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New Finite Volume Method for rotating channel flows involving boundary layers

Soumaya Ben Chaabane[2] · **Makram Hamouda**[1,3] · **Mahdi Tekitek**[3]

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Abstract We investigate in this article the boundary layers appearing in a fluid under moderate rotation when the viscosity is small. The fluid is modeled by the time-dependent rotating Stokes equations also known as the Stokes–Coriolis equations. The equations are considered in an infinite channel with periodicity on the lateral boundary and Dirichlet boundary conditions on the top and bottom of the channel. First, we analytically derive the correctors which describe the sharp variations at large Reynolds number (*i.e.* small viscosity). Second, thanks to a modified finite volume method (MFVM) we give the numerical solutions of the Stokes–Coriolis system at small viscosity (10^{-3}–10^{-10}). We follow the common idea which consists of adding the corrector functions to the Galerkin basis or its analogous for the classical Finite Volume Method, see Gie et al. (Discrete Contin Dyn Syst 36(5):2521–2583, 2016), Gie and Temam (Int J Numer Anal Model 12(3):536–566, 2015), Shih and Bruce (SIAM J Math Anal 18(5):1467–1511, 1987). The MFVM introduced here can be applied to a large class of singular perturbation problems.

Mathematics Subject Classification 76D10 · 76D17 · 65L11 · 76L05 · 68U120

✉ Makram Hamouda
mahamoud@indiana.edu

[1] The Institute for Scientific Computing and Applied Mathematics, Indiana University, 831 E. 3rd St., Rawles Hall, Bloomington, IN 47405, USA

[2] Department of Mathematics, Faculty of Sciences of Bizerte, University of Carthage, Zarzouna, Tunisia

[3] Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunis, Tunisia

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1 Introduction

We are interested in this article in the study of boundary layers of a time-dependent rotating fluid when the viscosity is small and the boundary is characteristic; this occurs for example when the boundary is solid and at rest. The boundary conditions are then homogeneous of Dirichlet type. More precisely, we consider a $3D$ flow which verifies the following system:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}^\varepsilon}{\partial t} - \varepsilon\Delta \boldsymbol{u}^\varepsilon + \boldsymbol{\omega} \times \boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f}, \quad \text{ in } \quad \Omega_\infty \times (0, T), \\ \operatorname{div} \boldsymbol{u}^\varepsilon = 0, \quad \text{in } \Omega_\infty \times (0, T), \\ \boldsymbol{u}^\varepsilon = 0, \quad \text{on } \partial\Omega_\infty \\ \boldsymbol{u}^\varepsilon \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions}, \\ \boldsymbol{u}^\varepsilon_{|t=0} = \boldsymbol{u}_0. \end{cases} \tag{1.1}$$

Here $\boldsymbol{\omega} = \alpha \boldsymbol{e_3}$ where $\boldsymbol{e_3}$ is the unit vector in the canonical basis of $\mathbb{R}^3$, $\Omega_\infty = \mathbb{R}^2 \times (0, h)$ is the relevant domain, $\partial\Omega_\infty = \mathbb{R}^2 \times \{0, h\}$ its boundary. The functions $\boldsymbol{u_0}$ and $\boldsymbol{f}$ are given and supposed to be as regular as necessary. Without loss of generality, the constant h will be taken from now equal to 1. For more details about the theory of rotating fluids, see [4] and [14] and the references therein.

The solutions $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ of the system (1.1) are such that $\boldsymbol{u}^\varepsilon(t; x, y, z) = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) \in \mathbb{R}^3$ and $p^\varepsilon \in \mathbb{R}$, the coefficient ε is a positive constant representing the inverse of the Reynolds number or the viscosity of the fluid. Throughout this paper the coefficient $\varepsilon > 0$ is intended to be small $\varepsilon \ll 1$. Because of the periodicity conditions $(1.1)_4$ we will consider a portion of the channel Ω_∞ that we denote by $\Omega = (0, 2\pi) \times (0, 2\pi) \times (0, 1)$ and its boundary $\Gamma = \partial\Omega = (0, 2\pi) \times (0, 2\pi) \times \{0, 1\}$ on which all our calculations will be done.

By standard energy estimates, it is easy to see that $\boldsymbol{u}^\varepsilon$, the solution of (1.1), is bounded in $L^\infty(0, T; L^2(\Omega))$. Hence, it is now natural to look for the limit as $\varepsilon \to 0$. Formally, the limit solution corresponding to the system (1.1), that we denote here by $\boldsymbol{u}^0$, is simply obtained by setting $\varepsilon = 0$ in (1.1). Hence, we have

$$\begin{cases} \dfrac{\partial \boldsymbol{u}^0}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u}^0 + \nabla p^0 = \boldsymbol{f}, \quad \text{in } \Omega \times (0, T), \\ \operatorname{div} \boldsymbol{u}^0 = 0, \quad \text{in } \ \Omega \times (0, T), \\ u_3^0 = 0, \quad \text{on } \partial\Omega, \\ \boldsymbol{u}^0 \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions}, \\ \boldsymbol{u}^0_{|t=0} = \boldsymbol{u}_0. \end{cases} \tag{1.2}$$

The absence in the limit system of the Laplacian term $(-\varepsilon\Delta \boldsymbol{u}^\varepsilon)$ which is a regularizing term, generates a loss of regularity for the limit solution $\boldsymbol{u}^0$. Thus some discrepancies between the viscous and inviscid solutions appear near the boundary of the domain, that is here $z = 0, 1$ as it is mentioned in $(1.1)_3$ and $(1.2)_3$. These thin regions are called *boundary layers* and where the convergence of $\boldsymbol{u}^\varepsilon$ to $\boldsymbol{u}^0$ is not expected at least in some Sobolev spaces as we will see later on. Hence, we introduce some correcting term called *correctors* for which the equations must be of course simpler than the ones in the original problem, namely (1.1). See e.g. [7,17,18,21] for more details about the theory of correctors.

The rest of the article is organized as follows. In Sect. 2, we derive the analytical expression of the correctors in addition to several estimates useful for the asymptotic analysis later on. Then, in Sect. 3, we prove the main theoretical result of this article which rigourously confirms the choice of the correctors. From the numerical point of view, we recall in Sect. 4 the CFVM discretization of the solution of (1.1) which is inherited from [11]. Afterwards, we introduce the MFVM in Sect. 5 and we numerically prove its accuracy in Sect. 6. Finally, in Sect. 7, we end the article with the conclusion and some future research directions.

2 The corrector equations

To study the asymptotic behavior of $\boldsymbol{u}^\varepsilon$, when $\varepsilon \to 0$, we propose the following asymptotic expansion of $\boldsymbol{u}^\varepsilon$:

$$\boldsymbol{u}^\varepsilon \simeq \boldsymbol{u}^0 + \boldsymbol{\varphi}^\varepsilon,$$

where $\boldsymbol{\varphi}^\varepsilon$ is the corrector function that will be introduced to correct the difference $\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0$ at $z = 0, 1$. The equations verified by $\boldsymbol{\varphi}^\varepsilon$ are as follows:

$$\begin{cases} \dfrac{\partial \boldsymbol{\varphi}^\varepsilon}{\partial t} - \varepsilon \dfrac{\partial^2 \boldsymbol{\varphi}^\varepsilon}{\partial z^2} + \boldsymbol{\omega} \times \boldsymbol{\varphi}^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \\ \operatorname{div} \boldsymbol{\varphi}^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \\ \boldsymbol{\varphi}^\varepsilon_{|z=0,1} = -\boldsymbol{u}^0_{|z=0,1}, \\ \boldsymbol{\varphi}^\varepsilon \quad \text{is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions}, \\ \boldsymbol{\varphi}^\varepsilon_{|t=0} = 0. \end{cases} \tag{2.1}$$

We now introduce an approximate function $\check{\boldsymbol{\varphi}}^\varepsilon$ of $\boldsymbol{\varphi}^\varepsilon$ defined as the sum of $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$ and $\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}$ the correctors that we propose to solve the boundary layers at the boundaries $z = 0$ and $z = 1$, respectively,

$$\check{\boldsymbol{\varphi}}^\varepsilon(t, x, y, z) = \overline{\boldsymbol{\varphi}}^{0,\varepsilon}\left(t, x, y, \frac{z}{\sqrt{\varepsilon}}\right) + \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}\left(t, x, y, \frac{1-z}{\sqrt{\varepsilon}}\right).$$

Omitting for instance the incompressibility condition $(2.1)_2$ and considering the boundary conditions $(2.1)_3$ separately at $z = 0$ and $z = 1$, then the system verified by $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$ is given by:

$$\begin{cases} \dfrac{\partial \overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial t} - \dfrac{\partial^2 \overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial \overline{z}^2} + \boldsymbol{\omega} \times \overline{\boldsymbol{\varphi}}^{0,\varepsilon} = 0, \quad \text{in } \widetilde{\Omega} \times (0, T), \\ \overline{\boldsymbol{\varphi}}^{0,\varepsilon}(\overline{z} = 0) = -\boldsymbol{u}^0(\overline{z} = 0), \\ \overline{\boldsymbol{\varphi}}^{0,\varepsilon} \to 0 \text{ as } \overline{z} \to \infty, \\ \overline{\boldsymbol{\varphi}}^{0,\varepsilon} \quad \text{is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions}, \\ \overline{\boldsymbol{\varphi}}^{0,\varepsilon}_{|t=0} = 0, \end{cases} \tag{2.2}$$

where $\bar{z} = \dfrac{z}{\sqrt{\varepsilon}}$, and we denoted by $\widetilde{\Omega}$ the stretched domain, i.e. $\widetilde{\Omega} = (0, 2\pi) \times (0, 2\pi) \times (0, +\infty)$.

Similarly $\widetilde{\varphi}^{1,\varepsilon}$ satisfies the following system:

$$\begin{cases} \dfrac{\partial \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial t} - \dfrac{\partial^2 \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial \widetilde{z}^2} + \boldsymbol{\omega} \times \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon} = 0, \quad \text{in } \widetilde{\Omega} \times (0, T), \\ \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}(\widetilde{z} = 0) = -\boldsymbol{u}^0(\widetilde{z} = 0), \\ \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon} \to 0 \quad \text{as } \widetilde{z} \to \infty, \\ \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon} \quad \text{is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions}, \\ \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}_{|t=0} = 0, \end{cases} \tag{2.3}$$

where $\widetilde{z} = \dfrac{1 - z}{\sqrt{\varepsilon}}$.

In the following we will derive the expressions of the solutions of the systems (2.2) and (2.3). For that purpose, we need the following proposition where we used the techniques borrowed from [22] to prove the result stated below.

Proposition 2.1 *Let $u = u(t; x, y, z)$ be the solution of the following problem:*

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \dfrac{\partial^2 \boldsymbol{u}}{\partial z^2} + \boldsymbol{\omega} \times \boldsymbol{u} = 0, \quad \text{in } \widetilde{\Omega} \times (0, T), \\ \boldsymbol{u} = \boldsymbol{g}, \quad \text{at } z = 0, \\ \boldsymbol{u} \to 0, \quad \text{as } z \to +\infty, \\ \boldsymbol{u} = 0, \quad \text{at } t = 0. \end{cases} \tag{2.4}$$

where $\boldsymbol{g} = (g_1, g_2, 0)$ is a continuous function in $\widetilde{\Omega} \times (0, T)$ and $\boldsymbol{w} = \alpha \boldsymbol{e_3}$.

Then, the explicit expression of $\boldsymbol{u}$ is given by:

$$\begin{aligned} \boldsymbol{u}(t; x, y, z) = -\int_0^t \frac{\partial K}{\partial z}(t - \tau, z)[(\boldsymbol{g} - i(\boldsymbol{e_3} \times \boldsymbol{g}))(\tau, x, y, 0)e^{i\alpha(\tau - t)} \\ + (\boldsymbol{g} + i(\boldsymbol{e_3} \times \boldsymbol{g}))(\tau, x, y, 0)e^{i\alpha(t - \tau)}]d\tau, \end{aligned}$$

where i is the complex number s.t. $i^2 = -1$, and K is the fundamental solution of the heat equation:

$$K(t, z) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-z^2}{4t}}.$$

Proof Let $\boldsymbol{u} = (u_1, u_2, u_3)$ be the solution of (2.4). We have $g_3 = 0$, hence $\boldsymbol{u} = (u_1, u_2, 0)$, i.e. $u_3 = 0$. Taking the cross product of $(2.4)_1$ with $\boldsymbol{e_3}$, we find:

$$\partial_t(\boldsymbol{e_3} \times \boldsymbol{u}) - \partial_z^2(\boldsymbol{e_3} \times \boldsymbol{u}) - \alpha \boldsymbol{u} = 0.$$

We then set $\boldsymbol{C}^{\pm} = \boldsymbol{u} \mp i(\boldsymbol{e_3} \times \boldsymbol{u})$, we obtain:

$$\partial_t \boldsymbol{C}^{\pm} - \partial_z^2 C^{\pm} \pm i\alpha \boldsymbol{C}^{\pm} = 0.$$

Denoting by $\boldsymbol{H}^{\pm} = \boldsymbol{C}^{\pm} e^{\pm i\alpha t}$, one arrives to the following system:

$$\begin{cases} \dfrac{\partial \boldsymbol{H}^{\pm}}{\partial t} - \dfrac{\partial^2 \boldsymbol{H}^{\pm}}{\partial z^2} = 0, \quad \text{in } \widetilde{\Omega} \times (0, T), \\ \boldsymbol{H}^{\pm}(z = 0) = (\boldsymbol{g}(z = 0) \mp i(\boldsymbol{e_3} \times \boldsymbol{g}(z = 0))e^{\pm i\alpha t}, \\ \boldsymbol{H}^{\pm} \to 0, \quad \text{as } z \to +\infty, \\ \boldsymbol{H}^{\pm}|_{t=0} = 0. \end{cases} \tag{2.5}$$

Hence $\boldsymbol{H}^{\pm}$ satisfies a heat equation with non-homogeneous boundary conditions, then it has the following expression [3]:

$$\boldsymbol{H}^{\pm} = -2 \int_0^t \frac{\partial K}{\partial z}(t - \tau, z)[(\boldsymbol{g} \mp i(\boldsymbol{e_3} \times \boldsymbol{g}))(\tau; x, y, 0)]e^{\pm i\alpha\tau} d\tau.$$

Then, we infer that:

$$\boldsymbol{C}^{\pm} = -2 \int_0^t \frac{\partial K}{\partial z}(t - \tau, z)[(\boldsymbol{g} \mp i(\boldsymbol{e_3} \times \boldsymbol{g}))(\tau; x, y, 0)]e^{\pm i\alpha(\tau - t)} d\tau.$$

Coming back to $\boldsymbol{u}$ we have:

$$\boldsymbol{u} = \frac{1}{2}(\boldsymbol{C}^{+} + \boldsymbol{C}^{-}),$$

hence we deduce the explicit expression of the solution of (2.4):

$$\begin{aligned} \boldsymbol{u} = -\int_0^t \frac{\partial K}{\partial z}(t - \tau, z) \times \{&[(\boldsymbol{g} - i(\boldsymbol{e_3} \times \boldsymbol{g}))(\tau, x, y, 0)]e^{i\alpha(\tau - t)} \\ &+ [(\boldsymbol{g} + i(\boldsymbol{e_3} \times \boldsymbol{g}))(\tau, x, y, 0)]e^{i\alpha(t - \tau)}\} d\tau. \end{aligned}$$

Now, according to Proposition 2.1, the solution of (2.2) $\overline{\boldsymbol{\varphi}}^{0,\varepsilon} = (\overline{\varphi}_1^{0,\varepsilon}, \overline{\varphi}_2^{0,\varepsilon}, \overline{\varphi}_3^{0,\varepsilon})$ has the following expression:

$$\begin{aligned} \overline{\varphi}_j^{0,\varepsilon} = -\int_0^t \frac{1}{\sqrt{4\pi(t - \tau)}} \frac{z}{2\sqrt{\varepsilon(t - \tau)}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{&2u_j^0(\tau, x, y, 0)\cos(\alpha(\tau - t)) \\ &+ 2(\boldsymbol{e_3} \times \boldsymbol{u}^0)_j(\tau, x, y, 0)\sin(\alpha(\tau - t))\} d\tau, \; j = 1, 2, \end{aligned} \tag{2.6}$$

for the two tangential components of $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$, and the normal component of $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$ is simply deduced using the incompressibility condition:

$$\begin{aligned} \overline{\varphi}_3^{0,\varepsilon} = -\int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t - \tau)}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{&-2\partial_z u_3^0(\tau, x, y, 0)\cos(\alpha(\tau - t)) \\ &-2(\partial_x u_2^0 - \partial_y u_1^0)(\tau, x, y, 0)\sin(\alpha(\tau - t))\} d\tau \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\
&-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau. \qquad (2.7)
\end{aligned}
$$

Then we write the system satisfied by $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$ which reads as follows:

$$
\begin{cases}
\dfrac{\partial\overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial t} - \varepsilon\dfrac{\partial^2\overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial z^2} + \boldsymbol{\omega}\times\overline{\boldsymbol{\varphi}}^{0,\varepsilon} = \left(0,0,\dfrac{\partial\overline{\varphi}_3^{0,\varepsilon}}{\partial t} - \varepsilon\dfrac{\partial^2\overline{\varphi}_3^{0,\varepsilon}}{\partial z^2}\right), \text{ in } \Omega\times(0,T),\\
\operatorname{div}\overline{\boldsymbol{\varphi}}^{0,\varepsilon} = 0, \quad \text{in } \Omega\times(0,T),\\
\overline{\boldsymbol{\varphi}}^{0,\varepsilon}(z=0) = (-u_1^0(z=0), -u_2^0(z=0), \overline{\varphi}_3^{0,\varepsilon}(z=0)),\\
\overline{\boldsymbol{\varphi}}^{0,\varepsilon}(z=1) = (\overline{\varphi}_1^{0,\varepsilon}(z=1), \overline{\varphi}_2^{0,\varepsilon}(z=1), 0),\\
\overline{\boldsymbol{\varphi}}^{0,\varepsilon} \ \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions},\\
\overline{\boldsymbol{\varphi}}^{0,\varepsilon}(t=0) = 0.
\end{cases}
\qquad (2.8)
$$

Now, we have to calculate the right-hand side (denoted hereafter RHS) of $(2.8)_1$. First, by differentiating (2.7) with respect to the time variable t, we obtain:

$$
\begin{aligned}
\frac{\partial\overline{\varphi}_3^{0,\varepsilon}}{\partial t} = &\int_0^t \frac{\sqrt{\varepsilon}}{4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\
&-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau\\
&-\int_0^t \frac{z^2}{8\sqrt{\varepsilon}\sqrt{\pi}(t-\tau)^{\frac{5}{2}}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\
&-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau\\
&-\int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{-2\alpha\partial_z u_3^0(\tau,x,y,0)\sin(\alpha(\tau-t))\\
&+2\alpha(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\cos(\alpha(\tau-t))\}d\tau\\
&-\int_0^t \frac{\sqrt{\varepsilon}}{4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\
&-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau\\
&+\int_0^t \frac{1}{8\sqrt{\varepsilon}\sqrt{\pi}(t-\tau)^{\frac{5}{2}}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\
&-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau\\
&+\int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\alpha\partial_z u_3^0(\tau,x,y,0)\sin(\alpha(\tau-t))\\
&+2\alpha(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\cos(\alpha(\tau-t))\}d\tau. \qquad (2.9)
\end{aligned}
$$

Then by differentiating $\overline{\varphi}_3^{0,\varepsilon}$ with respect to the normal variable z, we obtain:

$$\begin{aligned}\varepsilon\frac{\partial^2\overline{\varphi}_3^{0,\varepsilon}}{\partial z^2} &= -\int_0^t \frac{\sqrt{\varepsilon}}{4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\ &+2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau\\ &-\int_0^t \frac{z^2}{8\sqrt{\pi\varepsilon}(t-\tau)^{\frac{5}{2}}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\ &-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau. \end{aligned} \tag{2.10}$$

Therefore we deduce from (2.9) and (2.10):

$$\begin{aligned}\frac{\partial\overline{\varphi}_3^{0,\varepsilon}}{\partial t}-\varepsilon\frac{\partial^2\overline{\varphi}_3^{0,\varepsilon}}{\partial z^2} &= -\int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{-2\alpha\partial_z u_3^0(\tau,x,y,0)\sin(\alpha(\tau-t))\\ &+2\alpha(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\cos(\alpha(\tau-t))\}d\tau\\ &-\int_0^t \frac{\sqrt{\varepsilon}}{4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\ &-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau\\ &+\int_0^t \frac{1}{8\sqrt{\varepsilon}\sqrt{\pi}(t-\tau)^{\frac{5}{2}}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau,x,y,0)\cos(\alpha(\tau-t))\\ &-2(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\sin(\alpha(\tau-t))\}d\tau\\ &+\int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\alpha\partial_z u_3^0(\tau,x,y,0)\sin(\alpha(\tau-t))\\ &+2\alpha(\partial_x u_2^0-\partial_y u_1^0)(\tau,x,y,0)\cos(\alpha(\tau-t))\}d\tau. \end{aligned} \tag{2.11}$$

We denote by $J_1+\cdots+J_4$ the sum of the terms in the RHS of (2.11).

Then, estimating $|J_1|$, we get:

$$\begin{aligned}|J_1| &\le k\sqrt{\varepsilon}\int_0^t \frac{1}{\sqrt{t-\tau}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} d\tau\\ &\le k\sqrt{\varepsilon}\int_0^t \frac{1}{\sqrt{t-\tau}} d\tau\, e^{\frac{-z^2}{8\varepsilon T}}, \end{aligned} \tag{2.12}$$

and we obtain the L^2-norm of the term J_1:

$$\begin{aligned}\|J_1\|_{L^2(\Omega)}^2 &\le k\varepsilon\int_0^1 e^{\frac{-z^2}{4\varepsilon T}} dz\\ &\le k\varepsilon\int_0^1 e^{\frac{-cz}{\sqrt{\varepsilon T}}} dz, c>0\\ &\le k\varepsilon^{3/2}. \end{aligned}$$

Hence, we obtain

$$\|J_1\|_{L^2(\Omega)} \leq k\varepsilon^{3/4}. \tag{2.13}$$

Finally, combining (2.13) and the fact that J_2, J_2 and J_4 are e.s.t. (where *e.s.t.* stands for quantities which are exponentially small terms in all $H^m((0,T)\times\Omega),\ m\geq 0$), we conclude that:

$$\left\|\frac{\partial\overline{\varphi}_3^{0,\varepsilon}}{\partial t} - \varepsilon\frac{\partial\overline{\varphi}_3^{0,\varepsilon}}{\partial z^2}\right\|_{L^2(\Omega)} \leq k\varepsilon^{3/4}. \tag{2.14}$$

Remark 1 By symmetry the corrector $\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}$ has the same expression as $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$ with z replaced by $1-z$. Hence, all the estimates satisfied by $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$ remain valid for $\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}$.

3 Convergence result

In this section we prove the main theoretical result of this article.

Theorem 3.1 *The solution $\boldsymbol{u}^\varepsilon$ of (1.1), with $\boldsymbol{u}_0$ and $\boldsymbol{f}$ supposed to be sufficiently smooth, satisfies the following estimates:*

$$\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0 - \overline{\boldsymbol{\varphi}}^{0,\varepsilon} - \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))} \leq k\varepsilon^{3/4}, \tag{3.1}$$

$$\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0 - \overline{\boldsymbol{\varphi}}^{0,\varepsilon} - \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}\|_{L^2(0,T,\boldsymbol{H}^1(\Omega))} \leq k\varepsilon^{1/4}, \tag{3.2}$$

where k is a positive constant depending on the data but not ε and $\boldsymbol{u}^0$, $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$, and $\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}$ are defined respectively by (1.2), (2.8) and as in Remark 1. Here we denoted by $\boldsymbol{L}^2(\Omega)=(L^2(\Omega))^3$ and $\boldsymbol{H}^1(\Omega)=(H^1(\Omega))^3$.

Proof First we observe that the corrector $\boldsymbol{\varphi}^\varepsilon$ does not satisfy the desired boundary conditions as given by $(2.1)_3$, this is due to the choice of a corrector in a simpler form. To overcome this difficulty we introduce additional (small) correctors $\overline{\boldsymbol{\theta}}^\varepsilon$ and $\widetilde{\boldsymbol{\theta}}^\varepsilon$ as follows:

$$\begin{cases} -\varepsilon\Delta\overline{\boldsymbol{\theta}}^\varepsilon + \nabla\Pi^\varepsilon = 0, \quad \text{in } \ \Omega\times(0,T),\\ \mathrm{div}\overline{\boldsymbol{\theta}}^\varepsilon = 0,\\ \overline{\boldsymbol{\theta}}^\varepsilon|_{z=0} = (0,0,-\overline{\varphi}_3^{0,\varepsilon}|_{z=0}),\\ \overline{\boldsymbol{\theta}}^\varepsilon|_{z=1} = (-\overline{\varphi}_1^{0,\varepsilon}|_{z=1}, -\overline{\varphi}_2^{0,\varepsilon}|_{z=1}, 0),\\ \overline{\boldsymbol{\theta}}^\varepsilon \ \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions,} \end{cases} \tag{3.3}$$

and

$$\begin{cases} -\varepsilon\Delta\widetilde{\boldsymbol{\theta}}^\varepsilon + \nabla Q^\varepsilon = 0, \text{ in } \ \Omega\times(0,T),\\ \mathrm{div}\widetilde{\boldsymbol{\theta}}^\varepsilon = 0,\\ \widetilde{\boldsymbol{\theta}}^\varepsilon|_{z=1} = (0,0,-\widetilde{\varphi}_3^{0,\varepsilon}|_{z=1}),\\ \widetilde{\boldsymbol{\theta}}^\varepsilon|_{z=0} = (-\widetilde{\varphi}_1^{0,\varepsilon}|_{z=0}, -\widetilde{\varphi}_2^{0,\varepsilon}|_{z=0}, 0),\\ \widetilde{\boldsymbol{\theta}}^\varepsilon \ \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions.} \end{cases} \tag{3.4}$$

$\square$

Remark 2 The boundary values of $\overline{\boldsymbol{\theta}}^{\varepsilon}$ and $\widetilde{\boldsymbol{\theta}}^{\varepsilon}$ satisfy the compatibility condition $\int_{z=0,1}\overline{\boldsymbol{\theta}}^{\varepsilon}\cdot\boldsymbol{n}\,d\Gamma=0$ and $\int_{z=0,1}\widetilde{\boldsymbol{\theta}}^{\varepsilon}\cdot\boldsymbol{n}\,d\Gamma=0$, thanks to the 2π-periodicity in x and y of u_1^0 and u_2^0.

To estimate the L^2- norm of the additional correctors, we set $\overline{\boldsymbol{\theta}}^{\varepsilon}=\sqrt{\varepsilon}\widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}$, $\Pi^{\varepsilon}=\varepsilon^{3/2}\widetilde{\Pi}^{\varepsilon}$, hence $\widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}$ satisfies the following system:

$$\begin{cases} -\Delta\widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}+\nabla\widetilde{\Pi}^{\varepsilon}=0, \quad \text{in } \Omega\times(0,T) \\ \operatorname{div}\widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}=0, \\ \widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}|_{z=0}=\left(0,0,-\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=0}\right), \\ \widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}|_{z=1}=\left(-\frac{\overline{\varphi}_1^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1},-\frac{\overline{\varphi}_2^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1},0\right), \\ \widetilde{\widetilde{\boldsymbol{\theta}}} \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions.} \end{cases} \tag{3.5}$$

Then we deduce from the direct estimates of the Stokes problem (see [1]) that:

$$\begin{aligned} \|\widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}\|_{L^2(\Omega)} &\le k\left\|\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=0}\right\|_{H^{-1/2}(\Gamma)}+k\left\|\frac{\overline{\varphi}_1^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1}\right\|_{H^{-1/2}(\Gamma)}+k\left\|\frac{\overline{\varphi}_2^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1}\right\|_{H^{-1/2}(\Gamma)} \\ &\le k\left\|\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}\right\|_{L^2(\Omega)}+e.s.t. \end{aligned}$$

Now we will estimate the L^2- norm of $\dfrac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}$, hence we have:

$$\begin{aligned} \left|\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}\right| &\le k\int_0^t\frac{1}{\sqrt{t-\tau}}e^{\frac{-z^2}{4\varepsilon(t-\tau)}}d\tau \\ &\le k\int_0^t\frac{1}{\sqrt{t-\tau}}d\tau\, e^{\frac{-z^2}{8\varepsilon T}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \left\|\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}\right\|_{L^2(\Omega)}^2 &\le k\int_0^1 e^{\frac{-z^2}{4\varepsilon T}}dz \\ &\le k\int_0^1 e^{\frac{-cz}{\sqrt{2\varepsilon T}}}dz,\ c>0 \\ &\le k\sqrt{\varepsilon}. \end{aligned}$$

Hence, we infer that

$$\|\widetilde{\widetilde{\boldsymbol{\theta}}}^{\varepsilon}\|_{L^2(\Omega)}\le k\varepsilon^{1/4}.$$

Finally, we get

$$\|\overline{\boldsymbol{\theta}}^{\varepsilon}\|_{L^2(\Omega)} \leq k\varepsilon^{3/4}. \tag{3.6}$$

In the following we will estimate the $L^2(\Omega)$ norm of the gradient of $\widetilde{\overline{\boldsymbol{\theta}}}^{\varepsilon}$, hence we find:

$$\begin{aligned}
\|\nabla\widetilde{\overline{\boldsymbol{\theta}}}^{\varepsilon}\|_{L^2(\Omega)} &\leq k\left\|\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=0}\right\|_{H^{1/2}(\Gamma)} + k\left\|\frac{\overline{\varphi}_1^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1}\right\|_{H^{1/2}(\Gamma)} + k\left\|\frac{\overline{\varphi}_2^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1}\right\|_{H^{1/2}(\Gamma)}\\
&\leq k\left\|\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}\right\|_{H^1(\Omega)} + e.s.t\\
&\leq k\varepsilon^{-1/4}.
\end{aligned}$$

Thus we deduce that:

$$\|\nabla\overline{\boldsymbol{\theta}}^{\varepsilon}\|_{L^2(\Omega)} \leq k\varepsilon^{1/4}. \tag{3.7}$$

We notice that the estimate (3.6) also holds for the time derivative of $\overline{\theta}^{\varepsilon}$, i.e.,

$$\left\|\frac{\partial\overline{\boldsymbol{\theta}}^{\varepsilon}}{\partial t}\right\|_{L^2(\Omega)} \leq k\varepsilon^{3/4}. \tag{3.8}$$

Remark 3 Note that by symmetry all the estimates satisfied by $\overline{\boldsymbol{\theta}}^{\varepsilon}$ remain valid for $\widetilde{\boldsymbol{\theta}}^{\varepsilon}$.

We now define $\boldsymbol{w}^{\varepsilon} = \boldsymbol{u}^{\varepsilon} - \boldsymbol{u}^0 - \overline{\boldsymbol{\varphi}}^{0,\varepsilon} - \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon} - \overline{\boldsymbol{\theta}}^{\varepsilon} - \widetilde{\boldsymbol{\theta}}^{\varepsilon}$, and according to (1.1), (1.2), (2.8), (3.3) and (3.4), $\boldsymbol{w}^{\varepsilon}$ verifies:

$$\begin{cases}
\dfrac{\partial\boldsymbol{w}^{\varepsilon}}{\partial t} - \varepsilon\Delta\boldsymbol{w}^{\varepsilon} + \boldsymbol{\omega}\times\boldsymbol{w}^{\varepsilon} + \nabla(p^{\varepsilon} - p^0 - \Pi^{\varepsilon} - Q^{\varepsilon}) = \varepsilon\dfrac{\partial^2\overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial x^2} + \varepsilon\dfrac{\partial^2\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial x^2}\\
+\varepsilon\dfrac{\partial^2\overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial y^2} + \varepsilon\dfrac{\partial^2\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial y^2} + \varepsilon\Delta\boldsymbol{u}^0 - \boldsymbol{\omega}\times\overline{\boldsymbol{\theta}}^{\varepsilon} - \boldsymbol{\omega}\times\widetilde{\boldsymbol{\theta}}^{\varepsilon} - \dfrac{\partial\overline{\boldsymbol{\theta}}^{\varepsilon}}{\partial t} - \dfrac{\partial\widetilde{\boldsymbol{\theta}}^{\varepsilon}}{\partial t}\\
+\left(0, 0, \dfrac{\partial\overline{\varphi}_3^{0,\varepsilon}}{\partial t} - \varepsilon\dfrac{\partial^2\overline{\varphi}_3^{0,\varepsilon}}{\partial z^2}\right) + \left(0, 0, \dfrac{\partial\widetilde{\varphi}_3^{1,\varepsilon}}{\partial t} - \varepsilon\dfrac{\partial^2\widetilde{\varphi}_3^{1,\varepsilon}}{\partial z^2}\right), \quad \text{in} \quad \Omega\times(0,T),\\
\operatorname{div}\boldsymbol{w}^{\varepsilon} = 0, \quad \text{in} \quad \Omega\times(0,T),\\
\boldsymbol{w}^{\varepsilon} = 0, \text{at } z = 0, 1,\\
\boldsymbol{w}^{\varepsilon} \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions},\\
\boldsymbol{w}^{\varepsilon}|_{t=0} = 0.
\end{cases} \tag{3.9}$$

We multiply $(3.9)_1$ by $\boldsymbol{w}^{\varepsilon}$, integrate over Ω, and apply the Cauchy–Shwarz inequality, we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d\|\boldsymbol{w}^{\varepsilon}\|^2}{dt} + \varepsilon\|\nabla\boldsymbol{w}^{\varepsilon}\|^2 &\leq \varepsilon\left\|\frac{\partial^2\overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial x^2}\right\|\|\boldsymbol{w}^{\varepsilon}\| + \varepsilon\left\|\frac{\partial^2\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial x^2}\right\|\|\boldsymbol{w}^{\varepsilon}\| + \varepsilon\left\|\frac{\partial^2\overline{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial y^2}\right\|\|\boldsymbol{w}^{\varepsilon}\|\\
&\quad + \varepsilon\left\|\frac{\partial^2\widetilde{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial y^2}\right\|\|\boldsymbol{w}^{\varepsilon}\| + \varepsilon\|\Delta u^0\|\|\boldsymbol{w}^{\varepsilon}\| + \|\overline{\boldsymbol{\theta}}^{\varepsilon}\|\|\boldsymbol{w}^{\varepsilon}\| + \|\widetilde{\boldsymbol{\theta}}^{\varepsilon}\|\|\boldsymbol{w}^{\varepsilon}\|
\end{aligned}$$

$$+\left\|\frac{\partial\overline{\boldsymbol{\theta}}^{\varepsilon}}{\partial t}\right\|\|\boldsymbol{w}^{\varepsilon}\|+\left\|\frac{\partial\widetilde{\boldsymbol{\theta}}^{\varepsilon}}{\partial t}\right\|\|\boldsymbol{w}^{\varepsilon}\|+\left\|\frac{\partial\overline{\varphi}_3^{0,\varepsilon}}{\partial t}-\varepsilon\frac{\partial^2\overline{\varphi}_3^{0,\varepsilon}}{\partial^2 z}\right\|\|w^{\varepsilon}\|$$
$$+\left\|\frac{\partial\widetilde{\varphi}_3^{1,\varepsilon}}{\partial t}-\varepsilon\frac{\partial^2\widetilde{\varphi}_3^{1,\varepsilon}}{\partial^2 z}\right\|\|\boldsymbol{w}^{\varepsilon}\|.$$

Hence according to (2.14), (3.6) and (3.8), we have:

$$\frac{1}{2}\frac{d\|\boldsymbol{w}^{\varepsilon}\|^2}{dt}+\varepsilon\|\nabla\boldsymbol{w}^{\varepsilon}\|^2\le\frac{1}{2}\|\boldsymbol{w}^{\varepsilon}\|^2+k\varepsilon^{3/2}.$$

Using the Gronwall inequality, we obtain

$$\|\boldsymbol{w}^{\varepsilon}\|_{L^{\infty}(0,T;\boldsymbol{L}^2(\Omega))}\le k\varepsilon^{3/4}\quad\text{and}\quad\|\nabla\boldsymbol{w}^{\varepsilon}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\le k\varepsilon^{1/4}.$$

Hence, according to (3.6), (3.7) and the triangular inequality, we deduce (3.1) and (3.2). This concludes the proof of Theorem 3.1.

4 A collocated finite volume scheme with a splitting method for the time discretization

We follow here the notations of [11] that we recall in this section for the reader convenience. In the following, we uniformly discretize the domain Ω by using cube finite volumes of dimensions $\Delta x\Delta y\Delta z$:

$$K_{i,j,k}=[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]\times[y_{j-\frac{1}{2}},y_{j+\frac{1}{2}}]\times[z_{k-\frac{1}{2}},z_{k+\frac{1}{2}}],$$

where:

$$x_{i+\frac{1}{2}}=i\Delta x,\quad y_{j+\frac{1}{2}}=j\Delta y,\quad z_{k+\frac{1}{2}}=k\Delta z,$$

$$\forall\, i=0,\dots,M,\quad\forall\, j=0,\dots,N,\quad\forall\, k=0,\dots,L.$$

The edges of the control volumes are defined by:

$$\Gamma_{i+1/2,j,k}=\{(x,y,z);\,x=x_{i+1/2},\,y\in[y_{j-1/2},y_{j+1/2}],\,z\in[z_{k-1/2},z_{k+1/2}]\},$$

$$\Gamma_{i,j+1/2,k}=\{(x,y,z);\,x\in[x_{i-1/2},x_{i+1/2}],\,y=y_{j+1/2},\,z\in[z_{k-1/2},z_{k+1/2}]\},$$

$$\Gamma_{i,j,k+1/2}=\{(x,y,z);\,x\in[x_{i-1/2},x_{i+1/2}],\,y\in[y_{j-1/2},y_{j+1/2}],\,z=z_{k+1/2}\},$$

$$\forall\, i=0,\dots,M,\forall\, j=0,\dots,N,\forall\, k=0,\dots,L.$$

The velocity and the pressure are approximated in the center of the cells as follows:

$$\boldsymbol{u}_{i,j,k}(t) \simeq \frac{1}{\Delta x \Delta y \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \boldsymbol{u}(x, y, z, t) dx dy dz,$$

$$p_{i,j,k}(t) \simeq \frac{1}{\Delta x \Delta y \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} p(x, y, z, t) dx dy dz,$$

where $\boldsymbol{u} = (u, v, w)$ and p are the solutions of the system (1.1). We consider this notation instead of $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and p^ε, introduced in Sect. 1, since we aim here to simplify our presentation when we discretize the system (1.1).
We also define the velocity fluxes:

$$F_{u_{i+\frac{1}{2},j,k}} \simeq \frac{1}{\Delta y \Delta z} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} u(x_{i+\frac{1}{2}}, y, z, t) dy dz,$$

$$F_{v_{i,j+\frac{1}{2},k}} \simeq \frac{1}{\Delta x \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} v(x, y_{j+\frac{1}{2}}, z, t) dx dz,$$

$$F_{w_{i,j,k+\frac{1}{2}}} \simeq \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} w(x, y, z_{k+\frac{1}{2}}, t) dx dy.$$

4.1 Time discretization

For the time discretization of the system (1.1), let Δt be the time step such that $\Delta t = T/N_t$, where N_t is an integer and $T > 0$ is the final time ($t \in [0, T]$). Then, we define $\boldsymbol{u}^k$ as the approximate solution of $\boldsymbol{u}$ at the time $t_k = k\Delta t$ for $k = 0, \ldots, N_t$. Therefore, we define the time discretization of $(1.1)_1$ as follows:

$$\frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \varepsilon \Delta \boldsymbol{u}^{n+1} + 2\boldsymbol{\omega} \times \boldsymbol{u}^n - \boldsymbol{\omega} \times \boldsymbol{u}^{n-1} + 2\nabla p^n - \nabla p^{n-1} = \boldsymbol{f}^{n+1}. \tag{4.1}$$

Thanks to (4.1) we are able to compute the new velocity $\boldsymbol{u}^{n+1}$.

Hence, to obtain the pressure, we take the divergence of $(1.1)_1$ and use the incompressibility condition $(1.1)_2$ we find:

$$\Delta p = \operatorname{div}(\boldsymbol{f} + \varepsilon \Delta \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}). \tag{4.2}$$

Thus we discretize (4.2) as follows:

$$\Delta p^{n+1} = \operatorname{div}(\boldsymbol{f}^{n+1} + \varepsilon \Delta \boldsymbol{u}^{n+1} - 2\boldsymbol{\omega} \times \boldsymbol{u}^n + \boldsymbol{\omega} \times \boldsymbol{u}^{n-1}). \tag{4.3}$$

By replacing Δ by $-\nabla \times \nabla\times$ (see [11] and [15]), we rewrite (4.3) as below:

$$\Delta p^{n+1} = \mathrm{div}(\boldsymbol{f}^{n+1} - \varepsilon\nabla \times \nabla \times \boldsymbol{u}^{n+1} - 2\boldsymbol{\omega} \times \boldsymbol{u}^n + \boldsymbol{\omega} \times \boldsymbol{u}^{n-1}). \tag{4.4}$$

Now, by using the relation $\Delta\boldsymbol{u}^{n+1} = \nabla\mathrm{div}\boldsymbol{u}^{n+1} - \nabla\times\nabla\times\boldsymbol{u}^{n+1}$, then (4.1) becomes:

$$\begin{aligned}&\boldsymbol{f}^{n+1} - \varepsilon\nabla \times \nabla \times \boldsymbol{u}^{n+1} - 2\omega \times \boldsymbol{u}^n + \boldsymbol{\omega} \times \boldsymbol{u}^{n-1}\\ &= \frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \varepsilon\nabla\mathrm{div}\boldsymbol{u}^{n+1} + 2\nabla p^n - \nabla p^{n-1}.\end{aligned}$$

Hence, we deduce from (4.4) that

$$\Delta p^{n+1} = \mathrm{div}\left(\frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \varepsilon\nabla\mathrm{div}\boldsymbol{u}^{n+1} + 2\nabla p^n - \nabla p^{n-1}\right). \tag{4.5}$$

Thus, we obtain

$$\Delta(p^{n+1} - 2p^n + p^{n-1} + \varepsilon\mathrm{div}\boldsymbol{u}^{n+1}) = \mathrm{div}\left(\frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t}\right). \tag{4.6}$$

Then we compute the pressure from

$$\begin{cases}\Delta\psi^{n+1} = \mathrm{div}\left(\dfrac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t}\right),\\ \dfrac{\partial\psi^{n+1}}{\partial n} = 0,\end{cases} \tag{4.7}$$

and

$$p^{n+1} = \psi^{n+1} + 2p^n - p^{n-1} - \varepsilon\mathrm{div}\boldsymbol{u}^{n+1}. \tag{4.8}$$

Concerning the boundary conditions, we have the periodicity in the x and y directions and the Dirichlet boundary conditions in the z direction for u^{n+1}:

$$\boldsymbol{u}^{n+1}_{0,j,k} = \boldsymbol{u}^{n+1}_{M,j,k}, \qquad \boldsymbol{u}^{n+1}_{M+1,j,k} = \boldsymbol{u}^{n+1}_{1,j,k},$$

$$\boldsymbol{u}^{n+1}_{i,0,k} = \boldsymbol{u}^{n+1}_{i,N,k}, \qquad \boldsymbol{u}^{n+1}_{i,N+1,k} = \boldsymbol{u}^{n+1}_{i,1,k},$$

$$\frac{\boldsymbol{u}^{n+1}_{i,j,L+1} + \boldsymbol{u}^{n+1}_{i,j,L}}{2} = 0, \qquad \frac{\boldsymbol{u}^{n+1}_{i,j,0} + \boldsymbol{u}^{n+1}_{i,j,1}}{2} = 0.$$

The Neumann boundary conditions are imposed for ψ^{n+1} in the z direction and the periodicity in x and y directions. Thus, we have

$$\psi^{n+1}_{0,j,k} = \psi^{n+1}_{M,j,k}, \qquad \psi^{n+1}_{M+1,j,k} = \psi^{n+1}_{1,j,k},$$

$$\psi^{n+1}_{i,0,k} = \psi^{n+1}_{i,N,k}, \qquad \psi^{n+1}_{i,N+1,k} = \psi^{n+1}_{i,1,k},$$

$$\psi^{n+1}_{i,j,L+1} = \psi^{n+1}_{i,j,L}, \qquad \psi^{n+1}_{i,j,0} = \psi^{n+1}_{i,j,1}.$$

The periodicity in x and y for the pressure yields:

$$p_{0,j,k} = p_{M,j,k}, \qquad p_{M+1,j,k} = p_{1,j,k},$$

$$p_{i,0,k} = p_{i,N,k}, \qquad p_{i,N+1,k} = p_{i,1,k},$$

and for the terms $p_{i,j,0}$ and $p_{i,j,L+1}$ we use the second order compact scheme to compute them:

$$p_{i,j,0} = \frac{5}{2}p_{i,j,1} - 2p_{i,j,2} + \frac{1}{2}p_{i,j,3}, \qquad p_{i,j,L+1} = \frac{5}{2}p_{i,j,L} - 2p_{i,j,L-1} + \frac{1}{2}p_{i,j,L-2}.$$

4.2 Finite volume discretization

To compute the velocity $\boldsymbol{u}^{n+1}$, we discretize (4.1) and we obtain:

$$\begin{aligned}
&\Delta x\Delta y\Delta z\frac{3\boldsymbol{u}^{n+1}_{i,j,k} - 4\boldsymbol{u}^{n}_{i,j,k} + \boldsymbol{u}^{n-1}_{i,j,k}}{2\Delta t} - \varepsilon\Bigg[\Delta x\Delta y\frac{\boldsymbol{u}^{n+1}_{i,j,k+1} - 2\boldsymbol{u}^{n+1}_{i,j,k} + \boldsymbol{u}^{n-1}_{i,j,k-1}}{\Delta z}\\
&\quad + \Delta y\Delta z\frac{\boldsymbol{u}^{n+1}_{i+1,j,k} - 2\boldsymbol{u}^{n+1}_{i,j,k} + \boldsymbol{u}^{n+1}_{i-1,j,k}}{\Delta x} + \Delta x\Delta z\frac{\boldsymbol{u}^{n+1}_{i,j+1,k} - 2\boldsymbol{u}^{n+1}_{i,j,k} + \boldsymbol{u}^{n+1}_{i,j-1,k}}{\Delta y}\Bigg]\\
&\quad + 2\begin{pmatrix} \frac{\Delta y\Delta z}{2}(p^{n}_{i+1,j,k} - p^{n}_{i-1,j,k})\\ \frac{\Delta x\Delta z}{2}(p^{n}_{i,j+1,k} - p^{n}_{i,j-1,k})\\ \frac{\Delta x\Delta y}{2}(p^{n}_{i,j,k+1} - p^{n}_{i-1,j,k-1})\end{pmatrix} - \begin{pmatrix} \frac{\Delta y\Delta z}{2}(p^{n-1}_{i+1,j,k} - p^{n-1}_{i-1,j,k})\\ \frac{\Delta x\Delta z}{2}(p^{n-1}_{i,j+1,k} - p^{n-1}_{i,j-1,k})\\ \frac{\Delta x\Delta y}{2}(p^{n-1}_{i,j,k+1} - p^{n-1}_{i-1,j,k-1})\end{pmatrix}\\
&\quad + \Delta x\Delta y\Delta z(\boldsymbol{\omega}\times(2\boldsymbol{u}^{n}_{i,j,k} - \boldsymbol{u}^{n-1}_{i,j,k})) = \Delta x\Delta y\Delta z\,\boldsymbol{f}^{n+1}_{i,j,k}.
\end{aligned} \tag{4.9}$$

To compute the pressure we first compute ψ^{n+1}:

$$\begin{aligned}
&\Delta x\Delta y\frac{\psi^{n+1}_{i,j,k+1} - 2\psi^{n+1}_{i,j,k} + \psi^{n+1}_{i,j,k-1}}{\Delta z} + \Delta y\Delta z\frac{\psi^{n+1}_{i+1,j,k} - 2\psi^{n+1}_{i,j,k} + \psi^{n+1}_{i-1,j,k}}{\Delta x}\\
&\quad + \Delta x\Delta z\frac{\psi^{n+1}_{i,j+1,k} - 2\psi^{n+1}_{i,j,k} + \psi^{n+1}_{i,j-1,k}}{\Delta y} = \frac{1}{2\Delta t}\Bigg[\Delta y\Delta z\Bigg[\bigg(3F_{u^{n+1}_{i+\frac{1}{2},j,k}} - 4F_{u^{n}_{i+\frac{1}{2},j,k}} + F_{u^{n-1}_{i+\frac{1}{2},j,k}}\bigg)\\
&\quad - \bigg(3F_{u^{n+1}_{i-\frac{1}{2},j,k}} - 4F_{u^{n}_{i-\frac{1}{2},j,k}} + F_{u^{n-1}_{i-\frac{1}{2},j,k}}\bigg)\Bigg] + \Delta x\Delta z\Bigg[\bigg(3F_{v^{n+1}_{i,j+\frac{1}{2},k}} - 4F_{v^{n}_{i,j+\frac{1}{2},k}} + F_{v^{n-1}_{i,j+\frac{1}{2},k}}\bigg)
\end{aligned}$$

$$- (3F_{v^{n+1}_{i,j-\frac{1}{2},k}} - 4F_{v^{n}_{i,j-\frac{1}{2},k}} + F_{v^{n-1}_{i,j-\frac{1}{2},k}})\Bigg] + \Delta x\Delta y\left[\left(3F_{w^{n+1}_{i,j,k+\frac{1}{2}}} - 4F_{w^{n}_{i,j,k+\frac{1}{2}}} + F_{w^{n-1}_{i,j,k+\frac{1}{2}}}\right)\right.$$
$$\left.\left. - \left(3F_{w^{n+1}_{i,j,k-\frac{1}{2}}} - 4F_{w^{n}_{i,j,k-\frac{1}{2}}} + F_{w^{n-1}_{i,j,k-\frac{1}{2}}}\right)\right]\right].$$

Then, we easily obtain the pressure:

$$p^{n+1}_{i,j,k} = \psi^{n+1}_{i,j,k} + 2p^{n}_{i,j,k} - p^{n-1}_{i,j,k} - \frac{\varepsilon}{\Delta x\Delta y\Delta z}\left[\Delta y\Delta z\left(F_{u^{n+1}_{i+\frac{1}{2},j,k}} - F_{u^{n+1}_{i-\frac{1}{2},j,k}}\right)\right.$$
$$\left. + \Delta x\Delta z\left(F_{v^{n+1}_{i,j+\frac{1}{2},k}} - F_{v^{n+1}_{i,j-\frac{1}{2},k}}\right) + \Delta x\Delta y\left(F_{w^{n+1}_{i,j,k+\frac{1}{2}}} - F_{w^{n+1}_{i,j,k-\frac{1}{2}}}\right)\right].$$

4.3 Computation of the fluxes

We recall here that the simplest method to compute the fluxes (linear interpolation) does not work when the viscosity ε is small. Hence the authors in [11] considered a modified interpolation method for the fluxes in two dimensional case. Now, since we aim here to study the boundary layers at small viscosity, we need, on the one hand, to adapt the discretization in [11] to the $3D$ dimensional case and, on the other hand, to introduce the correctors in the finite volume discretization basis that is the MFVM. Thus we first start by introducing the $3D$ fluxes inherited from [11]:

$$\begin{aligned}
F_{u^{n+1}_{i+\frac{1}{2},j,k}} &= \frac{u^{n+1}_{i+1,j,k} + u^{n+1}_{i,j,k}}{2} + \theta\frac{\Delta y\Delta z}{4a}(p^{n}_{i+2,j,k} - 2p^{n}_{i+1,j,k} + p^{n}_{i,j,k}) \\
&\quad - \theta\frac{\Delta y\Delta z}{4a}(p^{n}_{i+1,j,k} - 2p^{n}_{i,j,k} + p^{n}_{i-1,j,k}), \\
F_{v^{n+1}_{i+\frac{1}{2},j,k}} &= \frac{v^{n+1}_{i,j+1,k} + v^{n+1}_{i,j,k}}{2} + \theta\frac{\Delta x\Delta z}{4a}(p^{n}_{i,j+2,k} - 2p^{n}_{i,j+1,k} + p^{n}_{i,j,k}) \\
&\quad - \theta\frac{\Delta x\Delta z}{4a}(p^{n}_{i,j+1,k} - 2p^{n}_{i,j,k} + p^{n}_{i,j-1,k}), \\
F_{w^{n+1}_{i,j,k+\frac{1}{2}}} &= \frac{w^{n+1}_{i,j,k+1} + w^{n+1}_{i,j,k}}{2} + \theta\frac{\Delta x\Delta y}{4a}(p^{n}_{i,j,k+2} - 2p^{n}_{i,j,k+1} + p^{n}_{i,j,k}) \\
&\quad - \theta\frac{\Delta x\Delta y}{4a}(p^{n}_{i,j,k+1} - 2p^{n}_{i,j,k} + p^{n}_{i,j,k-1}), \\
&\quad \forall\, i = 0,\ldots,M, \forall\, j = 0,\ldots,N, \forall\, k = 0,\ldots,L,
\end{aligned}$$

where: θ is the relaxation coefficient and

$$a = \frac{3\Delta x\Delta y\Delta z}{2\Delta t} + 2\varepsilon\frac{\Delta x\Delta y}{\Delta z} + 2\varepsilon\frac{\Delta y\Delta z}{\Delta x} + 2\varepsilon\frac{\Delta x\Delta z}{\Delta y}.$$

5 Modified finite volume discretization

In this section we introduce a modified finite volume scheme, that is we approximate the solution of (1.1) by

$$\boldsymbol{u}_h = \sum_{i,j=1} \boldsymbol{r}_{i,j,0} \hat{\boldsymbol{\varphi}}^{0,\varepsilon} \chi_{i,j,0} + \sum_{i,j=1} \boldsymbol{r}_{\boldsymbol{i},\boldsymbol{j},\boldsymbol{L+1}} \hat{\tilde{\boldsymbol{\varphi}}}^{1,\varepsilon} \chi_{i,j,L+1} + \sum_{i,j,k} \boldsymbol{u}_{i,j,k} \chi_{i,j,k}, \quad (5.1)$$

where

$$\begin{aligned}
h &= \Delta z, \\
\boldsymbol{r}_{i,j,0} &= \frac{\boldsymbol{u}_{i,j,0} + \boldsymbol{u}_{i,j,1}}{2}, \\
\boldsymbol{r}_{i,j,L+1} &= \frac{\boldsymbol{u}_{i,j,L+1} + \boldsymbol{u}_{i,j,L}}{2}, \\
\chi_{i,j,0} &= \chi_{(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \times (0,h)}, \\
\chi_{i,j,L+1} &= \chi_{(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \times ((L-1)h, Lh)}, \\
\chi_{i,j,k} &= \chi_{(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \times (z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}})},
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varphi}_i^{0,\varepsilon} &= -\int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \frac{z}{2\sqrt{\varepsilon(t-\tau)}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \\
&\quad \times \{2\tau \cos(\alpha(\tau - t)) - 2\tau \sin(\alpha(\tau - t))\} d\tau, \quad \forall\, i = 1, 2,
\end{aligned}$$

$$\begin{aligned}
\hat{\tilde{\varphi}}_i^{1,\varepsilon} &= -\int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \frac{1-z}{2\sqrt{\varepsilon(t-\tau)}} e^{\frac{-(1-z)^2}{4\varepsilon(t-\tau)}} \\
&\quad \times \{2\tau \cos(\alpha(\tau - t)) - 2\tau \sin(\alpha(\tau - t))\} d\tau, \quad \forall\, i = 1, 2.
\end{aligned}$$

$$\hat{\varphi}_3^{0,\varepsilon} = \hat{\tilde{\varphi}}_3^{1,\varepsilon} = 0.$$

Multiplying $(1.1)_1$ by $\chi_{i,j,k}$, integrating over Ω, and replacing $\boldsymbol{u}^\varepsilon$ by $\boldsymbol{u}_h$ we find that the equations are the same as the classical finite volume scheme (4.9). Moreover the correctors verify $(2.2)_1$, hence they do not contribute to these equations. For the numerical simulations we do not use the modified boundary layer $\hat{\boldsymbol{\varphi}}^{0,\varepsilon}$ and $\hat{\tilde{\boldsymbol{\varphi}}}^{1,\varepsilon}$ directly. Instead we consider another approximate form which reads as follows:

$$\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}(t,z) = \left(-exp\left(\frac{-z^2}{4\varepsilon t}\right), -exp\left(\frac{-z^2}{4\varepsilon t}\right), 0 \right).$$

Indeed, the approximation $\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}$ is much easier to be implemented numerically than the theoretical corrector $\overline{\boldsymbol{\varphi}}^{\boldsymbol{0},\boldsymbol{\varepsilon}}$ obtained in Sect. 2 as in (2.6) and (2.7) (see Fig. 1).

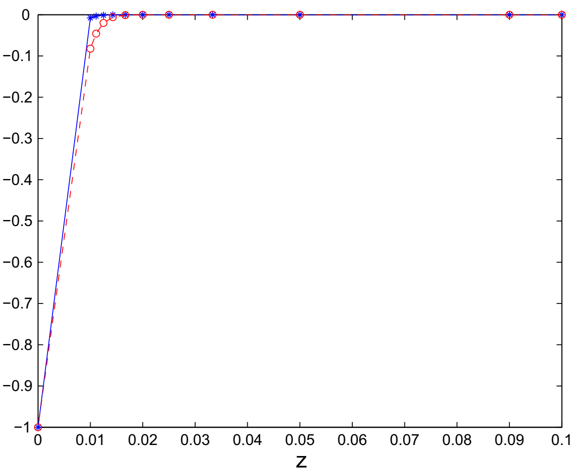

Fig. 1 The corrector $\hat{\boldsymbol{\varphi}}_{\mathbf{1}}^{\mathbf{0},\boldsymbol{\varepsilon}}$ (*asterisk*) and its approximation $\tilde{\tilde{\boldsymbol{\varphi}}}_1^{0,\varepsilon}$ (*circles*) at $t=1$, $\alpha=1$, $\varepsilon=10^{-5}$, near $z=0$

Due to the nodes $\boldsymbol{r}_{i,j,0}$ and $\boldsymbol{r}_{i,j,L+1}$, the linear system associated with this scheme is not closed. However, by adding the correctors, we are ensuring the closure of the linear system corresponding to the MFVM considered here. In the following, we will show how we handle this difficulty for the boundary layer at $z=0$, that is the coefficient $\boldsymbol{r}_{i,j,0}$, and we will skip the computations for the boundary layer at $z=1$, that is the coefficient $\boldsymbol{r}_{i,j,L+1}$, thanks to the symmetry. Hence, we multiply (4.1) by the corrector $\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}$ and integrate over $K_{i,j,1}$, we find:

$$\int_{K_{ij1}} \frac{3\boldsymbol{u}^{n+1}-4\boldsymbol{u}^{n}+\boldsymbol{u}^{n-1}}{2\Delta t}\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} - \varepsilon\int_{K_{ij1}} \Delta\boldsymbol{u}^{n+1}\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} + \int_{K_{ij1}} \boldsymbol{\omega}\times(2\boldsymbol{u}^{n}-\boldsymbol{u}^{n-1})\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}$$
$$+2\int_{K_{ij1}} \nabla p^{n}\tilde{\tilde{\varphi}}^{0,\varepsilon} - \int_{K_{ij1}} \nabla p^{n-1}\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} = \int_{K_{ij1}} \boldsymbol{f}^{n+1}\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}. \tag{5.2}$$

In the following we will calculate each term of (5.2). For the first term in the LHS (left-hand side) of (5.2) we find:

$$\int_{K_{ij1}} \frac{3\boldsymbol{u}^{n+1}-4\boldsymbol{u}^{n}+\boldsymbol{u}^{n-1}}{2\Delta t}\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}dxdydz = \frac{3\boldsymbol{u}_{i,j,1}^{n+1}-4\boldsymbol{u}_{i,j,1}^{n}+\boldsymbol{u}_{i,j,1}^{n-1}}{2\Delta t}\int_{K_{ij1}} \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}dxdydz.$$

For the second term in the LHS of (5.2), we obtain:

$$\begin{aligned}\int_{K_{ij1}} \Delta\boldsymbol{u}^{n+1}\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}dxdydz &= -\int_{K_{ij1}} \nabla\boldsymbol{u}^{n+1}\nabla\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}dxdydz + \int_{\partial K_{ij1}} \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}\frac{\partial\boldsymbol{u}^{n+1}}{\partial n}d\Gamma,\\ &= -\int_{K_{ij1}} \frac{\partial\boldsymbol{u}^{n+1}}{\partial z}\frac{\partial\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}}{\partial z}dxdydz + \int_{\partial K_{ij1}} \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon}\frac{\partial\boldsymbol{u}^{n+1}}{\partial n}d\Gamma.\end{aligned} \tag{5.3}$$

Now, we calculate the first term in the RHS of (5.3) and we find:

$$\begin{aligned}
\int_{K_{ij1}} \nabla \boldsymbol{u}^{n+1} \nabla \widetilde{\widetilde{\boldsymbol{\varphi}}}^{\mathbf{0},\boldsymbol{\varepsilon}} dxdydz &= \int_{K_{ij1}} \frac{\partial \boldsymbol{u}^{n+1}}{\partial z} \frac{\partial \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon}}{\partial z} dxdydz \\
&= \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{0}^{h/2} \frac{\partial \boldsymbol{u}^{n+1}}{\partial z} \frac{\partial \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon}}{\partial z} dxdydz \\
&\quad + \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{h/2}^{h} \frac{\partial \boldsymbol{u}^{n+1}}{\partial z} \frac{\partial \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon}}{\partial z} dxdydz \\
&= \frac{\boldsymbol{u}_{i,j,1}^{n+1} - \boldsymbol{r}_{i,j,1}^{n+1}}{\frac{h}{2}} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{0}^{h/2} \frac{\partial \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon}}{\partial z} dxdydz \\
&\quad + \frac{\boldsymbol{u}_{i,j,2}^{n+1} - \boldsymbol{u}_{i,j,1}^{n+1}}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{h/2}^{h} \frac{\partial \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon}}{\partial z} dxdydz \\
&= \frac{2}{h} (\boldsymbol{u}_{ij1}^{n+1} - \boldsymbol{r}_{ij0}^{n+1}) \Delta x \Delta y \left(\widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{h}{2} \right) - \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} (0) \right) \\
&\quad + \frac{\boldsymbol{u}_{i,j,2}^{n+1} - \boldsymbol{u}_{i,j,1}^{n+1}}{h} \Delta x \Delta y \left(\widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} (h) - \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{h}{2} \right) \right).
\end{aligned}$$

For the second term in the RHS of (5.3) we have

$$\begin{aligned}
\int_{\partial K_{ij1}} \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \frac{\partial \boldsymbol{u}^{n+1}}{\partial n} d\Gamma &= \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z=0} \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(-\frac{\partial \boldsymbol{u}}{\partial z} \right) d\Gamma \\
&\quad + \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z=h} \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{\partial \boldsymbol{u}}{\partial z} \right) d\Gamma \\
&\quad + \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{0}^{h} \int_{y=y_{j-1/2}} \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(-\frac{\partial \boldsymbol{u}}{\partial y} \right) d\Gamma \\
&\quad + \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{0}^{h} \int_{y=y_{j+1/2}} \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{\partial \boldsymbol{u}}{\partial y} \right) d\Gamma \\
&\quad + \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{0}^{h} \int_{x=x_{i-1/2}} \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(-\frac{\partial \boldsymbol{u}}{\partial x} \right) d\Gamma \\
&\quad + \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{0}^{h} \int_{x=x_{i+1/2}} \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{\partial \boldsymbol{u}}{\partial x} \right) d\Gamma.
\end{aligned}$$

Now, the third term in the LHS of (5.2) can be rewritten as below:

$$\int_{K_{ij1}} \boldsymbol{\omega} \times (2\boldsymbol{u}^n - \boldsymbol{u}^{n-1}) \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dxdydz = \boldsymbol{\omega} \times (2\boldsymbol{u}_{i,j,1}^n - \boldsymbol{u}_{i,j,1}^{n-1}) \Delta x \Delta y \int_0^h \widetilde{\widetilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dxdydz.$$

We calculate the first component of the fourth term in the LHS of (5.2) and we find:

$$\int_{K_{ij1}} \partial_x p^{n+1} \tilde{\tilde{\varphi}}_1^{0,\varepsilon} dxdydz = \frac{p_{i+1,j,1}^{n+1} - p_{i-1,j,1}^{n+1}}{2\Delta x} \Delta x \Delta y \int_0^h \tilde{\tilde{\varphi}}_1^{0,\varepsilon} dz.$$

For the second component of the fourth term in the LHS of (5.2) we have:

$$\int_{K_{ij1}} \partial_y p^{n+1} \tilde{\tilde{\varphi}}_2^{0,\varepsilon} dxdydz = \frac{p_{i,j+1,1}^{n+1} - p_{i,j-1,1}^{n+1}}{2\Delta y} \Delta x \Delta y \int_0^h \tilde{\tilde{\varphi}}_2^{0,\varepsilon} dz.$$

Concerning the first term on the RHS of (5.2), we obtain

$$\int_{K_{ij1}} \boldsymbol{f}^{n+1} \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dxdydz = \Delta x \Delta y \boldsymbol{f}_{i,j,1}^{n+1} \int_0^h \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dz.$$

Hence, we infer that

$$\begin{aligned}
&\frac{3\boldsymbol{u}_{i,j,1}^{n+1} - 4\boldsymbol{u}_{i,j,1}^{n} + \boldsymbol{u}_{i,j,1}^{n-1}}{2\Delta t} \int_0^h \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dz - \varepsilon \left[\frac{1}{h} \left(-3\tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{h}{2}\right) \boldsymbol{u}_{i,j,1}^{n+1} \right.\right. \\
&\left. + 2\boldsymbol{r}_{i,j,0}^{n+1} \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{h}{2}\right) + \boldsymbol{u}_{i,j,2}^{n+1} \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} \left(\frac{h}{2}\right) \right) - \left(\frac{1}{(\Delta x)^2} \left(\boldsymbol{u}_{i-1,j,1}^{n+1} - 2\boldsymbol{u}_{i,j,1}^{n+1} + \boldsymbol{u}_{i+1,j,1}^{n+1} \right) \right. \\
&\left.\left. + \frac{1}{(\Delta y)^2} \left(\boldsymbol{u}_{i,j-1,1}^{n+1} - 2\boldsymbol{u}_{i,j,1}^{n+1} + \boldsymbol{u}_{i,j+1,1}^{n+1} \right) \right) \int_0^h \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dz \right] + \boldsymbol{\omega} \times (2\boldsymbol{u}_{i,j,1}^{n} - \boldsymbol{u}_{i,j,1}^{n-1}) \int_0^h \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dz \\
&+ 2 \begin{pmatrix} \left(\frac{p_{i+1,j,1}^{n} - p_{i-1,j,1}^{n}}{2\Delta x}\right) \int_0^h \tilde{\tilde{\varphi}}_1^{0,\varepsilon} dz \\ \left(\frac{p_{i,j+1,1}^{n} - p_{i,j-1,1}^{n}}{2\Delta y}\right) \int_0^h \tilde{\tilde{\varphi}}_2^{0,\varepsilon} dz \\ 0 \end{pmatrix} - \begin{pmatrix} \left(\frac{p_{i+1,j,1}^{n-1} - p_{i-1,j,1}^{n-1}}{2\Delta x}\right) \int_0^h \tilde{\tilde{\varphi}}_1^{0,\varepsilon} dz \\ \left(\frac{p_{i,j+1,1}^{n-1} - p_{i,j-1,1}^{n-1}}{2\Delta y}\right) \int_0^h \tilde{\tilde{\varphi}}_2^{0,\varepsilon} dz \\ 0 \end{pmatrix} \\
&= \boldsymbol{f}_{i,j,1}^{n+1} \int_0^h \tilde{\tilde{\boldsymbol{\varphi}}}^{0,\varepsilon} dz.
\end{aligned}$$

6 Numerical results

In this section the error approximation is computed using the classical finite volume method and the modified finite volume method. For that purpose, the test solution for the pressure and the velocity are chosen as follows:

$$p(x, y, z, t) = t \cos(2\pi x) \cos(2\pi y) cos(\pi z),$$

$$u_1^\varepsilon(x, y, z, t) = t \sin(2\pi y) \left(1 - e^{\frac{-z}{\sqrt{\varepsilon}}} \cos\left(\frac{z}{\sqrt{\varepsilon}}\right)\right) \left(1 - e^{\frac{-(1-z)}{\sqrt{\varepsilon}}} \cos\left(\frac{1-z}{\sqrt{\varepsilon}}\right)\right),$$

Table 1 The L^2 norm of the velocity error with the CFVM and the MFVM for different values of ε at $t = 1$

N = M = L	ε	CFVM	MFVM
10		0.03206	0.12836
20	10^{-2}	0.00634	0.03893
30		0.00269	0.02553
10		0.09229	0.22647
20	10^{-3}	0.03372	0.15753
30		0.01331	0.08020
10		1.61660×10^3	0.04487
20	10^{-5}	0.08741	0.01030
30		0.11722	0.00460
10		$1, 10612 \times 10^{10}$	0.04490
20	10^{-6}	4.42881×10^6	0.01032
30		1.12960×10^3	0.00442
10		5.26218×10^{62}	0.04490
20	10^{-7}	1.16428×10^{29}	0.01032
30		6.72495×10^{17}	0.00443
10			0.04490
20	10^{-10}	The solution blows up	0.01032
30			0.00443

$$u_2^\varepsilon(x, y, z, t) = t \sin(2\pi x) \left(1 - e^{\frac{-z}{\sqrt{\varepsilon}}} \cos\left(\frac{z}{\sqrt{\varepsilon}}\right)\right) \left(1 - e^{\frac{-(1-z)}{\sqrt{\varepsilon}}} \cos\left(\frac{1-z}{\sqrt{\varepsilon}}\right)\right),$$

and

$$u_3^\varepsilon(x, y, z, t) = 0.$$

Note that the test solution given above satisfies the Eq. (1.1) with $\boldsymbol{u}_0 \equiv 0$, $\alpha = 1$[1], the periodicity condition in x and y with period 1 (instead of 2π for simplicity in the numerical simulations), and the resulting source function $\boldsymbol{f}$. More precisely, the source function is chosen using the test solution given above.

Now, to obtain the spatial accuracy of the schemes, we choose the time step $\Delta t = 10^{-2}$ and solve the system (1.1) with the above consideration for the data using the two methods (CFVM and MFVM) with different space step values $\Delta x = \Delta y = \Delta z = 1/10, 1/20, 1/30$. Moreover, the final time t is equal to 1 and the Reynolds number is taken in the range 10^2–10^{10}.

In what follows we will give some interpretations of the results obtained in Table 1 and Table 2. Let us start first by the velocity error stated in Table 1. By increasing

[1] Since we consider here a moderate rotation, we are not concerned with large values of α which ranges in this article between 1 and 50. Indeed, the case where α is large enough corresponds to the study of fast rotating fluids which are not the objective of this work, see e.g. [14,22,23].

Table 2 The L^2 norm of the pressure error with the CFVM and the MFVM for different values of ε at $t = 1$

N = M = L	ε	CFVM	MFVM
10		0.02493	0.03178
20	10^{-2}	0.00511	0.00920
30		0.00224	0.00533
10		0.02684	0.02771
20	10^{-3}	0.00553	0.00907
30		0.00238	0.00590
10		1.48996×10^2	0.02602
20	10^{-5}	0.00774	0.00539
30		0.00655	0.00238
10		1.01953×10^{16}	0.02601
20	10^{-6}	2.83861×10^5	0.00539
30		58.98117	0.00238
10		4.85027×10^{61}	0.02601
20	10^{-7}	7.46273×10^{27}	0.00539
30		3.51186×10^{16}	0.00238
10			0.02601
20	10^{-10}	The solution blows up	0.00539
30			0.00238

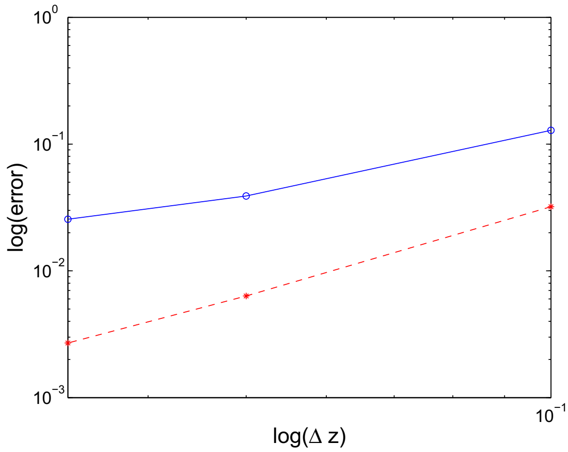

Fig. 2 The space discretization error on the velocity with CFVM (*asterisk*) and MFVM (*circles*) ($Re = 100, t = 1$)

the value of the Reynolds number, which is equivalent to decreasing ε, we can see that the MFVM attains better accuracy than the CFVM for $Re \gtrsim 10^5$, where $x \gtrsim y$ (respect. $x \lesssim y$) means $x \geq \mathcal{O}(y)$ (respect. $x \leq \mathcal{O}(y)$), whereas the CFVM does so when $Re \lesssim 10^3$. However, we noticed that, for $Re \gtrsim 10^5$, the CFVM becomes

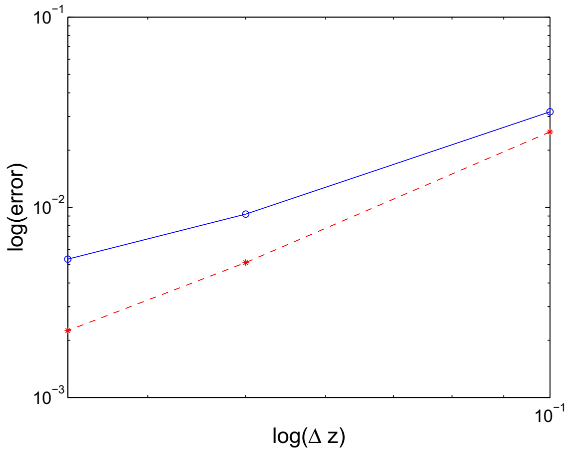

Fig. 3 The space discretization error on the pressure with CFVM (*asterisk*) and MFVM (*circles*) ($Re = 100, t = 1$)

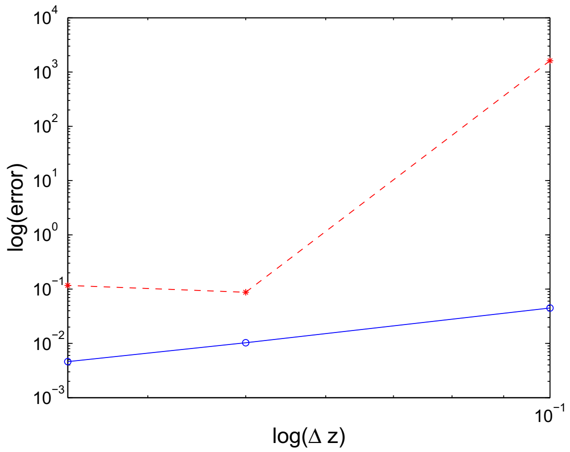

Fig. 4 The space discretization error on the velocity with CFVM (*asterisk*) and MFVM (*circles*)($Re = 10^5, t = 1$)

highly unstable and the MFVM does not. But, it is worth noting that the MFVM is less accurate than the CFVM when $Re \lesssim 10^3$ which is somehow natural because our new scheme MFVM is more designed for boundary layers, that is for high Reynolds number.

For the pressure error, as we observed for the velocity, the same conclusions deduced from Table 1 remain valid for Table 2.

Moreover, in Figs. 2 and 4 (respect. Figs. 3 and 5), we show the L^2-error on the velocity (respect. on the pressure) for both methods CFVM and MFVM at different values of the Reynolds number. More precisely, these errors are obtained for the

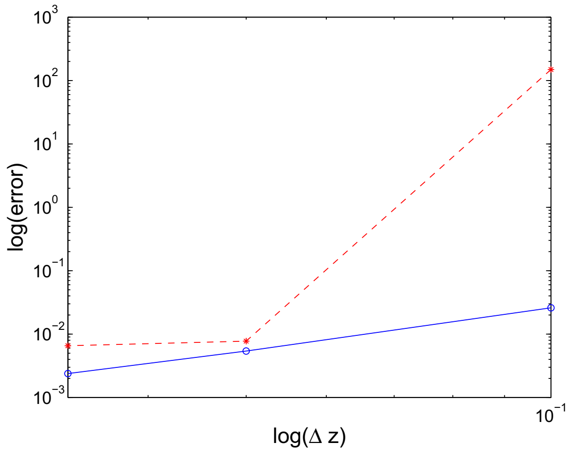

Fig. 5 The space discretization error on the pressure with CFVM (*asterisk*) and MFVM (*circles*) ($Re = 10^5, t = 1$)

velocity (respect. for the pressure) at $Re = 10^2$ in Fig. 2 (respect. Fig. 3), and in Fig. 4 (respect. Fig. 5) at $Re = 10^5$. According to Figs. 4 and 5, where we set the Reynolds number $Re = 10^5$, we observe that the errors values obtained from the MFVM are much smaller than the ones acquired from the CFVM, for both the velocity and the pressure.

7 Conclusion and future work

In this paper we have compared two different finite volume methods CFVM and MFVM when the viscosity is considered small and more precisely in the range of 10^{-2}–10^{-10}. To this end, we derived an approximate solution of the time-dependent rotating fluid in $3D$ channel using the splitting methods for the time discretization and colocated space discretization. One of the novelties of this article is that we propose a new numerical approach to treat the pressure and the incompressibility condition by introducing correctors which solve the boundary layers. We also showed that the MFVM is more performing than the CFVM *when the viscosity is small*, otherwise we showed that our MFVM still perform for very large Reynolds number. To the best of our knowledge, this is the first work which gives a modified finite volume scheme taking into account boundary layer without mesh refinement for the linearized Navier–Stokes equations. Note that the consideration of a physical viscosity in the numerical codes introduced in this article does not make the computations expensive (about tow hours when we consider $N = 30$ and $\varepsilon = 10^{-10}$). The method developed here may apply to many other problems and domains. This will be the subject of subsequent works.

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Appendix A

In this appendix, we give a sketch of the proof of the existence and regularity of the solution of the limit problem (1.2). For a complete study of the existence of solution of systems similar to (1.2) we refer the reader to [18], see also [5] and [16]. We first want to apply the Hille–Phillips–Yosida Theorem [28] to prove the existence and uniqueness of the solution of (1.2). Thus we start by introducing the adequate function spaces:

$$\begin{aligned} H = \{\boldsymbol{v} \in (L^2(\Omega))^3;\ \mathrm{div}\boldsymbol{v} = 0,\ v_3(z=0) = v_3(z=h) = 0, \\ \text{and } \boldsymbol{v} \text{ is } 2\pi \text{ periodic in the } x \text{ and } y \text{ directions}\}. \end{aligned}$$

$$D(A) = \{\boldsymbol{v} \in H;\ \exists\ p \in \mathcal{D}'(\Omega),\ \text{ such that } \boldsymbol{\omega} \times \boldsymbol{v} + \nabla p \in H\},$$

that we endow with the norm

$$\|\boldsymbol{v}\|_{D(A)} = (\|\boldsymbol{v}\|_H^2 + \|\boldsymbol{\omega} \times \boldsymbol{v} + \nabla p\|_H^2)^{1/2}. \tag{A.1}$$

Then for $\boldsymbol{v} \in D(A)$ we set $\mathcal{A}\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \nabla p$, thus we define an unbounded linear operator A which maps $D(A) \subset H$ onto H. Here, $\mathcal{A}$ denotes the differential operator associated with A.

Hence, we aim to apply the Hille–Phillips–Yosida Theorem for the system (1.2) which involves the operator A defined above. Here, we recall this well-known theorem.

Theorem A.1 (Hille–Phillips–Yosida Theorem) *Let H be a Hilbert space and let $B : D(B) \to H$ a linear unbounded operator, with domain $D(B) \subset H$ such that $D(B)$ is dense in H and $(-B)$ is m-dissipative. Then $(-B)$ is the infinitesimal generator of a contraction semigroup $\{S(t)\}_{t>0}$ in H, and the solution of the following system:*

$$\begin{cases} \dfrac{d\boldsymbol{v}}{dt} + B\boldsymbol{v} = \boldsymbol{f}, \\ \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0, \end{cases} \tag{A.2}$$

satisfies the following properties:

($\mathcal{P}_1$) If $\boldsymbol{v}_0$ and $f \in L^1(0, T; H)$ then $\boldsymbol{v} \in C^0([0, T]; H), \forall\, T > 0$.

($\mathcal{P}_2$) If $\boldsymbol{v}_0 \in D(B)$ and $f' \in L^1(0, T; H)$ then

$$\boldsymbol{v} \in C^1([0, T]; H) \cap C^0([0, T]; D(B)) \quad \textit{and} \quad \frac{d\boldsymbol{v}}{dt} \in L^\infty([0, T]; H), \forall\, T > 0.$$

The reader is referred to [28] and [2] for more details about the above result. Before proving that the operator A satisfies all the hypotheses of Theorem A.1, we first recall the definitions of dissipative and m-dissipative operators, see e.g. [8, Def. 3.13] and [6].

Definition A.1 A linear operator $A : D(A) \to H$ is called *dissipative* in H if and only if

$$\forall \boldsymbol{u} \in D(A), \forall \lambda > 0, \|\boldsymbol{u} - \lambda A\boldsymbol{u}\| \geq \|\boldsymbol{u}\|.$$

Definition A.2 A linear operator $A : D(A) \to H$ is called *m-dissipative* if A is dissipative and

$$\forall \boldsymbol{f} \in H, \forall \lambda > 0, \exists \boldsymbol{u} \in D(A), \boldsymbol{u} - \lambda A\boldsymbol{u} = \boldsymbol{f}.$$

Now, we can state and prove the existence result for the (1.2) as below.

Corollary A.1 *For given* $\boldsymbol{f} \in \mathcal{C}^1([0,T]; H)$ *and* $\mathbf{u}_0 \in D(A)$*, there exists a unique solution* $(\mathbf{u}^0, p^0)$ *to the system (1.2) with*

$$\begin{cases} \mathbf{u}^0 \in \mathcal{C}^1([0,T]; H) \cap \mathcal{C}^0([0,T]; D(A)), \\ \dfrac{d\mathbf{u}^0}{dt} \in \mathcal{C}^0([0,T]; H), \ \forall T > 0, \\ \nabla p^0 \in \mathcal{C}^0([0,T]; H), \ \forall T > 0. \end{cases} \tag{A.3}$$

Note that the pressure p^0 *is unique up to an additive constant.*

Proof First, using the operator A corresponding to (1.2) and introduced just before Theorem A.1, it is easy to see that the system (1.2) can be written in a similar setting as (A.2). Second, we now show that the operator $(-A)$ is m-dissipative. Hence, it is necessary to prove that the following system:

$$\begin{cases} \lambda\boldsymbol{\omega} \times \boldsymbol{u} + \lambda\nabla p + \boldsymbol{u} \ = \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} = 0, \\ u_3 = 0, \ \text{en } z = 0, 1, \end{cases} \tag{A.4}$$

has a unique solution in $D(A)$ for all $\boldsymbol{f} \in H$ and $\lambda > 0$, and in addition the solution of (A.4) satisfies the following estimate:

$$\|\boldsymbol{u}\|_H \leq \|\boldsymbol{f}\|_H, \quad \forall \boldsymbol{f} \in H. \tag{A.5}$$

For the existence issue, we will use the Lax–Milgram Theorem which necessitates the variational formulation of (A.4). Hence, we multiply (A.4) by $\boldsymbol{v} \in H$ and integrate over Ω, we find:

$$\lambda \int_\Omega (\boldsymbol{\omega} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, d\Omega + \lambda \int_\Omega \nabla p \cdot \boldsymbol{v} d\Omega + \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, d\Omega = \int_\Omega \boldsymbol{f}\boldsymbol{v}\, d\Omega.$$

Thanks to the fact that $\boldsymbol{v} \in H$, we have

$$\int_\Omega \nabla p \boldsymbol{v}\, d\Omega = -\int_\Omega p \operatorname{div} \boldsymbol{v}\, d\Omega + \int_{\partial\Omega} p\, \boldsymbol{v} \cdot n\, d\Gamma = 0.$$

Then, we set

$$a(\boldsymbol{u}, \boldsymbol{v}) = \lambda \int_{\Omega} (\boldsymbol{\omega} \times \boldsymbol{u}) \cdot \boldsymbol{v} \, d\Omega + \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{v} \, d\Omega,$$

and

$$F(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \boldsymbol{v} \, d\Omega.$$

Here a is a continuous and coercive bilinear form in $H \times H$. In fact we have:

$$\begin{aligned} |a(\boldsymbol{u}, \boldsymbol{v})| &\leq \lambda |\boldsymbol{u}|_H |\boldsymbol{v}|_H + |\boldsymbol{u}|_H |\boldsymbol{v}|_H, \\ &\leq k(\lambda) |u|_H |v|_H, \end{aligned}$$

and

$$|a(\boldsymbol{u}, \boldsymbol{u})| = |\boldsymbol{u}|_H^2.$$

Also $F(\boldsymbol{v})$ is a continuous linear form:

$$\int_{\Omega} \boldsymbol{f} \boldsymbol{v} \, d\Omega \leq |\boldsymbol{f}||\boldsymbol{v}|.$$

Hence, according to the Lax–Milgram Theorem, there exists a unique $\boldsymbol{u} \in H$ such that:

$$\lambda A u + \boldsymbol{u} = \boldsymbol{f},$$

that is,

$$\lambda \boldsymbol{\omega} \times \boldsymbol{u} + \lambda \nabla p + \boldsymbol{u} = \boldsymbol{f}.$$

Multiplying the above equation by $\boldsymbol{u}$ and integrating over Ω, we find:

$$\lambda \int_{\Omega} (\boldsymbol{\omega} \times \boldsymbol{u}) \boldsymbol{u} \, d\Omega + \lambda \int_{\Omega} \nabla p \boldsymbol{u} \, d\Omega + \int_{\Omega} \boldsymbol{u} \boldsymbol{u} \, d\Omega = \int_{\Omega} \boldsymbol{f} \boldsymbol{u} \, d\Omega,$$

then the solution $\boldsymbol{u}$ satisfies the estimate:

$$\|\boldsymbol{u}\|_H \leq \|\boldsymbol{f}\|_H.$$

Also we have:

$$\begin{aligned} \|\boldsymbol{u}\|_{D(A)} &= (\|\boldsymbol{u}\|_H^2 + \|\omega \times \boldsymbol{u} + \nabla p\|_H^2)^{1/2}, \\ &\leq \|\boldsymbol{u}\|_H + \|\boldsymbol{\omega} \times \boldsymbol{u} + \nabla p\|_H, \end{aligned}$$

$$\leq k(\lambda)\|\boldsymbol{f}\|_H.$$

Hence (-A) is a m-dissipative operator. Moreover we have $\boldsymbol{u}_0 \in H$, then according to the Hille–Yosida theorem the system (1.2) has a unique solution $\boldsymbol{u} \in C([0, \infty[, H)$. Furthermore, we have:

$$\begin{aligned}\|\nabla p\|_{H^{-1}} &\leq \frac{1}{\lambda}\|\boldsymbol{f}\|_{H^{-1}} + \frac{1}{\lambda}\|\boldsymbol{u}\|_{H^{-1}} + \|\boldsymbol{\omega} \times \boldsymbol{u}\|_{H^{-1}} \\ &\leq k(\lambda)\|\boldsymbol{f}\|_H.\end{aligned}$$

Then, we obtain

$$\|p\|_{L^2(\Omega)} \leq k(\lambda)\|\boldsymbol{f}\|_H.$$

□

Now, we end this appendix by stating and proving some regularity results for the solution of (1.2) which are straightforward obtained as a consequence of Theorem A.1. More precisely, we have the following.

Proposition A.1 *Let* $\boldsymbol{f} \in \mathcal{C}^1([0, T]; H \cap H^k(\Omega))$, $k \geq 1$ *and* $\mathbf{u}_0 \in D(A) \cap H^k(\Omega)$. *Then, the solution of (1.2) belongs to* $\mathcal{C}^0([0, T]; D(A) \cap H^k(\Omega))$.

Proof First, we observe that, since the rotation is assumed to be parallel to the $z-$direction, the Coriolis term vanishes in the normal direction, *i.e.* $(\boldsymbol{\omega} \times \boldsymbol{u}^0) \cdot \boldsymbol{e_3} = 0$. Second, we deduce the equation of the pressure p^0 by simply applying the divergence operator to $(1.2)_1$ and using $(1.2)_2$. Hence, we obtain

$$\Delta p^0 = \operatorname{div}(\boldsymbol{f} - \boldsymbol{\omega} \times \boldsymbol{u}^0). \tag{A.6}$$

Thanks to (A.3) and the regularity hypothesis on $\boldsymbol{f}$ as stated in Proposition A.1, we infer that $p^0 \in \mathcal{C}^1([0, T]; H^1(\Omega))$. Since $\boldsymbol{f}_x, \boldsymbol{f}_y \in \mathcal{C}^1([0, T]; H), k \geq 1$ and $\mathbf{u}_{0x}, \mathbf{u}_{0y} \in D(A)$ and using the invariance of the system (1.2) under differentiation in x and y, then Corollary A.1 implies that $p^0_x, p^0_y \in \mathcal{C}^1([0, T]; H^1(\Omega))$.

Third, we use the equation $(1.2)_1$ projected in the normal direction z and we infer that $p^0_z \in \mathcal{C}^0([0, T]; H^1(\Omega))$. Hence, we have $p^0 \in \mathcal{C}^0([0, T]; H^2(\Omega))$.

Using the two first equations in $(1.2)_1$ we deduce that $\mathbf{u}^0 \in \mathcal{C}^0([0, T]; D(A) \cap H^1(\Omega))$.

This mechanism allows us to prove a higher regularity for $\mathbf{u}^0$ since now we consider again Eq. (A.6) and we repeat the above steps.

This concludes the proof of Proposition A.1. □

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