della Scuola Normale Superiore di Pisa (Classe di Scienze)*, 2021: vol. XXII, 1.
- [GIP09] B. GRÉBERT, R. IMEKRAZ AND É. PATUREL, *Normal forms for semilinear quantum harmonic oscillators* *Commun. Math. Phys.* 291, 763–798 (2009).
- [Hor68] L. HÖRMANDER, *The spectral function of an elliptic operator*, *Acta Math* 121, 193-218, 1968.
- [IO22] R. IMEKRAZ AND E.M. OUHABAZ, *Bernstein inequalities via the heat semigroup* *Math. Ann.* 382, 783-819 (2022).
- [Mon24] F. MONZANI, *Normal form methods for some non linear Hamiltonian PDEs in higher dimension*, *Pdh thesis*, 2024.
- [RS80] M. REED AND B. SIMON, *Methods of Modern Mathematical Physics Volume 4*, Academic Press (1980).
- [YZ04] K. YAJIMA, G. ZHANG: *Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity.*, *J. Diff. Eq.* 202, 81-101, 2004.
- [YZ14] X. YUAN, J. ZHANG, *Long Time Stability of Hamiltonian Partial Differential Equations*, *SIAM Journal on Mathematical Analysis*, 2014 46:5, 3176-3222.

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