On numerical reflected waves in lattice Boltzmann schemes

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Abstract: In this contribution, we study the transmission of a one dimensional acoustic wave between two fluid media with the classical lattice Boltzmann scheme. The two media have the same hydrodynamic equations but different equilibrium distributions. We take here the case where the incident wave is normal to the interface. The theoretical modal study of this problem shows the presence of a reflected wave and Knudsen modes localised at the interface. This analysis leads to results in good agreement with numerical simulations.

Keywords: lattice Boltzmann; acoustic propagation; acoustic reflection.

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1 Introduction

Our motivation is to demonstrate that artificial phenomena can occur at the interface between two Lattice Boltzmann fluids that are used to simulate the same physical fluid in the large scale limit. We first briefly recall the Lattice Boltzmann Equation using moments, then we study theoretically an interface between two D2Q9 media. We find three families of waves: the acoustics, the Knudsen and the transverse ones. For normal incidence transverse and longitudinal waves decouple and we determine the spatial behaviour of the different waves and give a generalisation of Fresnel formula for the simple D1Q3 model. In section four we compare the theoretical results to the numerical one and analyse the Knudsen modes generated at the interface.

2 D2Q9 scheme

We analyse the LBE model (Benzi et al., 1992; Qian et al., 1992)

$$f_i(x + v_i\Delta t, t + \Delta t) = f_i(x, t) + Q_i(f)(x, t), \quad 0 \le i \le 8, \tag{1}$$

where $Q_i(f)(x,t) = \sum_{j=0}^{8} S_{i,j}(f_j - f_j^{eq})(x,t)$ and S is the matrix collision, using moments for the collision step. For the D2Q9 model, we consider a regular lattice $\mathcal{L}$ parametrised by a space step Δx, composed by a set $\mathcal{L}^0 \equiv \{x_j \in (\Delta x\mathbb{Z}) \times (\Delta x\mathbb{Z})\}$ of nodes or vertices. We define Δt as a small time step of the evolution of LBE and let the celerity $\lambda \equiv \frac{\Delta x}{\Delta t}$. We choose the velocities $v_i, i \in (0 \ldots 8)$ such that $v_i \equiv c_i \frac{\Delta x}{\Delta t} = c_i\lambda$, where the family of vectors c_i is defined by:

$$c_i = \begin{cases} (0,0), & i = 0, \\ \left(\cos\left((i-1)\frac{\pi}{2}\right), \sin\left((i-1)\frac{\pi}{2}\right)\right), & i = 1, \ldots, 4, \\ \left(\cos((2i-9)\frac{\pi}{4}\right), \sin\left((2i-9)\frac{\pi}{4}\right)\right), & i = 5, \ldots, 8. \end{cases}$$

We note that the LBE scheme given by (1) can be written as follows (Dubois, 2008):

$$f_i(x_j, t + \Delta t) = f_i^*(x_j - v_i\Delta t, t), \quad 0 \le i \le 8, \tag{2}$$

where the superscript $*$ denotes post-collision quantities. Therefore during each time increment Δt there are two fundamental steps: collision and advection.

- Following d'Humières (1992), the collision step is defined in the space of moments. We consider the moments obtained by orthogonalisation from the conserved moments: density (ρ), flux of linear momentum (j_x and j_y) and the non-conserved moments: energy (e), square of energy (ϵ), components of the stress tensor (p_{xx} and p_{xy}) and flux of kinetic energy (q_x and q_y). The above non-conserved moments relax following:

  $$m_k^* = (1 - s_k)m_k + s_k m_k^{eq}, \quad 3 \le k \le 8,$$

  where $s_k \equiv \frac{\Delta t}{\tau_k}$ is the relaxation ratio and τ_k is the relaxation time. The relaxation rates s_k are not necessarily identical as in the so called BGK case (Qian et al., 1992).
- The advection step describes the motion of a particle which has collided in node $x_j - v_i\Delta t$ having the velocity v_j and goes to the j^{th} neighbouring node x_j.

3 Interface between two D2Q9 media

We consider two domains $\Omega_1 \equiv \{(x,y); x < 0\}$, $\Omega_2 \equiv \{(x,y); x > 0\}$ and the interface $\Sigma \equiv \{(x,y); x = 0\}$. We suppose that we have the following classical acoustics problem in each domain:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div} j = 0, \\ \dfrac{\partial j_x}{\partial t} + c_s^2 \dfrac{\partial \rho}{\partial x} - \zeta \dfrac{\partial(\operatorname{div} j)}{\partial x} - \nu \triangle j_x = 0, \\ \dfrac{\partial j_y}{\partial t} + c_s^2 \dfrac{\partial \rho}{\partial y} - \zeta \dfrac{\partial(\operatorname{div} j)}{\partial y} - \nu \triangle j_y = 0, \end{cases} \tag{3}$$

where c_s is the celerity of sound and ζ, ν the bulk and shear kinematic viscosities and we neglect any nonlinear effects. To simulate this equation with LBE we have to fix the equilibrium moments as follows: $e^{eq} = -2\rho$, $\epsilon^{eq} = \alpha_\epsilon \rho$, $q_x = -j_x$, $q_y = -j_y$, $p_{xx}^{eq} = 0$, $p_{xy}^{eq} = 0$ and $s_{p_{xx}} = s_{p_{xy}}$. Hence we have the sound celerity $c_s^2 = \frac{\lambda^2}{3}$, the bulk viscosity $\zeta = \frac{\lambda^2\Delta t}{3}(\frac{1}{s_\epsilon} - \frac{1}{2})$ and shear viscosity $\nu = \frac{\lambda^2\Delta t}{3}(\frac{1}{s_{p_{xx}}} - \frac{1}{2})$ (d'Humières, 1992). We remark here that the coefficient α_ϵ of the moment ϵ equilibrium value does not appear in the hydrodynamics equations. That is why we shall study the transmission of an acoustic wave between the two media Ω_1 and Ω_2 with different coefficients α_ϵ. Intuitively no hydrodynamic reflected wave occurs. So we take $\epsilon^{eq} = \alpha_\epsilon \rho$ in Ω_1 and $\widetilde{\epsilon}^{eq} = \widetilde{\alpha}_\epsilon \rho$ in Ω_2.

To study theoretically this problem we perform a modal analysis of the LBE scheme for an harmonic solution.

3.1 Modal study

To simplify the analysis we consider the case of normal incidence to the interface Σ. The wave number $k \equiv (k_x, 0)$ is therefore parallel to the x-axis. Let $f(x,t) = e^{i(\omega t - k.x)}\phi$ be a solution of the LBE scheme. Equation (1) then becomes in Fourier space:

$$f(x, t + \Delta t) = e^{i\omega\Delta t} f(x,t) = A(I + M^{-1}CM)f(x,t), \tag{4}$$

where M is the moment matrix, C the collision operator and A the advection operator matrix represented by the following diagonal matrix:

$$A = \operatorname{diag}\left(1, p, 1, \frac{1}{p}, 1, p, \frac{1}{p}, \frac{1}{p}, p\right), \quad \text{where } p \equiv e^{ik\Delta x}.$$

Note that p is a phase factor that is unknown when simulating an acoustics situation. The above equation is a finite difference equation which has a general solution at time $t = n\Delta t$:

$$\phi(x = m\Delta x, t = n\Delta t) = K^m z^n \phi_0, \tag{5}$$

where ϕ_0 is the initial state. Equation (4) can be written as:

$$z\, f(x,t) = G(p) f(x,t) \tag{6}$$

where $z = \mathrm{e}^{(i\omega\Delta t)}$ and $G(p) \equiv A(I + M^{-1}CM)$ is the global operator of the LBE scheme. In our problem the frequency ω is imposed, so we search p solution of the following dispersion equation:

$$\det(G(p) - zI) = 0, \tag{7}$$

This equation is a polynomial function of degree 3 in $\left(p + \frac{1}{p}\right)$. We note that if we use the moment matrix $\widetilde{\mathrm{M}}$ defined by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\ -4 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 4 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix},$$

then the matrix $\widetilde{M}(G(p) - zI)\widetilde{M}^{-1}$ is a block diagonal matrix:

$$\widetilde{M}(G(p) - zI)\widetilde{M}^{-1} = \left(\begin{array}{c|c} D_1(p) & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & D_2(p) \end{array}\right)$$

Hence we have the property:

$$\begin{aligned} \det(G(p) - zI) &= \det(\widetilde{M}(G(p) - zI)\widetilde{M}^{-1}) \\ &= \det(D_1(p))\det(D_2(p)), \end{aligned}$$

so to solve Equation (7) we have to solve $\det(D_1(p)) = 0$ and $\det(D_2(p)) = 0$, which are polynomial functions of degree 2 and 1 in $\left(p + \frac{1}{p}\right)$ respectively. We find the six solutions: p_+ and p_-, $p_{K,1}$ and $p_{K,2}$, $p_{t,1}$ and $p_{t,2}$, which are functions of z, of the different parameters of the equilibrium and of the relaxation rates of the non-conserved moments. We also have the following asymptotic expansions in ω:

$$p_+ = 1 + i\frac{\omega}{c_s} + \mathrm{O}(\omega^2), \tag{8}$$

$$p_- = 1 - i\frac{\omega}{c_s} + \mathrm{O}(\omega^2), \tag{9}$$

$$p_{K,1} = \alpha_1 + \beta_1\omega + \mathrm{O}(\omega^2), \quad \text{where } \alpha_1 < -1, \tag{10}$$

$$p_{K,2} = \alpha_2 + \beta_2\omega + \mathrm{O}(\omega^2), \quad \text{where } -1 < \alpha_2 < 0, \tag{11}$$

$$p_{t,1} = 1 + (\alpha_{t,1} + i\beta_{t,1})\sqrt{\omega} + i\gamma_{t,1}\omega + \mathrm{O}(\omega\sqrt{\omega}), \tag{12}$$

$$p_{t,2} = 1 + (\alpha_{t,2} + i\beta_{t,2})\sqrt{\omega} + i\gamma_{t,2}\omega + \mathrm{O}(\omega\sqrt{\omega}). \tag{13}$$

Considering the ω-dependence of the previous solutions, we remark that p_+ and p_- are associated to acoustic waves which progress with speed $\pm c_s$, $p_{K,1}$ and $p_{K,2}$ are associated to Knudsen modes (Cornubert, 1991). The two solutions $p_{t,1}$ and $p_{t,2}$ are associated with transverse shear waves and they will play no role for the particular situation of incident acoustic waves normal to the boundary that are considered later. For the sake of completeness, we indicate that

$$(\alpha_{t,1} + i\beta_{t,1}) = \frac{1}{\sqrt{\nu}}\frac{(1+i)}{\sqrt{2}}$$

and

$$(\alpha_{t,2} + i\beta_{t,2}) = -\frac{1}{\sqrt{\nu}}\frac{(1+i)}{\sqrt{2}}.$$

The expression of the first terms in the expansion of the Knudsen phase factor $p_{K,1}$ and $p_{K,2}$, α_1 and α_2 are complex functions of relaxation rates s_i and α. But in the particular case of equal relaxation rates (i.e., $s_i = s$), we have:

$$p_{K,1} = \frac{1-s}{z} = (1-s) - i(1-s)\omega + \mathrm{O}(\omega^2),$$

$$p_{K,2} = \frac{z}{1-s} = \frac{1}{1-s} + i\frac{1}{1-s}\omega + \mathrm{O}(\omega^2).$$

From now on, we study cases where the wave vector of the incident wave k is normal to the interface so transverse waves do not contribute and thus will not be considered any more. Now we can find the eigenvectors ϕ_p of the matrix $G(p)$. Let ϕ_+ and ϕ_- be the eigenvectors associated to $p_\pm$ and $\phi_{K,1}$ and $\phi_{K,2}$ the eigenvectors associated to $p_{K,1}$ and $p_{K,2}$. Note that for general directions the simple separation of longitudinal and transverse modes does not exist.

3.2 *Analysis of the interface problem*

The solution f_l for the left hand side of the interface can be written as follows:

$$\begin{aligned} f_l = {} & z^n p_+^m \phi_+ + \beta_1 z^n p_-^m \phi_- + \eta_1 z^n p_{K,1}^m \phi_{K,1} \\ & + \delta_1 z^n p_{K,2}^m \phi_{K,2}, \end{aligned} \tag{14}$$

with incident and reflected acoustic waves. For the right hand side of the interface we have:

$$\begin{aligned} f_r = {} & \gamma_2 z^n \widetilde{p}_+^m \widetilde{\phi}_+ + \eta_2 z^n \widetilde{p}_{K,1}^m \widetilde{\phi}_{K,1} \\ & + \delta_2 z^n \widetilde{p}_{K,2}^m \widetilde{\phi}_{K,2}. \end{aligned} \tag{15}$$

with only transmitted waves. As we focus on an interface located at $x = 0$, we set $\eta_2 = 0$ and $\delta_1 = 0$ to prevent the existence of unphysical growing Knudsen modes on either sides of the boundary.

We have the following four unknown coefficients:

- β_1, γ_2 which determine the coefficients of reflection $r = \beta_1 \frac{\langle \phi_-, j_x \rangle}{\langle \phi_+, j_x \rangle}$ and transmission $t = \gamma_2 \frac{\langle \widetilde{\phi}_+, j_x \rangle}{\langle \phi_+, j_x \rangle}$
- η_1, δ_2 which determine the amplitude of the Knudsen modes.

The value of the above coefficients will be determined by studying one step of the LBE scheme in the nodes closest to the interface ($x_l = -\frac{\Delta x}{2}$ is the node on the left hand side of the interface and $x_r = \frac{\Delta x}{2}$ is the node on the right hand side of the interface).

So we write the advection part of the LBE scheme which is described by Equation (2) and we use Equations (14), (15) and $z\phi = A(I + M^{-1}CM)\phi$ (i.e., ϕ is an eigenvector). This leads to:

- For $i = 3$ and $i = 6$ in $x_l = -\frac{\Delta x}{2}$:

$$\sqrt{p_+}\phi_3^+ + \beta_1\sqrt{p_-}\phi_3^- + \eta_1\sqrt{p_{K,1}}\phi_3^{K,1} = \gamma_2\sqrt{\widetilde{p}_+}\widetilde{\phi}_3^+ + \delta_2\sqrt{\widetilde{p}_{K,2}}\widetilde{\phi}_3^{K,2}, \tag{16}$$

$$\sqrt{p_+}\phi_6^+ + \beta_1\sqrt{p_-}\phi_6^- + \eta_1\sqrt{p_{K,1}}\phi_6^{K,1} = \gamma_2\sqrt{\widetilde{p}_+}\widetilde{\phi}_6^+ + \delta_2\sqrt{\widetilde{p}_{K,2}}\widetilde{\phi}_6^{K,2}. \tag{17}$$

- For $i = 1$ and $i = 5$ in $x_r = \frac{\Delta x}{2}$:

$$\frac{1}{\sqrt{p_+}}\phi_1^+ + \beta_1\frac{1}{\sqrt{p_-}}\phi_1^- + \eta_1\frac{1}{\sqrt{p_{K,1}}}\phi_1^{K,1} = \gamma_2\frac{1}{\sqrt{\widetilde{p}_+}}\widetilde{\phi}_1^+ + \delta_2\frac{1}{\sqrt{\widetilde{p}_{K,2}}}\widetilde{\phi}_1^{K,2}, \tag{18}$$

$$\frac{1}{\sqrt{p_+}}\phi_5^+ + \beta_1\frac{1}{\sqrt{p_-}}\phi_5^- + \eta_1\frac{1}{\sqrt{p_{K,1}}}\phi_5^{K,1} = \gamma_2\frac{1}{\sqrt{\widetilde{p}_+}}\widetilde{\phi}_5^+ + \delta_2\frac{1}{\sqrt{\widetilde{p}_{K,2}}}\widetilde{\phi}_5^{K,2}. \tag{19}$$

So the system of Equations (16)–(19) provides the unknown coefficients $\beta_1, \gamma_2, \eta_1$ and δ_2 which are functions of the different eigenvectors and phase factors p. The expansion of β_1, γ_2, η_1 and δ_2 with respect to ω cannot be calculated analytically in the general case for Multiple Relaxation Time (MRT), except for some special cases (e.g., BGK or for D1Q3). Nevertheless, we can find the expansion with respect to ω for a fixed value of the different parameters for LBE scheme (i.e., relaxation rates, equilibrium moments ...) and by using various formal calculation software. To validate our theoretical calculation we shall compare it to the numerical results obtained by lattice Boltzmann automata.

3.3 *Interface of two D1Q3 media*

Let medium $\Omega_1 = \{x \in \mathbb{R}, x < 0\}$ and medium $\Omega_2 = \{x \in \mathbb{R}, x > 0\}$ be two domains with sound velocity and viscosity c_1, ν_1, and c_2, ν_2, respectively. So, if we have an incident wave f_i with a wave number k^+ in medium Ω_1, then there is a reflected wave f_r with a wave number k^- and a transmitted one f_t with a wave number $\widetilde{k}^+$ in medium Ω_2. The theoretical reflection coefficient is given by the Fresnel formula: $r_{th} = \frac{J_r}{J_i} = \frac{\widetilde{k}^+ - k^+}{k^+ + \widetilde{k}^+}$. With the help of the hydrodynamic modes of D1Q3, and the study of the LBE algorithm in the two nodes at left and right of the interface, we conduct an analysis similar to the one done above for a D2Q9 interface. We find the reflection coefficient (Tekitek, 2007)

$$r_{cal} = \frac{p_+ - \widetilde{p}_+}{1 - p_+\widetilde{p}_+}, \tag{20}$$

where $p_+ = e^{(ik^+\Delta x)}$ and $\widetilde{p}_+ = e^{(i\widetilde{k}^+\Delta x)}$.

It should be mentioned that in the D1Q3 scheme, the dispersion equation is a polynomial function of degree 1 in $\left(p + \frac{1}{p}\right)$ and there are only two modes of wave character (the acoustic waves). Hence we can calculate the coefficient of reflection and transmission.

If we do an asymptotic development of p_+ and $\widetilde{p_+}$ in ω, we find that

$$r_{th} = r_{cal} + \mathrm{O}(\omega^2) = \frac{c_1 - c_2}{c_1 + c_2} + \frac{i(\nu_1 c_2{}^2 - \nu_2 c_1^2)}{c_1 c_2 (c_1 + c_2)^2}\omega + \mathrm{O}(\omega^2).$$

4 Theoretical calculation vs. numerical results

In this section we compare the results obtained by the modal analysis method for D2Q9 scheme described in Section 2 and the results obtained by the numerical test of D2Q9 LBE scheme.

4.1 *Numerical tests*

We simulate the transmission of waves between two acoustic domains which are described by the same macroscopic problem (3) and have different equilibrium moments distribution with the D2Q9 LBE scheme. So let $\Omega = [0, l] \times [0, h]$, where $l = 4000$ and $h = 5$ be composed by $\Omega_- = [0, \frac{l}{2}] \times [0, h]$ and $\Omega_+ = [\frac{l}{2}, l] \times [0, h]$.

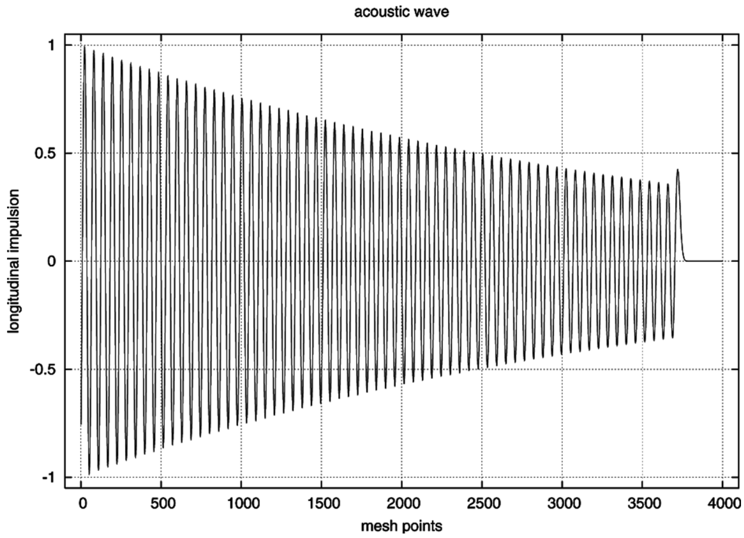
- In Ω_-, we take the following configuration for equilibrium moments:

$$e^{eq} = -2\rho, \quad \epsilon^{eq} = \alpha_\epsilon \rho, \quad q_x = -j_x, \quad q_y = -j_y, \quad p_{xx}^{eq} = 0, \quad p_{xy}^{eq} = 0.$$

- In Ω_+, we take the following configuration for equilibrium moments:

$$\widetilde{e}^{eq} = -2\rho, \quad \widetilde{\epsilon}^{eq} = \widetilde{\alpha}_\epsilon \rho, \quad \widetilde{q}_x = -j_x, \quad \widetilde{q}_y = -j_y, \quad \widetilde{p}_{xx}^{eq} = 0, \quad \widetilde{p}_{xy}^{eq} = 0.$$

Figure 1 J_x vs. N_x, wave transmission between $\Omega_- = \{x_i, i \in (0 \ldots 2000)\}$ where $\alpha_\epsilon = 1$ and $\Omega_+ = \{x_i, i \in (2000 \ldots 4000)\}$ where $\widetilde{\alpha}_\epsilon = \frac{1}{2}$ at time $T = 6464$



For the various relaxation rates s_i we first take the same values in the two domains. ($i.e$ $s_e = \widetilde{s}_e$, $s_\epsilon = \widetilde{s}_\epsilon$, $s_{q_x} = s_{q_y} = \widetilde{s}_{q_x} = \widetilde{s}_{q_y}$ and $s_{p_{xx}} = s_{p_{xy}} = \widetilde{s}_{p_{xx}} = \widetilde{s}_{p_{xy}}$). Here we take periodic boundary conditions for the y direction and a simple bounce back in the outer edges in $x = l$. In the inlet edges at $x = 0$ we impose an harmonic wave $J_x = \sin(\omega \Delta t)$ where $\omega = \frac{2\pi}{100}$ (implemented by bounce-back and application of $2J_x$ with appropriate weight factors for the velocities incoming in the computational domain). We take a fluid at rest for initial conditions and the total duration $T = n\Delta t$ of the simulations is chosen such that waves have not reached the outlet (see Fig. 1).

To determine the reflected wave and the Knudsen modes, we perform another simulation in the domain $\Omega_R = [0, l] \times [0, h]$. In this domain we take the same configuration as in the domain Ω_- with the same boundary conditions for the inlet edges at $x = 0$. This simulation gives us the reference solution. To see the reflected wave and the Knudsen modes we draw the difference between the flux J_x in Ω (the test case) and the flux J_x in Ω_R (the reference case) for the same number of time steps $= 6464$. It should be noted here that we have a small reflected wave between two hydrodynamically equivalent LBE. So in Figure 2 (for $x_i \in (1, 2, \ldots, 2000)$) we see a reflected hydrodynamic wave which has an amplitude of the order 10^{-6}. We also note that $J_x^{ref} - J_x^{test}$ is not null for $x > 2000$. Indeed there is a small change in the celerity of sound of order ω^2 due to the change of α_ϵ. Hence we have a slight difference between the spatial periods of J_x^{ref} and J_x^{test} which can be seen at $x > 2000$ in Figure 2. We have the same magnitude of reflected wave between two domains which have sound celerity variation $\Delta c_s \equiv (c_s - \widetilde{c}_s) = 1.6 \times 10^{-5}$ or shear viscosity variation $\Delta\nu \equiv (\nu - \widetilde{\nu}) = 1.75 \times 10^{-4}$.

To see the Knudsen modes we focus on the interface at $x = 2000$. These Knudsen modes are localised near the interface and decay with oscillations for successive space steps Δx as can be seen in Figure 3. Obviously this property

Figure 2 J_x vs. N_x, difference between the test case and reference case

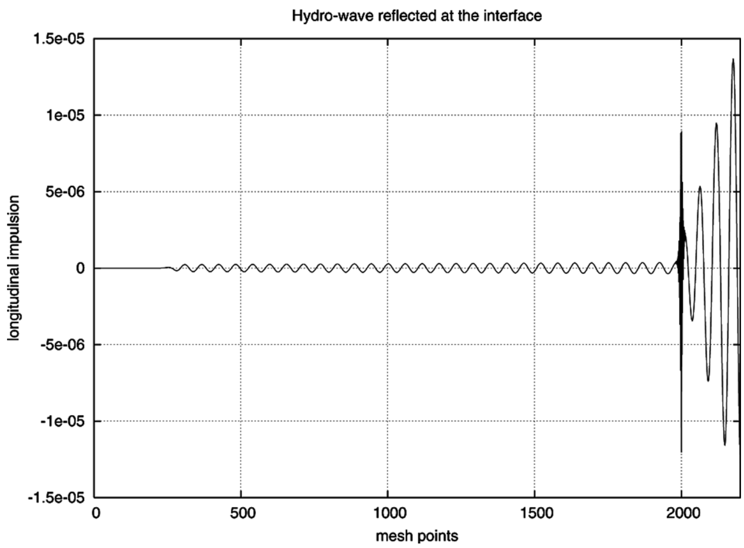


Figure 3 Details of Figure 2 showing Knudsen modes

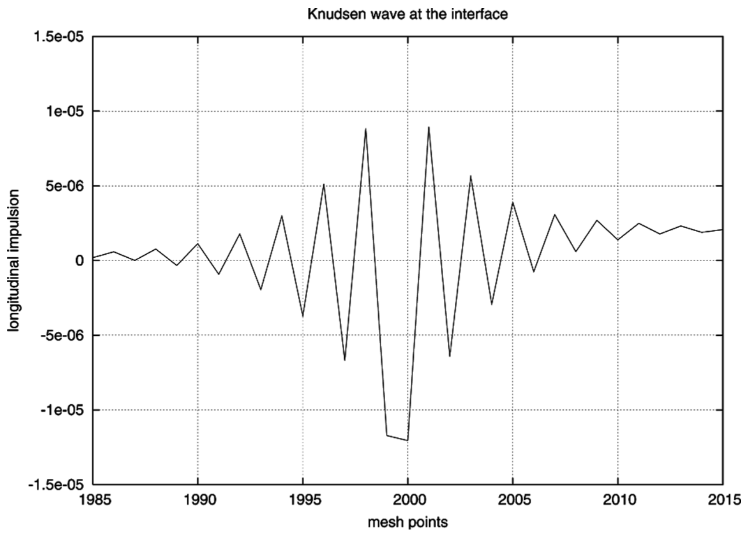


Figure 4 Knudsen amplitude η vs. $\widetilde{\alpha}_\epsilon$ equilibrium parameter for LBE in Ω_+, calculated by two methods

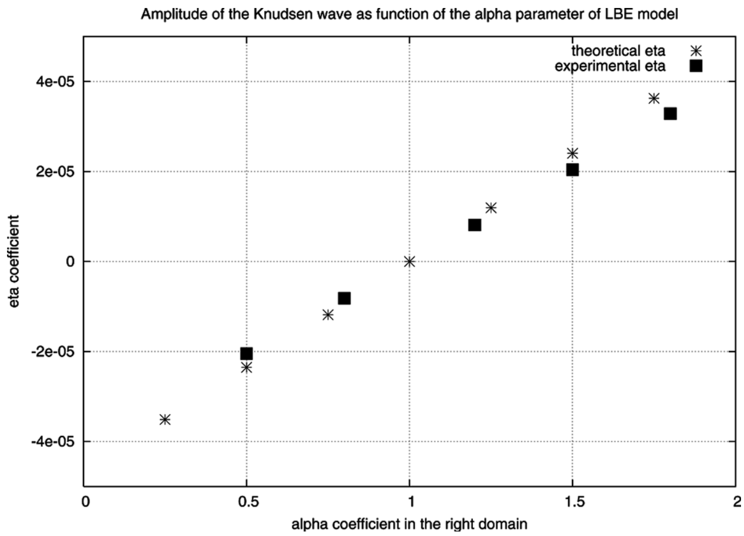
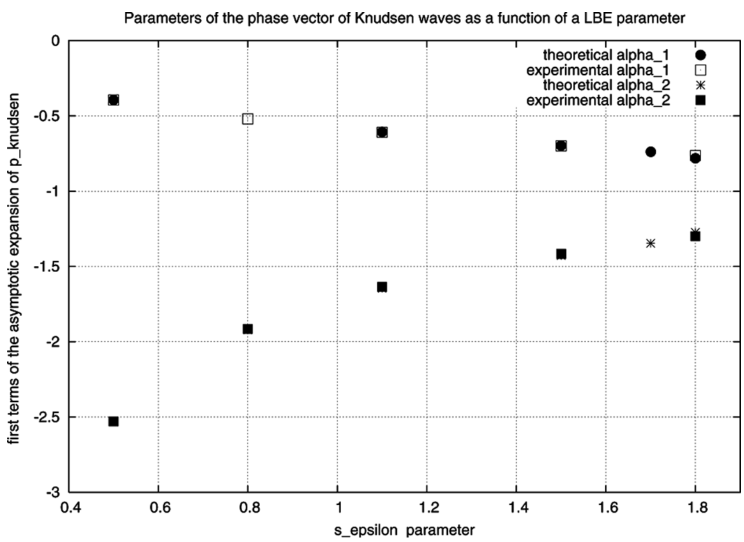


Figure 5 $\alpha_1,\ \alpha_2$ vs. the relaxation rate s_ϵ for $\epsilon^{eq} = \rho$ in Ω_- and $\widetilde{\epsilon}^{eq} = \frac{1}{2}\rho$ in Ω_+ calculated by two methods



of Knudsen modes is due to the first terms α_1 and α_2 of the asymptotic expansion (10) and (11) of $p_{K,1}$ and $p_{K,2}$.

4.2 *Comparison between numerical and theoretical results*

With the theoretical method introduced in Section 1, we have an estimation of the different coefficients of the reflected, transmitted and Knudsen modes. We compare the predicted values to the results obtained in the numerical tests.

In Figure 4 we plot the Knudsen amplitude vs. the coefficient $\widetilde{\alpha}_\epsilon$ of moment equilibrium $\widetilde{\epsilon}^{eq}$ in the domain Ω_+, for $\alpha_\epsilon = 1$ (i.e., $\epsilon^{eq} = \rho$ in the domain Ω_-). The curves show that the theoretical method is able to estimate the Knudsen amplitude. We find as shown in Figure 4, that the Knudsen amplitude η is a linear function of $\Delta\alpha_\epsilon \equiv (\alpha_\epsilon - \widetilde{\alpha}_\epsilon)$. For the particular case of equal relaxation rates, the modal analysis of the Equations (7), leads to the following Knudsen amplitude:

$$\eta_1 = \sqrt{1-s}(\alpha_\epsilon - \widetilde{\alpha}_\epsilon)\Delta t\omega + \mathrm{O}(\omega^2)$$

and

$$\delta_2 = -\sqrt{1-s}(\alpha_\epsilon - \widetilde{\alpha}_\epsilon)\Delta t\omega + \mathrm{O}(\omega^2).$$

We have also studied the dependence of the first terms α_1 and α_2 of the asymptotic expansion (10) and (11) of $p_{K,1}$ and $p_{K,2}$, in relaxation rates s_ϵ. Figure 5 plots α_1, α_2 vs. s_ϵ in the case where $\epsilon^{eq} = \rho$ (i.e., $\alpha_\epsilon = 1$) in Ω_- and $\widetilde{\epsilon}^{eq} = \frac{\rho}{2}$ (i.e., $\widetilde{\alpha}_\epsilon = 1$) in Ω_+. The curves show that the theoretical method gives a good estimate of the coefficients α_1 and α_2.

5 Conclusion

In this contribution, we have analysed the effect of a boundary between two lattice Boltzmann models that are hydrodynamically equivalent and have different equilibrium distribution. We show that the transmission of an acoustic wave between these two media generates a numerical hydrodynamics reflected wave and Knudsen modes.

We have used a theoretical method of analysis which is based on finding spatial modes of the Boltzmann scheme and a detailed study of the dynamical equations at local level at points close to the interface. Hence we get a connection formula. In the simple case of the D1Q3 model we find the generalisation of the classical Fresnel formula where we replace the wave number k by a phase factor $p = \mathrm{e}^{ik\Delta x}$. For D2Q9 models we have treated only incident waves normal to the interface that is chosen parallel to one of the microscopic velocities of the model and we have determined the amplitude of different waves generated at the interface. The extension for any incidence angle is more difficult due to the fact that the simple separation of longitudinal and transverse modes does not exist any more.

The results show that direct simulation of acoustic phenomena e.g., propagation in inhomogeneous media (as done in wave localisation studies) with LBE type techniques can be seriously affected by parasitic phenomena. They also show that work (Tekitek, 2007) aimed at transferring to LBE techniques ideas that are efficient in continuous CFD methods requires a detailed knowledge of the behavior of basic LBE models.

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