Towards perfectly matching layers for Lattice Boltzmann Equation

M.M. Tekitek [1], M. Bouzidi [2], F Dubois [1] [3] and P. Lallemand [*]

[1] *Numerical Analysis and Partial Differential Equations*

Department of Mathematics, Paris Sud University, Orsay, France.

[2] *Université Clermont 2, I.U.T., Ave. A. Briand 03107, Montluçon Cedex, France.*

[3] *Conservatoire National des Arts et Métiers, Paris, France.*

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Abstract

Following the efficient technique of Bérenger in classical computational fluid dynamics methods to avoid reflection of sound waves on the boundaries of the computational domain, we propose a new LBE scheme that behaves like a Bérenger medium for absorbing waves without reflection. This model is presented and its properties are discussed using the method of "equivalent equations". We also proposed a general method to introduce zero-order damping terms in Boltzmann schemes that are used to absorb the waves propagating in the Bérenger medium. Results of the simulation are discussed with theoretical interpretation in the case of waves incoming normal to the interface. We shall also show that the reflection of sound waves can be reduced simply by changing the "advection step" of the lattice Boltzmann algorithm on the nodes close to the interface.

Keywords: Lattice Boltzmann Equation, Bérenger medium, Perfectly Matched Layer, damping terms, reflected waves.

[*] mohamed-mahdi.tekitek@math.u-psud.fr, bouzidi@moniut.univ-bpclermont.fr, francois.dubois@math.u-psud.fr, pierre.lal@free.fr

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1 Introduction

Physical wave phenomena often take place in unbounded domains. The numerical study of such phenomena requires to create a finite computational region and thus to introduce artificial boundaries. The aim of these boundaries is to absorb all the waves and reduce the reflection of waves within the computational domain as much as possible.
Among the classical absorbing methodologies [3, 7, 1] we choose to simulate the perfectly matched layer method using the Lattice Boltzmann method.
The perfectly matched layer (PML) method was introduced by Bérenger [1] in the context of electromagnetic wave propagation by surrounding the truncated physical domain of interest with a buffer/sponge layer which has the property of absorbing all incoming waves without reflection for any frequency and any incident angle (see Fig. 1).

Figure 1: Left : Domain of interest Ω and buffer/sponge domain (PML), Right : Interface : Ω_- acoustics domain / Ω_+ PML domain

Hu [5] applies in (1996) the PML approach to aeroacoustic problem modeled with the linearized Euler equation for the domain of interest Ω_- (see Fig 1) :

$$
(1) \quad \left\{ \begin{array}{rcl}
\dfrac{\partial j_x}{\partial t} + \dfrac{\partial \rho}{\partial x} & = & 0, \\
\dfrac{\partial j_y}{\partial t} + \dfrac{\partial \rho}{\partial y} & = & 0, \\
\dfrac{\partial \rho}{\partial t} + \dfrac{\partial j_x}{\partial x} + \dfrac{\partial j_y}{\partial y} & = & 0,
\end{array} \right.
$$

where ρ is the fluid density and j_x, j_y are the flux of velocity components.

In the PML buffer Ω_+ (see Fig 1) we use the non-physical equations [5] :

$$(2) \quad \left\{ \begin{array}{rcl} \dfrac{\partial j_x}{\partial t} + \sigma j_x + \dfrac{\partial(\rho_x + \rho_y)}{\partial x} & = & 0, \\ \dfrac{\partial j_y}{\partial t} + \dfrac{\partial(\rho_x + \rho_y)}{\partial y} & = & 0, \\ \dfrac{\partial \rho_x}{\partial t} + \sigma \rho_x + \dfrac{\partial j_x}{\partial x} & = & 0, \\ \dfrac{\partial \rho_y}{\partial t} + \dfrac{\partial j_y}{\partial y} & = & 0, \end{array} \right.$$

where the coefficient σ is introduced for the absorption of waves in the PML. We will refer to it as zero-order damping term in this work and it will be assumed to be non negative. We note that when $\sigma = 0$, we are left with the original acoustics equations with : $\rho = \rho_x + \rho_y$.
We notice here that the mass ρ is assumed to be continuous at the interface between the domain of interest Ω_- and the PML Ω_+.

Our work is structured as follows. We first construct a Bérenger Lattice Boltzmann (BLB) scheme to model an absorbing medium without damping terms and we study the properties of this new model. Then we propose a method to simulate damping terms by changing the advection step. In section three we show numerical tests of an interface between classical D2Q9 medium and BLB medium. Finally in section five we propose a method to reduce reflected waves in the simple case of wave incident normal to the interface.

2 Bérenger Lattice Boltzmann scheme

In this section we construct the BLB scheme which has equations (2) as equivalent macroscopic equations up to order 1 relatively Δt (defined below). First we recall the classical D2Q9 [6] scheme.

2.1 Classical D2Q9 scheme

We consider the classical D2Q9 [8] model. Let $\mathcal{L}$ a regular lattice parametrized by a space step Δx, composed by a set $\mathcal{L}^0 \equiv \{x_j \in (\Delta x\mathbb{Z}) \times (\Delta x\mathbb{Z})\}$ of nodes or vertices. Δt is the time step of the evolution of LBE and $\lambda \equiv \frac{\Delta x}{\Delta t}$

is the elementary celerity. We choose the velocities v_i, $i \in (1\ldots9)$ such that $v_i \equiv c_i \frac{\Delta x}{\Delta t} = c_i\lambda$, where the family of vectors $\{c_i\}$ is defined by : $c = (0,0), (1,0), (0,1), (-1,0), (0,-1), (1,1), (-1,1), (-1,-1), (1,-1)$. The LBE is a mesoscopic method and deals with a small number of functions $\{f_i\}$ that can be interpreted as populations of fictitious "particles". The populations f_i evolve according to the LBE scheme which can be written as follows [2] :

$$(3) \quad f_i(x_j, t+\Delta t) = f_i^*(x_j - v_i\Delta t, t), \quad 1 \le i \le 9,$$

where the superscript $*$ denotes post-collision quantities. Therefore during each time increment Δt there are two fundamental steps : advection and collision.

• The advection step describes the motion of a particle which has collisioned in node $x_j - v_i\Delta t$ having the velocity v_j and goes to the j^{th} neighbouring node x_j.

• Following d'Humières [6], the collision step is defined in the space of moments. The 9 moments $\{m_\ell\}$ are obtained by a linear transformation of vectors f_j :

$$m_\ell = \sum_{j=1}^{9} M_{\ell j} f_j,$$

where the matrix $M \equiv (M_{\ell j})_{1\le \ell, j\le 9}$ is given by :

$$(4) \quad M = \begin{pmatrix} 0 & \lambda & 0 & -\lambda & 0 & \lambda & -\lambda & -\lambda & \lambda \\ 0 & 0 & \lambda & 0 & -\lambda & \lambda & \lambda & -\lambda & -\lambda \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ -4 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 4 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

The moments have an explicit physical significance [8] : $m_1 \equiv j_x$ and $m_2 \equiv j_y$ are x-momentum, y-momentum, $m_3 \equiv \rho$ is the density (density), m_4 and m_5 are diagonal stress and off-diagonal stress, m_6 is the energy, m_7 is related to

energy square, and m_8, m_9 are x-heat flux and y-heat flux. Note that we have changed the usual order of moments to simplify the introduction of the Bérenger Lattice Boltzmann scheme.
To simulate fluid problems, we conserve the flux momentum j_x, j_y and the density moment ρ in the collision step and obtain three macroscopic scalar equation. The other quantities (non-conserved moments) are assumed to relax towards equilibrium values m_ℓ^{eq} following :

$$(5)\quad m_\ell^* = (1 - s_\ell)m_\ell + s_\ell m_\ell^{eq}, \quad 4 \leq \ell \leq 9,$$

where s_ℓ ($s_\ell > 0$, for $\ell \geq 4$) are relaxation rates, not necessarily equal to a single value as in the so called BGK case [9]. The equilibrium values m_i^{eq} of the non conserved moments in equation (5) determine the macroscopic behavior of the scheme (*i. e.* equation (3)). Indeed with the following choice of equilibrium values (neglecting non-linear contributions) : $m_4^{eq} = 0$, $m_5^{eq} = 0$, $m_6^{eq} = -2\rho$, $m_7^{eq} = \rho$, $m_8^{eq} = -j_x$ and $m_9^{eq} = -j_y$ and using Taylor expansion [2] we find the acoustics equations up to order two in Δx :

$$(6)\quad \begin{aligned} \frac{\partial j_\alpha}{\partial t} + \frac{\lambda^2}{3}\frac{\partial \rho}{\partial x_\alpha} &= \lambda^2 \Delta t \,\frac{\sigma_6}{3}\,\frac{\partial(\mathrm{div} j)}{\partial x_\alpha} + \lambda^2 \Delta t \,\frac{\sigma_4}{3} \triangle j + \mathrm{O}(\Delta^2 x), \\ \frac{\partial \rho}{\partial t} + \mathrm{div} j &= \mathrm{O}(\Delta^2 x), \end{aligned}$$

where $\sigma_\ell \equiv \left(\frac{1}{s_\ell} - \frac{1}{2}\right)$, $\quad 4 \leq \ell \leq 9$, and in the case of $s_5 = s_4$. Values of the sound speed c_s, bulk viscosity ζ and shear viscosity ν are $c_s = \frac{\lambda}{\sqrt{3}}$, $\zeta = c_s^2 \Delta t \sigma_6$ and $\nu = \frac{\lambda^2 \Delta t}{3}\sigma_4$.

2.2 Bérenger Lattice Boltzmann scheme (BLB)

To have a perfectly matched layer for lattice Boltzmann method, we construct a Lattice Boltzmann scheme which models the buffer of Bérenger (BLB). At first we propose a scheme which has the acoustic PML equations (2) as macroscopic behavior without zero-order damping term (*i. e.* $\sigma = 0$). Later, we change the advection step of the BLB scheme to add the terms proportional to σ.
As there are four macroscopic equations (2) in the Bérenger scheme, we need to use four conserved quantities in the collision step. For simplicity, we keep

the classical D2Q9 velocity set (hopefully this will allow simple boundaries between the LBE and BLB domains), and we replace the list of moments generated with matrix M, by those generated with a new matrix M_B given below.

$$(7)\quad M_B = \begin{pmatrix} 0 & \lambda & 0 & -\lambda & 0 & \lambda & -\lambda & -\lambda & \lambda \\ 0 & 0 & \lambda & 0 & -\lambda & \lambda & \lambda & -\lambda & -\lambda \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ M^B_{41} & M^B_{42} & M^B_{43} & M^B_{44} & M^B_{45} & M^B_{46} & M^B_{47} & M^B_{48} & M^B_{49} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ -4 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 4 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \end{pmatrix},$$

Note that M and M_B differ only in the definition of the fourth moment, that we call $m_4^{'}$ and which will be conserved in collision (*i. e.* $s_4^{'} = 0$) to get a fourth macroscopic equation. Later we shall identify m_3 to $\rho \equiv \rho_x + \rho_y$ and $m_4^{'}$ to $\rho_x - \rho_y$.

To simplify later formula, we introduce coefficients $\gamma_{1 \dots 9}$ such that
$M^B_{41} = \gamma_3 - 4(\gamma_5 - \gamma_6)$,
$M^B_{42} = \lambda\gamma_1 + \gamma_3 + \gamma_4 - \gamma_6 - 2\gamma_7 - 2\gamma_8$,
$M^B_{43} = \lambda\gamma_2 + \gamma_3 - \gamma_4 - \gamma_6 - 2\gamma_7 - 2\gamma_9$,
$M^B_{44} = -\lambda\gamma_1 + \gamma_3 + \gamma_4 - \gamma_6 - 2\gamma_7 + 2\gamma_8$,
$M^B_{45} = -\lambda\gamma_2 + \gamma_3 - \gamma_4 - \gamma_6 - 2\gamma_7 + 2\gamma_9$,
$M^B_{46} = \lambda(\gamma_1 + \gamma_2) + \gamma_3 + \gamma_5 + 2\gamma_6 + \gamma_7 + \gamma_8 + \gamma_9$,
$M^B_{47} = \lambda(-\gamma_1 + \gamma_2) + \gamma_3 - \gamma_5 + 2\gamma_6 + \gamma_7 - \gamma_8 + \gamma_9$,
$M^B_{48} = -\lambda(\gamma_1 + \gamma_2) + \gamma_3 + \gamma_5 + 2\gamma_6 + \gamma_7 - \gamma_8 - \gamma_9$,
$M^B_{49} = \lambda(\gamma_1 - \gamma_2) + \gamma_3 - \gamma_5 + 2\gamma_6 + \gamma_7 + \gamma_8 - \gamma_9$.
We note that this corresponds to $M^B_{4\bullet} = M.(\gamma_1, \gamma_2, \dots, \gamma_9)^t$.

For the non conserved moments, we take new equilibrium values, $m_5^{eq} = 0$, $m_6^{eq} = a_x\rho_x + a_y\rho_y$, $m_7^{eq} = c_x\rho_x + c_y\rho_y$, $m_8^{eq} = \frac{c_1}{\lambda} j_x$ and $m_9^{eq} = \frac{c_2}{\lambda} j_y$.

We now determine the equivalent set of equations of the model defined above at first order in Δt and we try and identify these equations with the set of equations 2 with no linear damping ($\sigma = 0$). In addition we impose that the matrix M_B is invertible.

Using a first order Taylor expansion in Δt of the BLB scheme [2], we obtain

$$\frac{\partial j_x}{\partial t} + A_1\frac{\partial j_x}{\partial x} + A_2\frac{\partial j_y}{\partial x} + A_3\frac{\partial \rho}{\partial x} + A_4\frac{\partial(\rho_x-\rho_y)}{\partial x} = \mathrm{O}(\Delta t), \tag{8}$$

$$\frac{\partial j_y}{\partial t} + B_1\frac{\partial j_y}{\partial y} + B_2\frac{\partial j_x}{\partial y} + B_3\frac{\partial \rho}{\partial y} + B_4\frac{\partial(\rho_x-\rho_y)}{\partial y} = \mathrm{O}(\Delta t), \tag{9}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_x}{\partial y} = \mathrm{O}(\Delta t), \tag{10}$$

$$\begin{aligned}\frac{\partial(\rho_x-\rho_y)}{\partial t} + C_1\frac{\partial(\rho_x-\rho_y)}{\partial x} + C_2\frac{\partial(\rho_x-\rho_y)}{\partial y} + C_3\frac{\partial \rho}{\partial x} + C_4\frac{\partial \rho}{\partial y}\\ +C_5\frac{\partial j_x}{\partial x} + C_6\frac{\partial j_x}{\partial y} + C_7\frac{\partial j_y}{\partial x} + C_8\frac{\partial j_y}{\partial y} = \mathrm{O}(\Delta t).\end{aligned} \tag{11}$$

$$\text{where}\quad A_1 = \frac{-1}{2\gamma_4}\left(\gamma_1 + c_1\gamma_8\right), \quad A_2 = \frac{-1}{2\gamma_4}\left(\gamma_2 + c_2\gamma_9\right),$$

$$\begin{aligned}A_3 &= \frac{2}{3} - \frac{\gamma_3}{2\gamma_4} + \frac{a_x+a_y}{4}\left(\frac{1}{3} - \frac{\gamma_6}{\gamma_4}\right) - \frac{\gamma_7(c_x+c_y)}{4\gamma_4},\\ A_4 &= \frac{1}{2\gamma_4} + \frac{a_x-a_y}{4}\left(\frac{1}{3} - \frac{\gamma_6}{\gamma_4}\right) - \frac{\gamma_7(c_x-c_y)}{4\gamma_4},\end{aligned}$$

$$B_1 = \frac{1}{2\gamma_4}\left(\gamma_2 + c_2\gamma_9\right), \quad B_2 = \frac{1}{2\gamma_4}\left(\gamma_1 + c_1\gamma_8\right),$$

$$\begin{aligned}B_3 &= \frac{2}{3} + \frac{\gamma_3}{2\gamma_4} + \frac{a_x+a_y}{4}\left(\frac{1}{3} + \frac{\gamma_6}{\gamma_4}\right) + \frac{\gamma_7(c_x+c_y)}{4\gamma_4},\\ B_4 &= \frac{-1}{2\gamma_4} + \frac{a_x-a_y}{4}\left(\frac{1}{3} + \frac{\gamma_6}{\gamma_4}\right) + \frac{\gamma_7(c_x-c_y)}{4\gamma_4},\end{aligned}$$

$$C_1 = \frac{(a_x-a_y)}{2}\left(\frac{\gamma_1}{6} + \frac{\gamma_8}{3} + \frac{\gamma_6}{2\gamma_4}(2\gamma_8-\gamma_1)\right) + \frac{c_x-c_y}{2}\left(\frac{\gamma_8}{3} + \frac{\gamma_7}{2\gamma_4}(2\gamma_8-\gamma_1)\right),$$

$$C_2 = \frac{(a_x-a_y)}{2}\left(\frac{\gamma_2}{6} + \frac{\gamma_9}{3} + \frac{\gamma_6}{2\gamma_4}(\gamma_2-2\gamma_9)\right) + \frac{c_x-c_y}{2}\left(\frac{\gamma_9}{3} + \frac{\gamma_7}{2\gamma_4}(\gamma_2-2\gamma_9)\right),$$

$$
\begin{aligned}
C_3 = \frac{2\gamma_1}{3} + \frac{\gamma_3}{2\gamma_4}(2\gamma_8 - \gamma_1) \ &+ \ \frac{a_x + a_y}{2}\left(\frac{\gamma_8}{3} + \frac{\gamma_1}{6} + \frac{\gamma_6(2\gamma_8 - \gamma_1)}{2\gamma_4}\right) \\
&+ \ \frac{c_x + c_y}{2}\left(\frac{\gamma_8}{3} + \frac{\gamma_7}{2\gamma_4}(2\gamma_8 - \gamma_1)\right),
\end{aligned}
$$

$$
\begin{aligned}
C_4 = \frac{2\gamma_2}{3} + \frac{\gamma_3}{2\gamma_4}(-2\gamma_9 + \gamma_2) \ &+ \ \frac{a_x + a_y}{2}\left(\frac{\gamma_9}{3} + \frac{\gamma_2}{6} + \frac{\gamma_6(-2\gamma_9 + \gamma_2)}{2\gamma_4}\right) \\
&+ \ \frac{c_x + c_y}{2}\left(\frac{\gamma_9}{3} + \frac{\gamma_7}{2\gamma_4}(-2\gamma_9 + \gamma_2)\right),
\end{aligned}
$$

$$
C_5 = \gamma_3 + \gamma_6 + c_1(\gamma_6 + \gamma_7) + \frac{\gamma_4}{3}(1 - c_1) + \frac{\gamma_8\gamma_1}{2\gamma_4}(2 - c_1) + \frac{2c_1\gamma_8^2 - \gamma_1^2}{2\gamma_4},
$$

$$
C_6 = \frac{\gamma_5(2 + c_1)}{3} + \frac{1}{2\gamma_4}(c_1\gamma_8 + \gamma_1)(\gamma_2 - 2\gamma_9),
$$

$$
C_7 = \frac{\gamma_5(2 + c_2)}{3} - \frac{1}{2\gamma_4}(c_2\gamma_9 + \gamma_2)(\gamma_1 - 2\gamma_8),
$$

$$
C_8 = \gamma_3 + \gamma_6 + c_2(\gamma_6 + \gamma_7) - \frac{\gamma_4}{3}(1 - c_2) - \frac{\gamma_9\gamma_2}{2\gamma_4}(2 - c_1) - \frac{2c_2\gamma_9^2 - \gamma_2^2}{2\gamma_4}.
$$

The identification between a suitable linear combination of equations (8), (9), (10), (11) and the PML system (2) where $\sigma = 0$ leads to the following requirements :
$\gamma_1 = \gamma_2 = \gamma_8 = \gamma_9 = 0$,
$a_x = -4 + 6c_s^2$, $a_y = -4 + 6c_s^2$,
$c_x = \dfrac{(4\gamma_6 - 6\gamma_6 c_s^2 - \gamma_3 + 1)}{\gamma_7}$, $\quad c_y = \dfrac{(4\gamma_6 - 6\gamma_6 c_s^2 - \gamma_3 - 1)}{\gamma_7}$,
$c_1 = \dfrac{(3\gamma_3 + \gamma_4 + 3\gamma_6 - 3)}{(\gamma_4 - 3\gamma_6 - 3\gamma_7)}$, $\quad c_2 = \dfrac{(-3\gamma_3 + \gamma_4 - 3\gamma_6 - 3)}{(\gamma_4 + 3\gamma_6 + 3\gamma_7)}$.
For $\gamma_{3,4,5,6,7}$ we find two possible sets of solutions for $\gamma_{3,4,5,6,7}$:
i) $\gamma_3 = \gamma_6 + 2\gamma_7$, $\gamma_4 = 1$,
ii) $\gamma_5 = 0$.
Note that there are some free parameters left ($\gamma_{5,6,7}$ for the first case or $\gamma_{3,4,6,7}$ for the second one). To have a stable scheme, we have found that only the second is acceptable.

2.3 Dissipation properties of BLB scheme without damping terms

To study the dissipation properties of the BLB scheme without absorbing terms (*i. e.* $\sigma = 0$), we determine the macroscopic equations up to order 2 relatively to Δt.

Proposition 1

In the case where $s_6 = s_7$, $s_8 = s_9$, $c_s = \frac{\lambda}{\sqrt{3}}$ and $\gamma_5 = 0$, the BLB scheme models the following system of macroscopic equations up to order two on Δt :

$$\begin{aligned}
\frac{\partial j_x}{\partial t} + \frac{\lambda^2}{3}\frac{\partial(\rho_x+\rho_y)}{\partial x} + A_{xx}\frac{\partial^2 j_x}{\partial x^2} + A_{yy}\frac{\partial^2 j_x}{\partial y^2} + A_{xy}\frac{\partial^2 j_y}{\partial xy} &= \mathrm{O}(\Delta t^2),\\
\frac{\partial j_y}{\partial t} + \frac{\lambda^2}{3}\frac{\partial(\rho_x+\rho_y)}{\partial y} + B_{xx}\frac{\partial^2 j_y}{\partial x^2} + B_{yy}\frac{\partial^2 j_y}{\partial y^2} + B_{xy}\frac{\partial^2 j_x}{\partial xy} &= \mathrm{O}(\Delta t^2),\\
\frac{\partial \rho_x}{\partial t} + \frac{\partial j_x}{\partial x} + C_{xx}\frac{\partial^2 \rho_x}{\partial x^2} + C_{yy}\frac{\partial^2 \rho_x}{\partial y^2} + D_{xx}\frac{\partial^2 \rho_y}{\partial x^2} + D_{yy}\frac{\partial^2 \rho_y}{\partial y^2} &= \mathrm{O}(\Delta t^2),\\
\frac{\partial \rho_y}{\partial t} + \frac{\partial j_y}{\partial y} - C_{xx}\frac{\partial^2 \rho_x}{\partial x^2} - C_{yy}\frac{\partial^2 \rho_x}{\partial y^2} - D_{xx}\frac{\partial^2 \rho_y}{\partial x^2} - D_{yy}\frac{\partial^2 \rho_y}{\partial y^2} &= \mathrm{O}(\Delta t^2),
\end{aligned}$$

where $A_{xx} = -\frac{\lambda^2\Delta t(4\gamma_4-1)}{6\gamma_4}\sigma_6$, $A_{yy} = -\frac{\lambda^2\Delta t}{3}\frac{(3(\gamma_3-\gamma_6-2\gamma_7+\gamma_4)-1)}{\gamma_4-3(\gamma_6+\gamma_7)}\sigma_5$,
$A_{xy} = -\frac{\lambda^2\Delta t}{3}\left[\frac{3(\gamma_6+\gamma_7)+\gamma_4(6(\gamma_7-\gamma_3)+4\gamma_4-1)}{2\gamma_4(\gamma_4+3(\gamma_6+\gamma_7))}\,\sigma_6 + \frac{3(\gamma_6-\gamma_3+2\gamma_7+\gamma_4)-1}{\gamma_4+3(\gamma_6+\gamma_7)}\sigma_5\right]$,
$B_{xx} = -\frac{\lambda^2\Delta t}{3}\frac{(3(-\gamma_3+\gamma_6+3\gamma_7+\gamma_4)-1)}{\gamma_4+3(\gamma_6+\gamma_7)}\sigma_5$, $B_{yy} = -\frac{\lambda^2\Delta t(2\gamma_4+1)}{3\gamma_4}\sigma_6$,
$B_{xy} = -\frac{\lambda^2\Delta t}{3}\left[\frac{3(\gamma_6+\gamma_7)+\gamma_4(3(\gamma_3-\gamma_7)+2\gamma_4-2)}{\gamma_4(\gamma_4-3(\gamma_6+\gamma_7))}\,\sigma_6 + \frac{3(\gamma_3-\gamma_6-2\gamma_7+\gamma_4)-1}{\gamma_4-3(\gamma_6+\gamma_7)}\sigma_5\right]$,
$C_{xx} = \frac{\lambda^2\Delta t}{18}\sigma_8(3(\gamma_6+\gamma_7)-\gamma_4)(2(\gamma_7-\gamma_6)+\gamma_3-1)$,
$C_{yy} = \frac{\lambda^2\Delta t}{18}\sigma_8(3(\gamma_6+\gamma_7)+\gamma_4)(2(\gamma_7-\gamma_6)+\gamma_3-1)$,
$D_{xx} = \frac{\lambda^2\Delta t}{18}\sigma_8(3(\gamma_6+\gamma_7)-\gamma_4)(2(\gamma_7-\gamma_6)+\gamma_3+1)$
and $D_{yy} = \frac{\lambda^2\Delta t}{18}\sigma_8(3(\gamma_6+\gamma_7)+\gamma_4)(2(\gamma_7-\gamma_6)+\gamma_3+1)$.
We note that this model is not isotropic.

Proof of Proposition 1

To obtain the macroscopic equations we can use the usual Chapman-Enskog analysis [4] or Taylor expansion [2]. The details are given in [10]. In general the second order space derivatives in the preceding equations are not isotropic. To obtain isotropy, the following conditions have to be met :

$A_{xx} = B_{yy},\ A_{yy} = B_{xx},\ A_{xy} = B_{xy}$ and $A_{xx} - A_{xy} = A_{yy}$, where $A_{xx,yy,xy}$ and $B_{xx,yy,xy}$ are the coefficients appearing in the equivalent equations of the model BLB (see proposition 1). This can be satisfied only for $s_5 = 0$. This fact introduces a new conservation law which is incompatible with the Bérenger model. Therefore our model is not isotropic.

2.4 Stability analysis

We study numerically the stability of the BLB scheme by using the Von Neumann analysis. It consists in considering the solution of the scheme for a plane wave $f_j(x_i, t) = \phi_j e^{i(\omega t - k.x_i)}$ and by using the Fourier transform of the equation (3). We obtain the following equation :

(12) $f(x_i, t + \Delta t) = G(p, q) f(x_i, t)$,

where $p = e^{ik_x \Delta x}$, $q = e^{ik_y \Delta x}$, $(k_x, k_y) = k$ and $G(p, q) = A(p, q) M_B^{-1} C M_B$. The advection operator $A(p, q)$ can be written as follows :
$A = \mathrm{diag}\left(1, p, q, \frac{1}{p}, \frac{1}{q}, pq, \frac{q}{p}, \frac{1}{pq}, \frac{p}{q}\right)$, the moments matrix M_B is given by (7) and the collision matrix is given by :

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - s_5 & 0 & 0 & 0 & 0 \\ a_x s_6 & \frac{a_x - a_y}{2} s_6 & 0 & 0 & 0 & 1 - s_6 & 0 & 0 & 0 \\ c_x s_7 & \frac{c_x - c_y}{2} s_7 & 0 & 0 & 0 & 0 & 1 - s_7 & 0 & 0 \\ 0 & 0 & \frac{c_1}{\lambda} s_8 & 0 & 0 & 0 & 0 & 1 - s_8 & 0 \\ 0 & 0 & 0 & \frac{c_2}{\lambda} s_9 & 0 & 0 & 0 & 0 & 1 - s_9 \end{pmatrix}.$$

Let introduce $z = e^{i\omega \Delta t}$, then equation (12) becomes :

$$z f(x_i, t) = G(p, q) f(x_i, t).$$

So the stability relies on the eigenvalue problem for the operator G. Therefore we compute numerically the eigenvalues z_α and the stability occurs when $\mathrm{Re}(\mathrm{ln} z_\alpha) < 0$ (*i. e.* $|z_\alpha| < 1$) for all wave vector k.

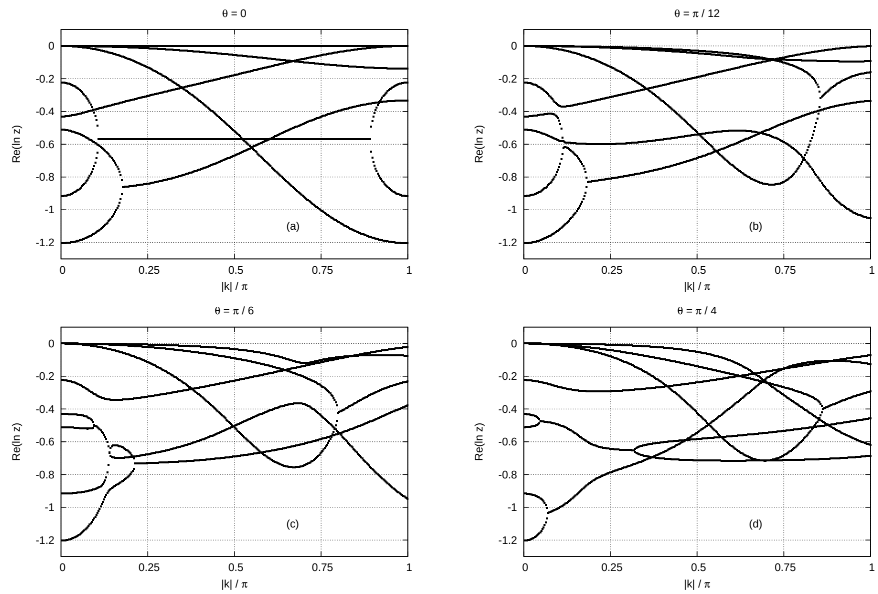

Figure 2: Real part of logarithmic eigenvalues of the BLB model *versus* $|k|$. The value of the parameters are $\gamma_3 = 7$, $\gamma_6 = 3$, $\gamma_7 = 2$, $\gamma_4 = 1$ and $c_s = \frac{1}{\sqrt{3}}$. The relaxation parameters are $s_5 = 1.4$, $s_6 = 1.6$, $s_7 = 1.65$, $s_8 = 1.3$ and $s_9 = 1.8$. (a) For $\theta = 0$ angle of wave vector k ($i.\ e.$ k is parallel to Ox). (b) for $\theta = \dfrac{\pi}{12}$. (c) for $\theta = \dfrac{\pi}{6}$. (d) for $\theta = \dfrac{\pi}{4}$.

For the case where sound speed $c_s = \frac{\lambda}{\sqrt{3}}$ we find that the BLB scheme is not stable for the first choice : $\gamma_5 \neq 0$, $\gamma_3 = \gamma_6 + 2\gamma_7$ and $\gamma_4 = 1$. So we take the second choice ($i.\ e.$ $\gamma_5 = 0$). We find that the BLB algorithm is stable for the following configuration : $\gamma_4 = 1$, $\gamma_3 = \gamma_6 + 2\gamma_7$, $\gamma_6 \in [0.88, 3.22]$, $\gamma_7 \in [0.77, 2.22]$, $s_5 \in]0, 1.6[$, $s_{6,7} \in]0, 1.66[$ and $s_{8,9} \in]0, 1.8[$. Figures 2(a), 2(b), 2(c) and 2(d) show the real part of logarithm of the eigenvalues as function of wave vector k. We see that for this choice of the parameters the BLB algorithm is stable. We note that we have not find situations where the attenuation is less 10^{-2} typically ($i.\ e.$ one order of magnitude greater than the classical D2Q9).

2.5 BLB with damping terms

Until now we studied the case of BLB without absorbing terms ($i.\ e.$ $\sigma = 0$ in the system of equations (2)) to represent only the non-reflecting property of the BLB scheme. To model the zero-order damping terms we propose to change the advection step of the BLB scheme as follows :

Proposition 2

If we modify the advection step of the BLB scheme as follows :

$$f_j(x_i, t+\Delta t) = f_j^*\left(x_i - v_j\Delta t, t\right) - \sum_{\ell=1}^{9} \widetilde{\sigma}^B_{\ell,j} f_\ell^*\left(x_i - v_\ell \Delta t, t\right), \quad 1 \le j \le 9.$$

where the matrix $\widetilde{\sigma}_B \equiv (\widetilde{\sigma}^B_{\ell,j})_{1\le\ell,j\le 9}$, is given by :
$\widetilde{\sigma}^B_{2,\bullet} = \frac{\sigma\Delta t}{4}(1+a_1, 4, 0, 0, 0, a_2+3, a_2-1, a_2-1, a_2+3)$,
$\widetilde{\sigma}^B_{4,\bullet} = \frac{\sigma\Delta t}{4}(1+a_1, 0, 0, 4, 0, a_2-1, a_2+3, a_2+3, a_2-1)$,
and $\widetilde{\sigma}^B_{\ell,j} = 0$ for $\ell \ne (2,4), 1 \le j \le 9$, where $a_1 = \gamma_3 - 4(\gamma_6 - \gamma_7)$ and $a_2 = \gamma_3 + 2\gamma_6 + \gamma_7$. We simulate the terms of damping proportional to σ in the PML system of equations (2). We note here that we give the matrix $\widetilde{\sigma}$ only for the case where the BLB scheme is stable.

Proof of Proposition 2

We use here the Taylor expansion [2] for the above equation to find the macroscopic equivalent equations (2). So we write the Taylor expansion up to order 2 on Δt of the BLB scheme equation (see Proposition 2) :

$$\begin{aligned} f_j(x_i,t) + \Delta t \partial_t f_j(x_i,t) &= \big(f_j^*(x_i,t) - \Delta t v_j \nabla f_j^*(x_i,t)\big) \\ &- \sum_{\ell=1}^{9} \widetilde{\sigma}^B_{j,\ell}\left(f_\ell^*(x_i,t) - \Delta t v_\ell \nabla f_\ell^*(x_i,t)\right) + \mathrm{O}(\Delta t^2), \end{aligned}$$

With the help of the moment matrix M_B, using the fact $f_j^* = f_j^{eq} + \mathrm{O}(\Delta t)$ and neglecting the terms in (Δt^2), we obtain :

$$m_\ell + \Delta t \partial_t m_\ell = m_\ell^* - \Delta t \sum_{j=1,9} M^B_{\ell,j} v_j^\beta \partial_\beta f_j^{eq} - \sum_{j=1}^{9} M^B_{\ell,j} \sum_{p=1}^{9} \widetilde{\sigma}^B_{j,p} f_p^{eq}(x,t) + \mathrm{O}(\Delta t^2),$$

We rewrite the above equation as follows :

$$m_\ell^* - m_\ell = \Delta t \partial_t m_\ell \; + \; \Delta t \sum_{j=1,9} M^B_{\ell,j} v_j^\beta \partial_\beta f_j^{eq} + \sum_{j=1}^{9} \Psi_{\ell,j} f_j^{eq}(x,t) + \mathrm{O}(\Delta t^2),$$

where the matrix $(\Psi_{\ell,j})_{1\le\ell,j,\le 9} = M_B.\widetilde{\sigma}_B$ is the product of matrix M_B and $\widetilde{\sigma}_B$. So with the help of the matrix Ψ we calculate the terms :
$\sum_{j=1}^{9} \Psi_{\ell,j} f_j^{eq}(x,t)$, for $\ell = 1..9$ which is equal to :

$\sigma\Delta t j_x$ for $\ell = 1$, 0 pour $\ell = 2$, $\sigma\Delta t\frac{\rho+(\rho_x-\rho_y)}{2} = \sigma\Delta t\rho_x$ for $\ell = 3$ and $\sigma\Delta t\frac{\rho+(\rho_x-\rho_y)}{2} = \sigma\Delta t\rho_x$ pour $\ell = 4$.
Now we write equation (13) for the four conserved moments (*i. e.* $\ell = \{1, 2, 3, 4\}$) and with the help of $m_\ell^* = m_\ell$ we obtain the PML system (2) with absorption.

3 Numerical test of interfaces

In this section we present numerical simulations for acoustic waves normally incident to an interface between a classical D2Q9 medium (on the left) and various situations on the right : first a BLB without absorption then BLB with absorption and finally classical D2Q9 with absorption.

3.1 Classical D2Q9/BLB without absorption

So let $\Omega = [0, l] \times [0, h]$, where $l = 4000$ and $h = 5$ be composed by $\Omega_- = [0, \frac{l}{2}] \times [0, h]$ and $\Omega_+ = [\frac{l}{2}, l] \times [0, h]$.
• In Ω_-, we use the classical D2Q9 scheme with the following relaxation rates : $s_4 = s_5 = 1.95$, $s_6 = 1.97$, $s_7 = 1.9$ and $s_8 = s_9 = 1.7$.
• In Ω_+, we use the BLB scheme without absorption and we take the following configuration for different parameters : $\gamma_3 = 7$, $\gamma_4 = 1$, $\gamma_6 = 3$, $\gamma_7 = 2$, $c_s = \frac{1}{\sqrt{3}}$, $s_5 = 1.8$, $s_6 = 1.6$, $s_7 = 1.6$ and $s_8 = s_9 = 1.7$.
Here we take periodic boundary conditions for the y direction and a simple bounce back in the outer edges in $x_i = l$. In the inlet edges at $x_i = 0$ we impose an harmonic wave $j_x = \sin(\omega\Delta t)$ where $\omega = \frac{2\pi}{100}$ (implemented by bounce-back and application of $2j_x$ with appropriate weight factors for the velocities incoming in the computational domain). We take a fluid at rest for initial conditions and the total duration $T = n\Delta t$ of the simulations is chosen such that waves have not reached the outlet (see Fig. 3(a)). We note here that the acoustic wave is more absorbed for $x_i > 2000$ Fig. 3(a), and this is due to the change of viscosity in the BLB medium.
To determine the reflected wave, we perform another simulation in the domain $\Omega_R = [0, l] \times [0, h]$. In this domain we take the same configuration as in the domain Ω_- with the same boundary conditions for the inlet edges at $x_i = 0$. This simulation gives us the reference solution. To see the reflected

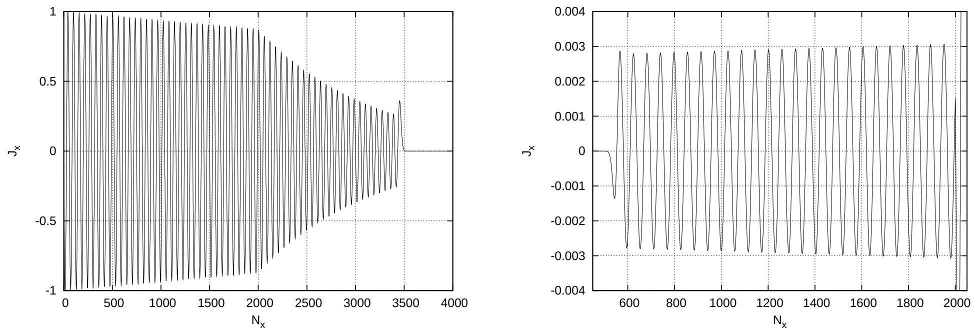

Figure 3: Interface test in the case of normal incidence between classical D2Q9 acoustic medium and BLB without absorption medium. Left: j_x^{test} *vs* N_x. Acoustic wave transmission between Ω_- (classical D2Q9 medium) and Ω_+ (BLB without absorption medium) at time $T = 6000$, interface at $x_i = 2000$. Right: $j_x^{test} - j_x^{ref}$ *vs* N_x, difference between the test and reference cases in the classical D2Q9 acoustic medium.

wave and the Knudsen modes that are generated at the interface we draw the difference between the flux j_x^{test} in Ω (the test case) and the flux j_x^{ref} in Ω_R (the reference case) for the same number of time steps = 6000. It should be noted here that we have a small reflected wave between classical D2Q9 acoustic medium and BLB without absorption medium. So in Fig. 3(b) (for $x_i \in (1, 2, \ldots .2000)$) we see a reflected acoustic wave which has an amplitude of the order 3.10^{-3}. This reflected acoustic wave is generated by the change in the viscosity between the two media. As indicated above, the BLB scheme is anisotropic and is not stable for parameters corresponding to a viscosity as small as that can be obtained with D2Q9 (for more details see [11]).

3.2 Classical D2Q9/BLB with absorption

To test this interface we make the same simulation as above, but now we only change the Ω_+ medium. Indeed in Ω_+ we use the BLB scheme with absorption (*i. e.* changing the advection step as described in proposition 2). We take the following parameters : $\gamma_3 = 7$, $\gamma_4 = 1$, $\gamma_6 = 3$, $\gamma_7 = 2$, $c_s = \frac{1}{\sqrt{3}}$, $s_5 = 1.8$, $s_6 = 1.6$, $s_7 = 1.6$, $s_8 = s_9 = 1.7$ and $\sigma(x_i) = 10^{-7}(x_i - 2000)^2$.

Figure 4(a) shows that the transmitted acoustic wave is absorbed (for $x_i > 2000$) in the BLB with absorption medium. We note also that the reflected acoustic wave (see Fig. 4(b)) in the D2Q9 medium has the same

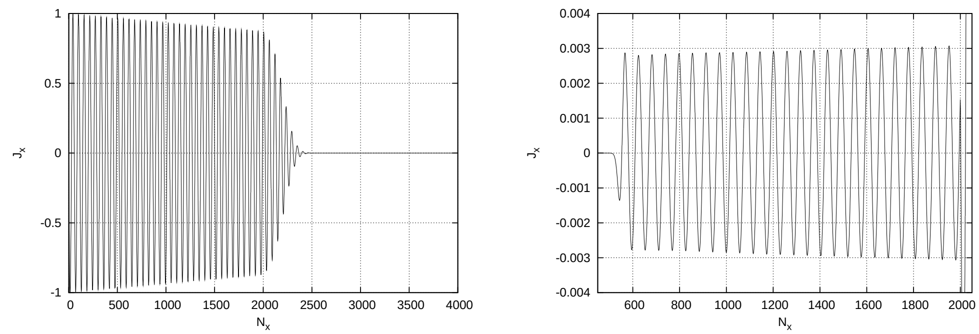

Figure 4: Interface test in the case of normal incidence between classical D2Q9 acoustic medium and BLB with absorption medium. Left: j_x^{test} *vs* N_x. acoustic wave transmission between Ω_- (D2Q9 medium) and Ω_+ (BLB with absorption medium) at time $T = 6000$ and interface at $x_i = 2000$. Right: $j_x^{test} - j_x^{ref}$ *vs* N_x Reflected wave in the classical D2Q9 acoustic medium: difference between the test and reference cases.

amplitude as in the case D2Q9/BLB without absorption.

3.3 Classical D2Q9/ Classical D2Q9 with absorption

Now to test the classical D2Q9/classical D2Q9 with absorption we only change the medium Ω_+. So we take the following D2Q9 scheme where we have only changed the advection step in Ω_+ :

$$f_j(x_i, t + \Delta t) = (\mathrm{Id} - \widetilde{\sigma}) f_j^* (x_i - v_j \Delta t, t) , \quad 1 \le j \le 9,$$

where the matrix $\widetilde{\sigma} \equiv (\widetilde{\sigma}_{\ell,j})_{1 \le \ell, j \le 9}$ is given by : $\widetilde{\sigma}_{2,\bullet} = \frac{\sigma \Delta t}{2}(1, 2, 1, 0, 1, 2, 0, 0, 2)$, $\widetilde{\sigma}_{4,\bullet} = \frac{\sigma \Delta t}{2}(1, 0, 1, 2, 1, 0, 2, 2, 0)$, and $\widetilde{\sigma}_{\ell,j} = 0$ for $\ell \neq (2, 4)$, $1 \le j \le 9$. This scheme has the following macroscopic equation up to order 1 in Δt :

$$\left\{ \begin{array}{ccc} \partial_t \rho + \sigma \rho + \partial_x j_x + \partial_y j_y & = & \mathrm{O}(\Delta t), \\ \partial_t j_x + \sigma j_x + c_s^2 \partial_x \rho & = & \mathrm{O}(\Delta t), \\ \partial_t j_y + c_s^2 \partial_y \rho & = & \mathrm{O}(\Delta t). \end{array} \right.$$

at $x_i = 2000$ and $T = 6000$.]

In Ω_+ we take the following conditions : $m_4^{eq} = m_5^{eq} = 0$, $m_6^{eq} = -2\rho$, $m_7^{eq} = \rho$, $m_8^{eq} = -j_x$, $m_9^{eq} = -j_y$, $s_4 = s_5 = 1.9$, $s_6 = 1.8$, $s_7 = 1.75$, $s_8 = s_9 = 1.7$, and $\sigma(x_i) = 10^{-7}(x_i - 2000)^2$. Figure 5(a) shows that the transmitted wave is absorbed (for $x_i > 2000$) in the D2Q9 with absorption medium. We note here that this interface generates a very small reflected

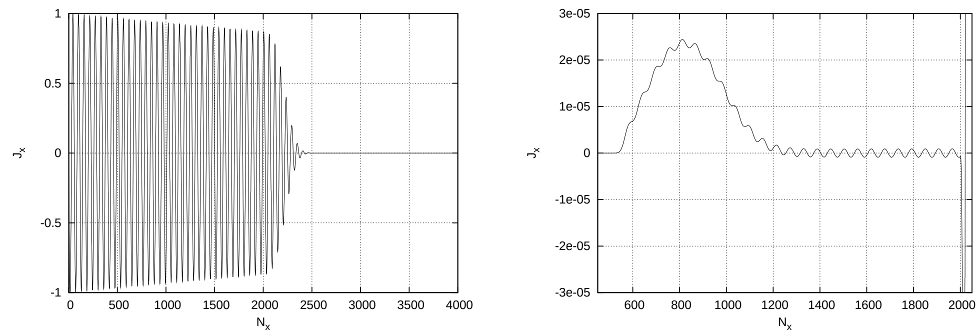

Figure 5: Interface test in the case of normal incidence between D2Q9 acoustic medium and D2Q9 with absorption medium. Left: j_x^{test} *vs* N_x wave transmission between Ω_- (D2Q9 medium) and Ω_+ (D2Q9 with absorption medium) at time $T = 6000$.j_x^{test} *vs* N_x : Acoustic wave, interface at $x_i = 2000$ and $T = 6000$. Right: $j_x^{test} - j_x^{ref}$ *vs* N_x, difference between the test and reference cases. $j_x^{test} - j_x^{ref}$ *vs* N_x : Reflected wave in the D2Q9 acoustic medium.

wave (see Fig. 5(b)) in normal incidence which is due to the change of the speed of sound in the two media (for more details see [10, 11]).

3.4 Comparison between numerical interfaces

The BLB without absorption scheme generates an undesired reflected acoustic wave in the domain of interest. The BLB with absorption scheme is stable and does not generate any additional reflected wave. Finally the classical D2Q9 scheme with absorption is more efficient but it generates a small reflected wave for normal incidence. Thus we propose a new method to cancel reflected wave.

4 Towards cancellation of reflected waves

Let Ω_-, Ω_+ be two one dimensional acoustic domains simulated by D1Q3 scheme with sound velocity and viscosity (c_s, ν) and $(\widetilde{c}_s, \widetilde{\nu})$ respectively. So we have the following reflection coefficient [11] :

$$(13)\quad r = \frac{p_+ - \widetilde{p}_+}{1 - p_+\widetilde{p}_+} = \frac{c_1 - c_2}{c_1 + c_2} + \frac{i(\nu_1 {c_2}^2 - \nu_2 c_1^2)}{c_1 c_2 (c_1 + c_2)^2}\omega + \mathrm{O}(\omega^2),$$

where $p_+ = e^{(ik^+\Delta x)}$, $\widetilde{p}_+ = e^{(i\widetilde{k}^+\Delta x)}$, ω is the frequency of incident wave and k^+, $\widetilde{k}^+$ are the progressive wave vectors in Ω_- and Ω_+ respectively.
In order to cancel the reflected wave we propose to change the advection step at the interface. Thus the new f_1 in node $x_r = \frac{\Delta x}{2}$ is a linear combination of f_1^* in node $x_l = -\frac{\Delta x}{2}$ and f_1^* in node $x_l - \Delta x$ (see Fig. 6). Whereas we keep the same advection step for f_2 which goes in the opposite direction. Thus we propose the following scheme at the interface :

$$\begin{aligned} f_1(t+\Delta t, x_i) &= \delta_1 f_1^*(t, x_i - \Delta x) + \delta_2 f_1^*(t, x_i - 2\Delta x), \qquad \text{in} \quad x_i = \frac{\Delta x}{2}, \\ f_2(t+\Delta t, x_i) &= f_2^*(t, x_i + \Delta x), \qquad \text{in} \quad x_i = -\frac{\Delta x}{2}, \end{aligned}$$

where δ_1 and δ_2 are two scalar coefficients which will be fixed in order to cancel the reflected wave.

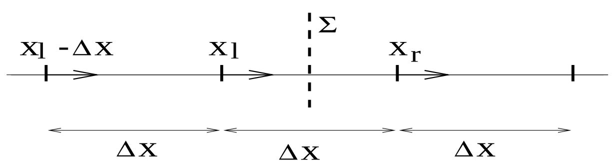

Figure 6: Connection at interface.

Proposition 3
For D1Q3 monodimensional acoustic interface, we can find coefficients δ_1 and δ_2 in order to cancel terms of order 0 and 1 in ω of the reflection coefficient given in equation (13).

Proof of Proposition 3
To find coefficients δ_1 and δ_2 we calculate the theoretical expression of the reflection coefficient taking into account the new advection step at interface. Then we resolve the equation $r = \mathrm{O}(\omega^2)$. (for more details see [10]).

• Numerical test : Let $\Omega_- = \{x_i,\ i = 1..1000\}$ and $\Omega_- = \{x_i,\ i = 1001..2000\}$ with sound velocity and viscosity ($c_s = 0.577, \nu = 0.001$) and ($\widetilde{c}_s = 0.479, \widetilde{\nu} = 0.2$). Figure 7(a) shows that there is a reflected wave which has an amplitude of the order 10^{-1}. By using the new proposed method (see proposition 3) we have reduced the reflected wave. In figure 7 (b) the reflected wave has an amplitude about 10^{-4}.

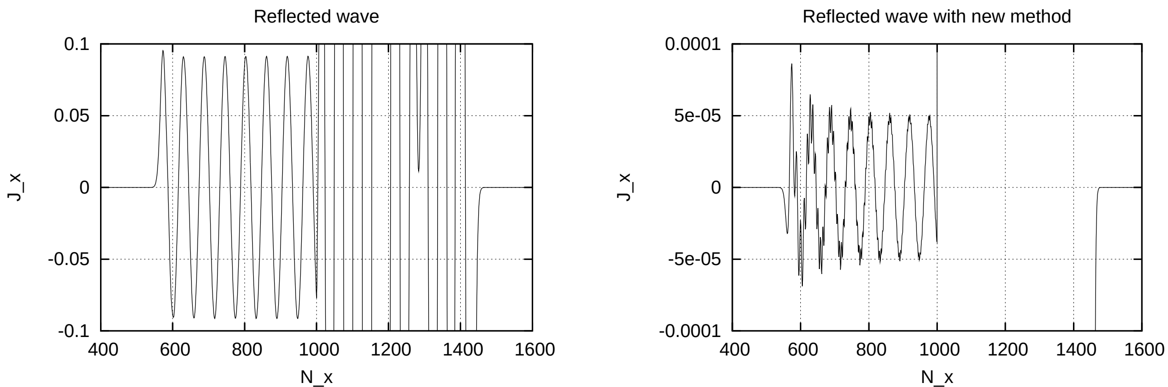

Figure 7: $j_x^{test} - j_x^{ref}$ *vs* N_x : difference between test and reference cases at $T = 1500$, (a) without changing the advection step at interface and (b) with interpolation of the advection step at the interface.

5 Conclusion

We have proposed a new scheme called BLB to model the perfeclty matched layer of Bérenger. Unfortunately this scheme generates a reflected wave in the domain of interest and this is due to the non isotropic property of BLB. The method used here to obtain a fourth macroscopic equation (as in the Bérenger scheme) needs to be tested for more complicated schemes than D2Q9 in order to model first order equations without obtaining unsatisfactory second order equations (by this we mean anisotropic viscous terms). We have also proposed a method to model the zero-order damping terms. This method consists in changing the advection scheme. This method is stable and does not generate a reflected wave.

We have proposed a new method to cancel the reflected wave for normal incidence based on a local modification of the propagation rules near the interface. Future work could be the extension of the this method for two and three dimensional interface and for any incidence angle.

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