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JID:YJDEQ AID:7869 /FLA [m1+; v1.205; Prn:9/05/2015; 20:13] P.1 (1-22)

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Journal of Differential Equations
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Mathematical analysis of a sharp–diffuse interfaces model for seawater intrusion

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Received 18 February 2015; revised 28 April 2015

Abstract

We consider a new model mixing sharp and diffuse interface approaches for seawater intrusion phenomena in free aquifers. More precisely, a phase field model is introduced in the boundary conditions on the virtual sharp interfaces. We thus include in the model the existence of diffuse transition zones but we preserve the simplified structure allowing front tracking. The three-dimensional problem then reduces to a two-dimensional model involving a strongly coupled system of partial differential equations of parabolic type describing the evolution of the depths of the two free surfaces, that is the interface between salt- and freshwater and the water table. We prove the existence of a weak solution for the model completed with initial and boundary conditions. We also prove that the depths of the two interfaces satisfy a coupled maximum principle.
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MSC: 35R35; 35K20; 76S05; 76T05

Keywords: Seawater intrusion; Free boundaries; Nonlinear parabolic partial differential equation; System of strongly coupled partial differential equations; Initial and boundary value problem

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http://dx.doi.org/10.1016/j.jde.2015.05.005
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Please cite this article in press as: C. Choquet et al., Mathematical analysis of a sharp–diffuse interfaces model for seawater intrusion, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.005

1. Introduction

Seawater intrusion in coastal aquifers is a major problem for water supply. It motivates the study of efficient and accurate models to simulate the displacement of saltwater front in unsaturated porous media for the optimal exploitation of fresh groundwater.

Observations show that near the coasts fresh and salty underground water tends to separate into two distinct layers. It was the motivation for the derivation of seawater intrusion models treating salt- and freshwater as immiscible fluids (see [6] in unsaturated media). Points where the salty phase disappears may be viewed as a sharp interface. Nevertheless the explicit tracking of the interfaces remains unworkable to implement without further assumptions. An additional assumption, the so-called Dupuit approximation, consists in considering that the hydraulic head is constant along each vertical direction. It allows to assume the existence of a smooth sharp interface. Classical sharp interface models are then obtained by vertical integration based on the assumption that no mass transfer occurs between the fresh and the salty area (see *e.g.* [3,8,14] and even the Ghyben–Herzberg static approximation). This class of models allows direct tracking of the salt front. Nevertheless the upscaling procedure perturbs the conservative form of the equations. In particular the maximum principle does not apply. Of course fresh and salty water are two miscible fluids. A physically correct approach shall include the existence of a transition zone characterized by the variations of the salt concentration. Moreover, the aquifer being a partially saturated porous medium, there is another transition zone between the completely saturated part and the dry part of the reservoir. But such a realism is very heavy from theoretical (degenerate equations) and numerical (full 3D problem) points of view ([6], see also [2] when further assuming a saturated medium; see [1] for numerical recipes).

In the present paper we choose a mixed approach. The model considered here takes advantage of the Dupuit approximation and thus reduces to a two-dimensional upscaled model. The three-dimensional character remains in the model through the free boundaries depths. We also superimpose a phase-field model, here an Allen–Cahn model in fluid/fluid context, for the modeling of the boundary conditions on the virtual sharp interfaces. We thus re-include in the model the existence of diffuse transition zones.

From a theoretical point of view, the addition of the two diffusive areas has the following advantages: The system has a parabolic structure, it is thus no longer necessary to introduce viscous terms in a preliminary fixed point step for avoiding degeneracy as is in the demonstration of [11]. But the main point is that we can demonstrate an efficient and logical maximum principle from the point of view of physics, which is not possible in the case of classical sharp interface approximation (see for instance [8,14]).

In the next section we present the model for the evolution of the depth h of the interface between freshwater and saltwater and of the depth h_1 of the interface between the saturated and unsaturated zone. The derivation of the model is based on the coupling of Darcy's law with the mass conservation principle written for freshwater and saltwater. After vertical upscaling a phase-field model is superimposed to mix the sharp and diffuse approaches. The resulting model consists in a system of strongly and nonlinearly coupled PDEs of parabolic type. The main result of the paper is presented in Section 3. We state an existence result of variational solutions for this model completed by initial and boundary conditions. Section 4 is devoted to the proof. We apply a Schauder fixed point strategy to a regularized problem penalized by the velocity of the fresh front. Then we establish uniform estimates allowing to turn back to the original problem.

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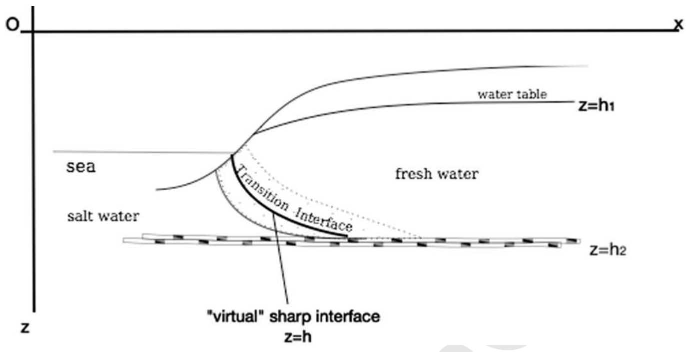

Fig. 1. Transition zone with variable salt concentration and corresponding virtual sharp interface.

2. The sharp–diffuse interfaces model

We mention the textbooks [3–5] for general informations about seawater intrusion problems. For the three-dimensional description, we denote by (x, z), $x = (x_1, x_2) \in \mathbb{R}^2$, $z \in \mathbb{R}$, the usual coordinates. Subscript f (respectively s) refers to quantities involved in the freshwater (resp. saltwater) domain.

Fluids and soil are considered to be incompressible. The basis of the modeling is the mass conservation law for each 'species' (fresh and salt water) coupled with the classical Darcy law for porous media:

$$\nabla \cdot q_i = Q_i, \quad q_i = -K_i \nabla \Phi_i,\ K_i = kg\rho_i/\mu_i, \quad i = f, s. \tag{1}$$

We have denoted by ϕ and k the porosity and the permeability of the medium. Density (resp. viscosity) of fluid i is ρ_i (resp. μ_i). Since $\rho_f \neq \rho_s$, the model is density driven. The gravitational acceleration constant is g. The hydraulic conductivity K_i expresses the ability of the underground to conduct fluid i. The hydraulic head of fluid i is denoted by Φ_i and its Darcy velocity by q_i. Quantities Q_i are generic source terms (for production and replenishment).

The next step consists in exploiting the slow dynamics of the displacement in the aquifer. It let the fluids tend to the picture described in Fig. 1. We assume that an abrupt interface separates two distinct domains, one for the saltwater and one for the freshwater. A sharp interface is also assumed separating the saturated and the dry parts of the aquifer, thus neglecting the thickness of the partially saturated zone. This approximation is justified because the thickness of the capillarity fringe is much smaller than the distance to the ground surface. We will alleviate these assumptions by re-including somehow mass transfers between layers. Before we integrate vertically equations (1), thus reducing the 3D problem to a 2D problem. We use Dupuit approximation of vertical equipotentials which is actually based on the very small slope of the observed interfacial layers.

The aquifer is represented by a three-dimensional domain $\Omega \times (h_2, h_{max})$, $\Omega \subset \mathbb{R}^2$, function h_2 (resp. h_{max}) describing its lower (resp. upper) topography. For the sake of simplicity, we assume that the upper surface of the aquifer is at constant depth, $h_{max} \in \mathbb{R}$, and moreover that $h_{max} = 0$.

We denote by h_1 (respectively h) the depth of the free interface separating the freshwater layer and the dry part of the aquifer (respectively the saltwater layer). Since we do not consider very deep geologic formations, we assume that the pressure is constant and equal to the atmospheric

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pressure P_a in the upper dry part of the aquifer, that is between $z = h_1$ and $z = 0$. We impose pressure equilibrium at the boundary of each area. More precisely, for the upper boundary:

$$\begin{cases} \text{If } h_1 < h_{max} = 0: \ \Phi_{f|z=h_1} = P_a/\rho_f g + h_1 - h_{ref}. \\ \text{If } h_1 = h_{max} = 0: \ \Phi_{f|z=h_{max}} = P_a/\rho_f g - h_{ref}. \end{cases} \tag{2}$$

It follows that the right quantity for the hydraulic head Φ_f to be meaningful in the whole aquifer is $h_1^- = \inf(0, h_1)$. The upper head equilibrium condition (2) reads $\Phi_{f|z=h_1^-} = P_a/\rho_f g + h_1^- - h_{ref}$. Similar elements on the depth of the salt interface h lead to introduce $h^- = \inf(0, h)$.

Now we perform the vertical integration of (1). We begin with the freshwater zone between depths h^- and h_1^-. We set:

$$\tilde{\Phi}_f = \frac{1}{B_f}\int_{h^-}^{h_1^-} \Phi_f \, dz, \quad Q_f = \frac{1}{B_f}\int_{h^-}^{h_1^-} Q_f \, dz,$$

$$\tilde{q}'_f = -\frac{1}{B_f}\int_{h^-}^{h_1^-} \left(K'_f \nabla' \Phi_f\right) dz = -\frac{1}{B_f}\int_{h^-}^{h_1^-} \left(K'_f \nabla' \tilde{\Phi}_f\right) dz = -\tilde{K}'_f \nabla' \tilde{\Phi}_f,$$

$$\tilde{K}'_f = \frac{1}{B_f}\int_{h^-}^{h_1^-} K'_f \, dz,$$

where $B_f = h_1^- - h^-$ is the thickness of the freshwater layer and $\nabla' = (\partial_{x_1}, \partial_{x_2})$. We apply Leibnitz rule and we use Dupuit's approximation, that is $\Phi_f(x_1, x_2, z) \simeq \tilde{\Phi}_f(x_1, x_2)$, $x = (x_1, x_2) \in \Omega$, $z \in (h^-, h_1^-)$. The averaged mass conservation law for the freshwater then reads

$$\nabla' \cdot (B_f \tilde{K}'_f \nabla' \tilde{\Phi}_f) - q_{f|z=h_1^-} \cdot \nabla(z - h_1^-) + q_{f|z=h^-} \cdot \nabla(z - h^-) + B_f \tilde{Q}_f = 0. \tag{3}$$

Similar computations in the saltwater layer give

$$\nabla' \cdot (B_s \tilde{K}'_s \nabla' \tilde{\Phi}_s) + q_{s|z=h_2} \cdot \nabla(z - h_2) - q_{s|z=h^-} \cdot \nabla(z - h^-) + B_s \tilde{Q}_s = 0, \tag{4}$$

where $B_s = h^- - h_2$. At this point, we have obtained an undetermined system of two PDEs (3)–(4) with four unknowns, $\tilde{\Phi}_i$, $i = f, s$, h_1^- and h^-.

We now include in the model the continuity and transfer properties across interfaces.

First, continuity relations for pressures on the interfaces allow to properly reduce the number of unknowns in Eqs. (3)–(4). Indeed Dupuit approximation $\tilde{\Phi}_f \simeq \Phi_{f|z=h_1^-}$, $\tilde{\Phi}_f \simeq \Phi_{f|z=0}$ and $\Phi_{s|z=h^-} \simeq \tilde{\Phi}_s$ combined with the definition of the hydraulic heads in term of pressures give

$$\tilde{\Phi}_f = \frac{P_a}{\rho_f g} + h_1^- - h_{ref}, \quad P_a \in \mathbb{R}, \tag{5}$$

$$(1 + \alpha)\,\tilde{\Phi}_s = \frac{P_a}{\rho_f g} + h_1^- + \alpha h^- - (1 + \alpha) h_{ref}, \quad \alpha = \frac{\rho_s}{\rho_f} - 1. \tag{6}$$

Here parameter α characterizes the densities contrast.

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The next step consists in the flux characterization. For the traditional sharp interface approach there is no mass transfer across the interface between fresh and salt water $z = h^-$:

$$\left(\frac{q_{f|z=h^-}}{\phi} - \vec{v}\right) \cdot \vec{n} = \left(\frac{q_{s|z=h^-}}{\phi} - \vec{v}\right) \cdot \vec{n} = 0, \tag{7}$$

where $\vec{n}$ denotes the normal unit vector to the interface, $\vec{n} = |\nabla(z - h^-)|^{-1}\nabla(z - h^-)$. Here we couple (7) with a tri-stable Allen–Cahn phase-field equation, one 'point' of stability being imposed on the sharp interface. Denoting by δ the characteristic size of the diffuse transition zone, the projection on the interface reads (see [7] for details):

$$-\partial_t h^- + \vec{v} \cdot \nabla(z - h^-) + \delta\Delta' h^- = 0. \tag{8}$$

Combining (7) and (8), we obtain the following regularized Stefan type boundary condition:

$$\begin{aligned} q_{f|z=h^-} \cdot \nabla(z - h^-) &= q_{s|z=h^-} \cdot \nabla(z - h^-) = \phi(\partial_t h^- - \delta\Delta' h^-) \\ &= \phi\big(\mathcal{X}_0(-h)\partial_t h - \delta\nabla' \cdot (\mathcal{X}_0(-h)\nabla' h)\big) \end{aligned} \tag{9}$$

where we set

$$\mathcal{X}_0(h) = \begin{cases} 0 & \text{if} \quad h \le 0 \\ 1 & \text{if} \quad h > 0 \end{cases}.$$

We perform the same reasoning for the upper capillary fringe. Likewise, we obtain

$$q_{f|z=h_1^-} \cdot \nabla(z - h_1^-) = \phi\big(\mathcal{X}_0(-h_1)\partial_t h_1 - \delta\nabla' \cdot \big(\mathcal{X}_0(-h_1)\nabla' h_1\big)\big) + \widetilde{q_{Lf}}, \tag{10}$$

where $\widetilde{q_{Lf}}$ is a fresh leakage term. For the lower boundary $z = h_2$ situation is more simple. Including a source term $\widetilde{q_{Ls}}$ accounting for leakage transfers coming from an eventual aquitard under the aquifer, we write

$$q_s(h_2) \cdot \nabla(z - h_2) = \widetilde{q_{Ls}}. \tag{11}$$

Finally we add some assumptions, essentially introduced for the sake of simplicity of the equations. The medium is supposed to be isotropic and the viscosity the same for the salt and fresh water. It follows from (1) and $\mu_f = \mu_s$ that

$$\tilde{K}'_s = (1 + \alpha)\tilde{K}'_f. \tag{12}$$

We choose to base the model on the salt mass conservation and on the total mass conservation. Rewriting (4) and summing up (3) and (4), we get

$$\begin{aligned} &-(1+\alpha)\nabla' \cdot \big(B_s \tilde{K}'_f \nabla' \tilde{\Phi}_s\big) + q_{s|z=h^-} \cdot \nabla\big(z - h^-\big) \\ &\qquad - q_{s|z=h_2} \cdot \nabla\big(z - h_2\big) = B_s \tilde{Q}_s, \\ &-\nabla' \cdot \big(B_f \tilde{K}'_f \nabla' \tilde{\Phi}_f\big) - (1+\alpha)\nabla' \cdot \big(B_s \tilde{K}'_f \nabla \tilde{\Phi}_s\big) + q_{f|z=h_1^-} \cdot \nabla\big(z - h_1^-\big) \\ &\qquad - q_{s|z=h_2} \cdot \nabla\big(z - h_2\big) = B_f \tilde{Q}_f + B_s \tilde{Q}_s. \end{aligned}$$

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We also reverse the vertical axis thus changing h_1 into $-h_1$, h into $-h$, h_2 into $-h_2$, z into $-z$. Bearing in mind that now $B_s = h_2 - h^+$, $B_f = h^+ - h_1^+$ and using (5)–(6), (9), (10) and (11), we write the latter system as:

$$(\mathcal{M})\quad \begin{cases} \phi\mathcal{X}_0(h)\partial_t h - \nabla'\cdot\big(\alpha\tilde{K}'_f(h_2-h^+)\nabla' h\big) - \nabla'\cdot\big(\delta\phi\mathcal{X}_0(h)\nabla' h\big) \\ \quad -\nabla'\cdot\big(\tilde{K}'_f\mathcal{X}_0(h_1)(h_2-h^+)\nabla' h_1\big) - q_{Ls}(x,h_1,h) = -\tilde{Q}_s(h_2-h^+), \\ \phi\mathcal{X}_0(h_1)\partial_t h_1 - \nabla'\cdot\big(\tilde{K}'_f\mathcal{X}_0(h_1)\big((h^+-h_1^+)+(h_2-h^+)\big)\nabla' h_1\big) \\ \quad -\nabla'\cdot\big(\delta\phi\tilde{K}'_f\mathcal{X}_0(h_1)\nabla' h_1\big) - \nabla'\cdot\big(\tilde{K}'_f\alpha(h_2-h^+)\mathcal{X}_0(h)\nabla' h\big) \\ \quad - q_{Lf}(x,h_1,h) - q_{Ls}(x,h_1,h) = -\tilde{Q}_f(h^+-h_1^+) - \tilde{Q}_s(h_2-h^+). \end{cases}$$

Leakage terms q_{Lf} and q_{Ls} are in the form (see [5])

$$\begin{aligned} q_{Lf}(x,h_1,h) &= \big(1-\chi_0(h_1)\big)\chi_0(h-h_1)Q_{Lf}(x), \\ q_{Ls}(x,h_1,h) &= \chi_0(h_2-h)Q_{Ls}(x)\big(R_{Ls}(x)+h_1/2+h/2\big). \end{aligned} \tag{13}$$

Indeed we specify that only fresh exchanges are allowed in q_{Lf}, thus the term $\chi_0(h-h_1)$, and that the semi-permeable zone is at depth $h_{max}=0$, thus the term $(1-\chi_0(h_1))$ (we consider here a phreatic aquifer: there is no leakage at the upper boundary unless the aquifer is fully saturated). Only salty exchanges occur at the bottom, thus the term $\chi_0(h_2-h)$ in q_{Ls}.

3. Mathematical setting and main results

We consider an open bounded domain Ω of $\mathbb{R}^2$ describing the projection of the aquifer on the horizontal plane. The boundary of Ω, assumed $\mathcal{C}^1$, is denoted by Γ. The time interval of interest is $(0,T)$, T being any nonnegative real number, and we set $\Omega_T=(0,T)\times\Omega$.

3.1. Some auxiliary results

For the sake of brevity we shall write $H^1(\Omega)=W^{1,2}(\Omega)$ and

$$V=H_0^1(\Omega),\ \ V'=H^{-1}(\Omega),\ \ H=L^2(\Omega).$$

The embeddings $V\subset H=H'\subset V'$ are dense and compact. For any $T>0$, let $W(0,T)$ denote the space

$$W(0,T):=\big\{\omega\in L^2(0,T;V),\ \partial_t\omega\in L^2(0,T;V')\big\}$$

endowed with the Hilbertian norm $\|\cdot\|_{W(0,T)}=\big(\|\cdot\|^2_{L^2(0,T;V)}+\|\partial_t\cdot\|^2_{L^2(0,T;V')}\big)^{1/2}$. The following embeddings are continuous [10, Prop. 2.1 and Thm. 3.1, Chapter 1]

$$W(0,T)\subset\mathcal{C}([0,T];[V,V']_{\frac12})=\mathcal{C}([0,T];H)$$

while the embedding

$$W(0,T)\subset L^2(0,T;H) \tag{14}$$

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is compact (Aubin's Lemma, see [13]). The following result by F. Mignot (see [9]) is used in the sequel.

Lemma 1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous and nondecreasing function such that* $\limsup_{|\lambda|\to+\infty} |f(\lambda)/\lambda| < +\infty$. *Let* $\omega \in L^2(0,T;H)$ *be such that* $\partial_t\omega \in L^2(0,T;V')$ *and* $f(\omega) \in L^2(0,T;V)$. *Then*

$$\langle \partial_t\omega, f(\omega)\rangle_{V',V} = \frac{d}{dt}\int_\Omega \Big(\int_0^{\omega(\cdot,y)} f(r)\,dr\Big)\,dy \text{ in } \mathcal{D}'(0,T).$$

Hence for all $0 \le t_1 < t_2 \le T$

$$\int_{t_1}^{t_2} \langle \partial_t\omega, f(\omega)\rangle_{V',V}\,dt = \int_\Omega \Big(\int_{\omega(t_1,y)}^{\omega(t_2,y)} f(r)\,dr\Big)\,dy.$$

3.2. Main results

We aim giving an existence result of physically admissible weak solutions for model $(\mathcal{M})$ completed by initial and boundary conditions.

First we re-write the model $(\mathcal{M})$ with some notational simplifications. The 'primes' are suppressed in the differentiation operators in $\mathbb{R}^2$. Source terms are denoted without 'tildes'. Permeability $\tilde{K}'_f$ is now denoted by K. We set $\alpha = 1$. We assume that depth h_2 is constant, $h_2 > 0$. We define some functions T_s and T_f by

$$T_s(u) = h_2 - u, \quad T_f(u) = u, \quad \text{for } u \in (0, h_2).$$

These functions are extended continuously and constantly outside $(0, h_2)$. We then consider the following set of equations in Ω_T:

$$\begin{aligned} &\phi\partial_t h - \nabla\cdot\big(KT_s(h)\mathcal{X}_0(h_1)\nabla h\big) - \nabla\cdot\big(\delta\phi\nabla h\big) - \nabla\cdot\big(KT_s(h)\mathcal{X}_0(h_1)\nabla h_1\big) \\ &\qquad - \phi q_{Ls}(x,h_1,h) = -Q_sT_s(h), \end{aligned} \tag{15}$$

$$\begin{aligned} &\phi\partial_t h_1 - \nabla\cdot\Big(K\big(T_f(h-h_1) + \mathcal{X}_0(h_1)T_s(h)\big)\nabla h_1\Big) - \nabla\cdot\big(\delta\phi K\nabla h_1\big) \\ &\qquad - \nabla\cdot\big(KT_s(h)\mathcal{X}_0(h_1)\nabla h\big) \\ &\qquad - \phi q_{Lf}(x,h_1,h) - \phi q_{Ls}(x,h_1,h) = -Q_fT_f(h-h_1) - Q_sT_s(h). \end{aligned} \tag{16}$$

Notice that we do not use $h^+ = \sup(0,h)$ and $h_1^+ = \sup(0,h_1)$ in functions T_s and T_f because a maximum principle will ensure that these supremums are useless. Likewise, we have canceled the terms $\mathcal{X}_0(h)$ (resp. $\mathcal{X}_0(h_1)$) in front of $\partial_t h$ and ∇h (resp. $\partial_t h_1$). Substitution of all the terms in the form $\nabla\cdot\big(KT_s(h)\nabla h\big)$ by $\nabla\cdot\big(\mathcal{X}_0(h_1)KT_s(h)\nabla h\big)$ does not change the physical meaning of the problem. System (16) is completed by the following boundary and initial conditions:

$$h = h_D, \quad h_1 = h_{1,D} \quad \text{in } \Gamma\times(0,T), \tag{17}$$

$$h(0,x) = h_0(x), \quad h_1(0,x) = h_{1,0}(x) \quad \text{in } \Omega, \tag{18}$$

with the compatibility conditions

$$h_0(x) = h_D(0, x), \quad h_{1,0}(x) = h_{1,D}(0, x), \quad x \in \Gamma.$$

Let us now detail the mathematical assumptions. We begin with the characteristics of the porous structure. We assume the existence of two positive real numbers K_- and K_+ such that the hydraulic conductivity tensor is a bounded elliptic and uniformly positive definite tensor:

$$0 < K_-|\xi|^2 \le \sum_{i,j=1,2} K_{i,j}(x)\xi_i\xi_j \le K_+|\xi|^2 < \infty \quad x \in \Omega,\ \xi \in \mathbb{R}^2,\ \xi \neq 0.$$

We assume that porosity is constant in the aquifer. Indeed, in the field envisaged here, the effects due to variations in ϕ are negligible compared with those due to density contrasts. From a mathematical point of view, these assumptions do not change the complexity of the analysis but rather avoid cumbersome computations.

Source terms Q_f and Q_s are given functions of $L^2(0, T; H)$. Leakage terms q_{Lf} and q_{Ls} are defined by (13) where Q_{Lf}, Q_{Ls} and $Q_{Ls}R_{Ls}$ are functions of $L^2(0, T; H)$ such that

$$Q_{Lf} \ge 0,\ Q_{Ls} \ge 0,\ R_{Ls} \ge 0 \text{ a.e. in } \Omega \times (0, T). \tag{19}$$

Assumption $Q_{Lf} \ge 0$ a.e. in Ω_T means that the leakage through the aquitard is upwards (indeed leakage occurs from low to high piezometric head, see [5]). We also assume

$$-\big(\max(Q_f, 0) + \max(Q_s, 0)\big)h_2 + Q_{Lf} + Q_{Ls}R_{Ls} \ge 0 \text{ a.e. in } \Omega \times (0, T). \tag{20}$$

This assumption which could appear rather technical is actually introduced because the aquifer's depth is at most h_2. All the source terms thus have to compensate somehow. Assumption (20) is the mathematical companion of the common-sense principle 'a filled box can no more be filled in'. Notice for instance that pumping of freshwater corresponds to assumption $Q_f \le 0$ a.e. in $\Omega \times (0, T)$. Functions h_D and $h_{1,D}$ belong to the space $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))')$ while functions h_0 and $h_{1,0}$ are in $H^1(\Omega)$. Finally, we assume that the boundary and initial data satisfy physically realistic conditions on the hierarchy of interfaces depths:

$$0 \le h_{1,D} \le h_D \le h_2 \text{ a.e. in } \Gamma \times (0, T), \quad 0 \le h_{1,0} \le h_0 \le h_2 \text{ a.e. in } \Omega.$$

We state and prove the following existence result.

Theorem 1. *Assume a low spatial heterogeneity for the hydraulic conductivity tensor:*

$$K_- \le K_+ \le 2K_-.$$

Then for any $T > 0$, problem (15)–(18) admits a weak solution (h, h_1) satisfying $(h - h_D, h_1 - h_{1,D}) \in W(0, T) \times W(0, T)$. Furthermore the following maximum principle holds true:

$$0 \le h_1(t, x) \le h(t, x) \le h_2 \quad \textit{for a.e. } x \in \Omega \textit{ and for any } t \in (0, T).$$

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Next section is devoted to the proof of Theorem 1. Let us sketch our strategy. First step consists in using a Schauder fixed point theorem for proving an existence result for an auxiliary regularized and truncated problem. More precisely we regularize the step function $\mathcal{X}_0$ with a parameter $\epsilon > 0$ and we introduce a weight based on the velocity of the freshwater front in the equation of the upper free interface. Subsequent difficulty is that the mapping used for the fixed point approach has to be continuous in the strong topology of $L^2(0, T; H^1(\Omega))$. We then prove that we have sufficient control on the velocity of the fresh front to ignore the latter weight. We show that the regularized solution satisfies the maximum principles announced in Theorem 1. We finally show sufficient uniform estimates to let the regularization ϵ tend to zero.

4. Proof

Without lost of generality, we can simplify the equations by taking null leakage terms $q_{Lf} = q_{Ls} = 0$ for the existence proof. The leakage terms will be re-inserted when stating the maximum principle results. Let $\epsilon > 0$ and pick a constant $M > 0$ that we will precise later. For any $x \in \mathbb{R}_+^*$, we set

$$L_M(x) = \min\Big(1, \frac{M}{x}\Big).$$

Such a truncation L_M was originally introduced in [12]. It allows to use the following point in the estimates hereafter. For $(g, g_1) \in (L^\infty(0, T; H^1(\Omega)))^2$, setting (here $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(0,T;H)}$)

$$d(g, g_1) = -T_s(g) L_M\big(||\nabla g_1||_{L^2}\big)\nabla g_1,$$

we have

$$\|d(g, g_1)\|_{L^2(0,T;H)} \leq M h_2.$$

We also define a regularized step function for $\mathcal{X}_0$ by

$$\mathcal{X}_0(h_1) = \begin{cases} 0 & \text{if} \quad h_1 \leq 0 \\ 1 & \text{if} \quad h_1 > 0 \end{cases}, \qquad \mathcal{X}_0^\epsilon(h_1) = \begin{cases} 0 & \text{if} \quad h_1 \leq 0 \\ h_1/(h_1^2 + \epsilon)^{1/2} & \text{if} \quad h_1 > 0. \end{cases}$$

Then $0 \leq \mathcal{X}_0^\epsilon \leq 1$ and $\mathcal{X}_0^\epsilon \to \mathcal{X}_0$ as $\epsilon \to 0$. Introducing the regularization $\mathcal{X}_0^\epsilon$ of $\mathcal{X}_0$, we replace system (16) by the following one:

$$\begin{aligned}
&\phi\partial_t h^\epsilon - \nabla\cdot(\delta\phi\nabla h^\epsilon) - \nabla\cdot\big(K T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla h^\epsilon\big) \\
&\qquad - \nabla\cdot\big(K T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon) L_M\big(||\nabla h_1^\epsilon||_{L^2}\big)\nabla h_1^\epsilon\big) = -Q_s T_s(h^\epsilon), \\
&\phi\partial_t h_1^\epsilon - \nabla\cdot(\delta\phi\nabla h_1^\epsilon) - \nabla\cdot\Big(K\big(T_f(h^\epsilon - h_1^\epsilon) + T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\big)\nabla h_1^\epsilon\Big) \\
&\qquad - \nabla\cdot\big(K T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla h^\epsilon\big) = -Q_f T_f(h^\epsilon - h_1^\epsilon) - Q_s T_s(h^\epsilon).
\end{aligned}$$

The proof is outlined as follows: In the first step, using the Schauder theorem, we prove that for every $T > 0$ and every $\epsilon > 0$, the above regularized system completed by the initial and boundary conditions

$$h^\epsilon = h_D,\ h_1^\epsilon = h_{1,D} \text{ in } \Gamma \times (0,T),$$
$$h^\epsilon(0,x) = h_0(x),\ h_1^\epsilon(0,x) = h_{1,0}(x) \text{ a.e. in } \Omega,$$

has a solution $(h^\epsilon, h_1^\epsilon)$ such that $(h^\epsilon - h_D, h_1^\epsilon - h_{1,D}) \in W(0,T) \times W(0,T)$. We observe that the sequence $(h^\epsilon - h_D,\ h_1^\epsilon - h_{1,D})$ is uniformly bounded in $(L^2(0,T;V))^2$ and we show the maximum principle $0 \le h_1^\epsilon(t,x) \le h^\epsilon(t,x) \le h_2$ a.e. in Ω_T for every $\epsilon > 0$. Finally we prove that any (weak) limit (h, h_1) in $(L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V'))^2$ of the sequence $(h^\epsilon,\ h_1^\epsilon)$ is a solution of the original problem.

4.1. Step 1: existence for the regularized system

We now omit ϵ for the sake of simplicity in the notations. Then the weak formulation of the latter problem reads: for any $w \in V$,

$$\begin{aligned}
&\int_0^T \phi \langle \partial_t h, w\rangle_{V',V} dt + \int_{\Omega_T} \delta\phi \nabla h \cdot \nabla w \, dx dt + \int_{\Omega_T} K T_s(h) \mathcal{X}_0^\epsilon(h_1) \nabla h \cdot \nabla w \, dx dt \\
&\quad + \int_{\Omega_T} K T_s(h) \mathcal{X}_0^\epsilon(h_1) L_M\big(||\nabla h_1||_{L^2}\big) \nabla h_1 \cdot \nabla w \, dx dt \\
&\quad + \int_{\Omega_T} Q_s T_s(h) w \, dx dt = 0, \qquad (21)
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \phi \langle \partial_t h_1, w\rangle_{V',V} dt + \int_{\Omega_T} \delta\phi \nabla h_1 \cdot \nabla w \, dx dt \\
&\quad + \int_{\Omega_T} K\Big(\big(\mathcal{X}_0^\epsilon(h_1) T_s(h) + T_f(h-h_1)\big)\nabla h_1 + T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\Big) \cdot \nabla w \, dx dt \\
&\quad + \int_{\Omega_T} \big(Q_f T_f(h-h_1) + Q_s T_s(h)\big) w \, dx dt = 0. \qquad (22)
\end{aligned}$$

For the fixed point strategy, we define the application $\mathcal{F}$ by

$$\begin{aligned}
\mathcal{F} : (L^2(0,T;H^1(\Omega)))^2 &\longrightarrow (L^2(0,T;H^1(\Omega)))^2 \\
(\bar{h}, \bar{h}_1) &\longmapsto \mathcal{F}(\bar{h}, \bar{h}_1) = \big(\mathcal{F}_1(\bar{h}, \bar{h}_1) = h, \mathcal{F}_2(\bar{h}, \bar{h}_1) = h_1\big),
\end{aligned}$$

where (h, h_1) is the solution of the following variational problem:

$$\begin{aligned}
&\int_0^T \phi \langle \partial_t h, w\rangle_{V',V} + \int_{\Omega_T} \delta\phi \nabla h \cdot \nabla w + \int_{\Omega_T} K T_s(\bar{h}) \mathcal{X}_0^\epsilon(\bar{h}_1) \nabla h \cdot \nabla w \\
&\quad + \int_{\Omega_T} K T_s(\bar{h}) L_M\big(||\nabla \bar{h}_1||_{L^2}\big) \mathcal{X}_0^\epsilon(\bar{h}_1) \nabla \bar{h}_1 \cdot \nabla w + \int_{\Omega_T} Q_s T_s(\bar{h}) w = 0, \qquad (23)
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \phi \langle \partial_t h_1, w\rangle_{V',V} + \int_{\Omega_T} \delta\phi \nabla h_1 \cdot \nabla w + \int_{\Omega_T} K\big(T_s(\bar{h})\mathcal{X}_0^\epsilon(\bar{h}_1) + T_f(\bar{h} - \bar{h}_1)\big) \nabla h_1 \cdot \nabla w \\
&\quad + \int_{\Omega_T} K T_s(\bar{h}) \mathcal{X}_0^\epsilon(\bar{h}_1) \nabla h \cdot \nabla w + \int_{\Omega_T} \big(Q_f T_f(\bar{h} - \bar{h}_1) + Q_s T_s(\bar{h})\big) w = 0, \qquad (24)
\end{aligned}$$

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for any $w \in V$. Indeed we know from classical parabolic theory (see *e.g.* [10]) that the linear variational system (23)–(24) admits a unique solution. The end of the present subsection is devoted to the proof of a fixed point property for application $\mathcal{F}$.

Continuity of $\mathcal{F}_1$: Let $(\bar{h^n}, \bar{h_1^n})$ be a sequence of functions of $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ and $(\bar{h}, \bar{h}_1)$ be a function of $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ such that

$$(\bar{h^n}, \bar{h_1^n}) \to (\bar{h}, \bar{h}_1) \text{ in } L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)).$$

We set $h_n = \mathcal{F}_1(\bar{h^n}, \bar{h_1^n})$ and $h = \mathcal{F}_1(\bar{h}, \bar{h}_1)$. We aim showing that $h_n \to h$ in $L^2(0,T;H^1(\Omega))$.

For all $n \in \mathbb{N}$, h_n satisfies (23). Choosing $w = h_n - h_D$ in the n-dependent counterpart of (23) yields:

$$\begin{aligned} &\int_0^T \phi \langle \partial_t (h_n - h_D), (h_n - h_D) \rangle_{V',V} \, dt + \int_{\Omega_T} \big(\delta\phi + K T_s(\bar{h}^n) \mathcal{X}_0^\epsilon(\bar{h}_1^n)\big) \nabla h_n \cdot \nabla h_n \, dx dt \\ &= - \int_{\Omega_T} K T_s(\bar{h}^n) L_M\big(||\nabla \bar{h}_1^n||_{L^2}\big) \mathcal{X}_0^\epsilon(\bar{h}_1^n) \nabla \bar{h}_1^n \cdot \nabla (h_n - h_D) \, dx dt \\ &\quad + \int_{\Omega_T} \big(\delta\phi + K T_s(\bar{h}^n) \mathcal{X}_0^\epsilon(\bar{h}_1^n)\big) \nabla h_n \cdot \nabla h_D \, dx dt \\ &\quad - \int_{\Omega_T} Q_s T_s(\bar{h}^n)(h_n - h_D) \, dx dt - \int_0^T \phi \langle \partial_t h_D, (h_n - h_D) \rangle_{V',V} \, dt. \end{aligned} \tag{25}$$

Function $h_n - h_D$ belongs to $L^2(0,T;V) \cap H^1(0,T;V')$ and then to $\mathcal{C}(0,T;L^2(\Omega))$. Thus, thanks moreover to Lemma 1, we write

$$\int_0^T \phi \langle \partial_t (h_n - h_D), (h_n - h_D) \rangle_{V',V} \, dt = \frac{\phi}{2} ||h_n(\cdot, T) - h_D||_H^2 - \frac{\phi}{2} ||h_0 - h_{D|t=0}||_H^2.$$

Besides

$$\int_{\Omega_T} \big(\delta\phi + K T_s(\bar{h}^n) \mathcal{X}_0^\epsilon(\bar{h}_1^n)\big) \nabla h_n \cdot \nabla h_n \, dx dt \ge \delta\phi ||\nabla h_n||^2_{L^2(0,T;H)}.$$

Then applying the Cauchy–Schwarz and Young inequalities, we get for any $\epsilon_1 > 0$

$$\begin{aligned} &\Big| \int_{\Omega_T} \big(\delta\phi + K T_s(\bar{h}^n) \mathcal{X}_0^\epsilon(\bar{h}_1^n)\big) \nabla h_n \cdot \nabla h_D \Big| \\ &\qquad \le (\delta\phi + K_+ h_2) ||\nabla h_n||_{L^2(0,T;H)} ||\nabla h_D||_{L^2(0,T;H)} \\ &\qquad \le \frac{\varepsilon_1}{2} ||\nabla h_n||^2_{L^2(0,T;H)} + \frac{(\delta\phi + K_+ h_2)^2}{2\varepsilon_1} ||\nabla h_D||^2_{L^2(0,T;H)}, \end{aligned}$$

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$$\begin{aligned}\Big| -\int_{\Omega_T} KT_s(\bar{h}^n)L_M\big(\|\nabla\bar{h}_1^n\|_{L^2}\big)\mathcal{X}_0^\epsilon(\bar{h}_1^n)\nabla\bar{h}_1^n\cdot\nabla h_n\Big| \\ \le K_+\|d(\bar{h}^n,\bar{h}_1^n)\|_{L^2(0,T;H)}\|\nabla h_n\|_{L^2(0,T;H)} \\ \le MK_+h_2\|\nabla h_n\|_{L^2(0,T;H)} \le \frac{K_+^2M^2}{2\varepsilon_1}h_2^2+\frac{\varepsilon_1}{2}\|\nabla h_n\|^2_{L^2(0,T;H)}.\end{aligned}$$

Since it depends on h_D, the next term is simply estimated by

$$\begin{aligned}\Big|\int_{\Omega_T} KT_s(\bar{h}^n)L_M\big(\|\nabla\bar{h}_1^n\|_{L^2}\big)\mathcal{X}_0^\epsilon(\bar{h}_1^n)\nabla\bar{h}_1^n\cdot\nabla h_D\,dxdt\Big| \\ \le K_+\|d(\bar{h}^n,\bar{h}_1^n)\|_{L^2(0,T;H)}\|h_D\|_{L^2(0,T;H^1)} \le MK_+h_2\|h_D\|_{L^2(0,T;H^1)}.\end{aligned}$$

Finally we have

$$\begin{aligned}\Big|-\int_0^T \phi\langle\partial_t h_D,(h_n-h_D)\rangle_{V',V}\,dt\Big| \\ \le \frac{\phi}{2\delta}\|\partial_t h_D\|^2_{L^2(0,T;(H^1(\Omega))')}+\frac{\delta\phi}{2}\|h_n\|^2_{L^2(0,T;H^1)}+\frac{\delta\phi}{2}\|h_D\|^2_{L^2(0,T;H^1)},\end{aligned}$$

and

$$\Big|-\int_{\Omega_T} Q_sT_s(\bar{h}^n)(h_n-h_D)\,dxdt\Big| \le \frac{\|Q_s\|^2_{L^2(0,T;H)}}{2\phi}h_2^2+\frac{\phi}{2}\|h_n-h_D\|^2_{L^2(0,T;H)}.$$

Using all the latter estimates in (25), we get after simplifications

$$\begin{aligned}&\frac{\phi}{2}\|h_n(\cdot,T)-h_D\|_H^2+\Big(\frac{\delta\phi}{2}-\varepsilon_1\Big)\|\nabla h_n\|^2_{L^2(0,T;H)} \\ &\quad\le \frac{\phi}{2}\|h_0-h_{D|t=0}\|_H^2+\Big(\frac{\|Q_s\|^2_{L^2(0,T;H)}}{2\phi}+\frac{K_+^2M^2}{2\varepsilon_1}\Big)h_2^2 \\ &\qquad+\frac{(\delta\phi(1+\varepsilon_1)+K_+h_2)^2}{2\varepsilon_1}\|h_D\|^2_{L^2(0,T;H^1)}+\frac{\phi}{2\delta}\|\partial_t h_D\|^2_{L^2(0,T;(H^1(\Omega))')} \\ &\qquad+\frac{\phi}{2}\int_0^T\|h_n-h_D\|_H^2\,dt+\frac{\delta\phi}{2}\int_0^T\|h_n\|_H^2\,dt \\ &\qquad+MK_+h_2\|h_D\|_{L^2(0,T;H^1)}+\frac{\phi}{2}\|h_D\|^2_{L^2(0,T;H)}. \qquad (26)\end{aligned}$$

We choose ε_1 such that $\delta\phi/2-\varepsilon_1\ge\epsilon_0>0$ for some $\epsilon_0>0$. Relation (26) with the Gronwall lemma enables to conclude that there exist real numbers $A_M=A_M(\phi,\delta,K,h_0,h_D,h_2,Q_s,M)$ and $B_M=B_M(\phi,\delta,K,h_0,h_D,h_2,Q_s,M)$ depending only on the data of the problem such that

$$\|h_n\|_{L^\infty(0,T;H)}\le A_M, \qquad \|h_n\|_{L^2(0,T;H^1)}\le B_M. \qquad (27)$$

Please cite this article in press as: C. Choquet et al., Mathematical analysis of a sharp–diffuse interfaces model for seawater intrusion, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.005

Hence the sequence $(h_n)_n$ is uniformly bounded in $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;H)$. Notice that the estimate in $L^\infty(0,T;H)$ is justified by the fact that we could make the same computations replacing T by any $\tau \le T$ in the time integration. In the sequel, we set

$$C_M = \max(A_M, B_M).$$

We now prove that $(\partial_t(h_n - h_D))_n$ is bounded in $L^2(0,T;V')$. Due to the assumption $h_D \in H^1(0,T;(H^1(\Omega))')$, it will follow that $(h_n)_n$ is uniformly bounded in $H^1(0,T;V')$. We have

$$\begin{aligned}
&||\partial_t(h_n - h_D)||_{L^2(0,T;V')}\\
&\quad= \sup_{||w||_{L^2(0,T;V)}\le 1} \Big| \int_0^T \langle \partial_t(h_n - h_D), w\rangle_{V',V} \Big|\\
&\quad= \sup_{||w||_{L^2(0,T;V)}\le 1} \Big| \int_0^T -\langle \partial_t h_D, w\rangle_{V',V} - \frac{1}{\phi}\Big(\int_{\Omega_T} \big(\delta\phi + KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla w\\
&\qquad + \int_{\Omega_T} KT_s(\bar h^n)L_M\big(||\nabla \bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla \bar h_1^n\cdot\nabla w + \int_{\Omega_T} Q_s T_s(\bar h^n) w\Big)\Big|.
\end{aligned}$$

Since

$$\begin{aligned}
&\Big|\int_{\Omega_T} \big(\delta\phi + KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla w\Big|\\
&\qquad\le \big(\delta\phi + K_+ h_2\big)||h_n||_{L^2(0,T;H^1(\Omega))}||w||_{L^2(0,T;V)},
\end{aligned}$$

and since h_n is uniformly bounded in $L^2(0,T;H^1(\Omega))$, we write

$$\Big|\int_{\Omega_T} \big(\delta\phi + KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla w\,dxdt\Big| \le \big(\delta\phi + K_+h_2\big)C_M||w||_{L^2(0,T;V)}. \tag{28}$$

Furthermore we have

$$\Big|\int_{\Omega_T} T_s(\bar h^n)L_M\big(||\nabla \bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla \bar h_1^n\cdot\nabla w\,dxdt\Big| \le Mh_2||w||_{L^2(0,T;V)} \tag{29}$$

and

$$\Big|\int_{\Omega_T} Q_s T_s(\bar h^n) w\,dxdt\Big| \le \|Q_s\|_{L^2(0,T;H)} h_2 ||w||_{L^2(0,T;V)}. \tag{30}$$

Summing up (28)–(30), we conclude that

$$||\partial_t(h_n - h_D)||_{L^2(0,T;V')} \le D_M, \tag{31}$$

$$D_M = ||\partial_t h_D||^2_{L^2(0,T;(H^1(\Omega))')} + \delta C_M + \frac{h_2}{\phi}(K_+C_M + M + \|Q_s\|_{L^2(0,T;H)}).$$

We have proved that the sequence $\big(h_n\big)_n$ is uniformly bounded in the space $L^2(0,T;H^1(\Omega))\cap H^1(0,T;V')$. Using Aubin's lemma, we extract a subsequence, not relabeled for convenience, $(h_n)_n$, converging strongly in $L^2(\Omega_T)$ and weakly in $L^2(0,T;H^1(\Omega))\cap H^1(0,T;V')$ to some limit denoted by ℓ. Using in particular the strong convergence in $L^2(\Omega_T)$ and thus the convergence a.e. in Ω_T, we check that ℓ is a solution of Eq. (23). The solution of (23) being unique, we have $\ell = h$.

It remains to prove that $(h_n)_n$ actually tends to h strongly in space $L^2(0,T;H^1(\Omega))$. Subtracting the weak formulation (23) to its n-dependent counterpart for the test function $w = h_n - h$, we get

$$\begin{aligned}
&\int_0^T \phi\langle\partial_t(h_n-h), h_n-h\rangle_{V',V} \\
&+\int_{\Omega_T}\big(\delta\phi + KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla(h_n-h)\cdot\nabla(h_n-h) \\
&-\int_{\Omega_T} K\big(T_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n) - T_s(\bar h)\mathcal{X}_0^\epsilon(\bar h_1)\big)\nabla(h_n-h)\cdot\nabla h \\
&+\int_{\Omega_T} K\Big(T_s(\bar h^n)L_M\big(||\nabla \bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla\bar h_1^n - T_s(\bar h)L_M\big(||\nabla\bar h_1||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1)\nabla\bar h_1\Big)\cdot\nabla(h_n-h) \\
&+\int_{\Omega_T} Q_s\big(T_s(\bar h^n)-T_s(\bar h)\big)(h_n-h) = 0.
\end{aligned}\tag{32}$$

Using assumption $(\bar h^n, \bar h_1^n)\to(\bar h,\bar h_1)$ in $L^2(0,T;H^1(\Omega))\times L^2(0,T;H^1(\Omega))$ and the above results of convergence for h_n, the limit as $n\to\infty$ in (32) reduces to

$$\lim_{n\to\infty}\Big(\int_{\Omega_T}\big(\delta\phi + KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla(h_n-h)\cdot\nabla(h_n-h)\,dxdt\Big) = 0.$$

Due to the positiveness of K, we infer from the latter relation that

$$\lim_{n\to\infty}\Big(\int_{\Omega_T}\delta\phi|\nabla(h_n-h)|^2\,dxdt + \int_{\Omega_T}K_-T_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)|\nabla(h_n-h)|^2\,dxdt\Big)\le 0.$$

Hence $\nabla h_n\to\nabla h$ strongly in $L^2(0,T;H)$. Continuity of $\mathcal{F}_1$ for the strong topology of $L^2(0,T;H^1(\Omega))$ is proved.

Continuity of $\mathcal{F}_2$: Likewise, we prove the continuity of $\mathcal{F}_2$ by setting $h_{1,n} = \mathcal{F}_2(\bar h^n, \bar h_1^n)$ and $h_1 = \mathcal{F}_2(\bar h, \bar h_1)$ and showing that $h_{1,n}\to h_1$ in $L^2(0,T;H^1(\Omega))$. The key estimates are obtained using the same type of arguments than in the proof of the continuity of $\mathcal{F}_1$. We thus do not detail the computations. Let us only emphasize that we can now use the estimate (27) previously derived for h^n, thus the dependence with regard to C_M in the following estimates:

$$||h_{1,n}||_{L^\infty(H)}\le E_M = E_M\big(\phi,\delta,K,h_{1,0},h_{1,D},h_2,Q_s,Q_f,M,C_M\big),\tag{33}$$

$$||h_{1,n}||_{L^2(0,T;H^1)}\le F_M = F_M\big(\phi,\delta,K,h_{1,0},h_{1,D},h_2,Q_s,Q_f,M,C_M\big).\tag{34}$$

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We set

$$C_{1,M} = \max(E_M, F_M).$$

One also computes that

$$||\partial_t h_{1,n}||_{L^2(0,T;V')} \le D_{1,M}, \tag{35}$$
$$D_{1,M} = \frac{1}{\phi}\Big(\delta\phi + \Big(2K_+ C_M + K_+ C_{1,M} + 2\big(\|Q_f\|_{L^2(0,T;H)} + \|Q_s\|_{L^2(0,T;H)}\big)\Big)h_2\Big).$$

Conclusion. $\mathcal{F}$ is continuous in $(L^2(0,T;H^1(\Omega)))^2$ because its two components $\mathcal{F}_1$ and $\mathcal{F}_2$ are. Furthermore, let $A \in \mathbb{R}_+^*$ be the real number defined by

$$A = \max(C_M, D_M, C_{1,M}, D_{1,M}),$$

and W be the nonempty (strongly) closed convex bounded subset of space $(L^2(0,T;H^1(\Omega)))^2$ defined by

$$W = \Big\{(g, g_1) \in \big(L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')\big)^2;\ (g(0), g_1(0)) = (h_0, h_{1,0}),$$
$$(g_{|\Gamma}, g_{1|\Gamma}) = (h_D, h_{1,D}),\ ||(g, g_1)||_{(L^2(0,T;H^1(\Omega))\cap H^1(0,T;V'))^2} \le A\Big\}.$$

We have shown that $\mathcal{F}(W) \subset W$. It follows from the Schauder theorem [15, Cor. 9.7] that there exists $(h, h_1) \in W$ such that $\mathcal{F}(h, h_1) = (h, h_1)$. This fixed point for $\mathcal{F}$ is a weak solution of problem (21)–(22).

4.2. Step 2: elimination of the auxiliary function L_M

We now claim that there exists a real number $B > 0$, not depending on ϵ neither on M, such that any weak solution $(h, h_1) \in W$ of problem (21)–(22) satisfies

$$||\nabla h||_{L^2(0,T;H)} \le B \quad \text{and} \quad ||\nabla h_1||_{L^2(0,T;H)} \le B. \tag{36}$$

Taking $w = h - h_D$ (resp. $w = h_1 - h_{1,D}$) in (21) (resp. (22)) leads to

$$\begin{aligned}
&\int_0^T \phi\langle\partial_t h, h - h_D\rangle_{V',V} + \int_{\Omega_T} \delta\phi \nabla h \cdot \nabla(h - h_D) \\
&\quad + \int_{\Omega_T} K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h \cdot \nabla(h - h_D) \\
&= -\int_{\Omega_T} K T_s(h)\mathcal{X}_0^\epsilon(h_1) L_M\big(||\nabla h_1||_{L^2}\big)\nabla h_1 \cdot \nabla(h - h_D) \\
&\quad - \int_{\Omega_T} Q_s T_s(h)(h - h_D)
\end{aligned} \tag{37}$$

and

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$$\begin{aligned}
&\int_0^T \phi\langle\partial_t h_1, h_1 - h_{1,D}\rangle_{V',V} + \int_{\Omega_T} \delta\phi\nabla h_1 \cdot \nabla(h_1 - h_{1,D}) \\
&\quad + \int_{\Omega_T} K\big(T_s(h)\mathcal{X}_0^\epsilon(h_1) + T_f(h-h_1)\big)\nabla h_1 \cdot \nabla(h_1 - h_{1,D}) \\
&= -\int_{\Omega_T} K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h \cdot \nabla(h_1 - h_{1,D}) \\
&\quad - \int_{\Omega_T} \big(Q_f T_f(h-h_1) + Q_s T_s(h)\big)(h_1 - h_{1,D}).
\end{aligned} \tag{38}$$

Summing up relations (37) and (38), and using the decomposition

$$\begin{aligned}
&K\nabla h\cdot\nabla h + K L_M\big(||\nabla h_1||_{L^2}\big)\nabla h_1\cdot\nabla h + K\nabla h_1\cdot\nabla h_1 + K\nabla h\cdot\nabla h_1 \\
&= K\nabla(h+h_1)\cdot\nabla(h+h_1) + K\big(1 - L_M\big(||\nabla h_1||_{L^2}\big)\big)\nabla h_1\cdot\nabla h_1 \\
&\quad - K\big(1 - L_M\big(||\nabla h_1||_{L^2}\big)\big)\nabla h_1\cdot\nabla(h+h_1),
\end{aligned}$$

we write

$$\begin{aligned}
&\int_0^T \phi\big(\langle\partial_t(h-h_D), h-h_D\rangle_{V',V} + \langle\partial_t(h_1-h_{1,D}), h_1-h_{1,D}\rangle_{V',V}\big) \\
&\quad + \int_{\Omega_T}\delta\phi\big(\nabla h\cdot\nabla h + \nabla h_1\cdot\nabla h_1\big) + \int_{\Omega_T} K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla(h+h_1)\cdot\nabla(h+h_1) \\
&\quad + \int_{\Omega_T} K\Big(\big(1 - L_M(||\nabla h_1||_{L^2})\big)T_s(h)\mathcal{X}_0^\epsilon(h_1) + T_f(h-h_1)\Big)\nabla h_1\cdot\nabla h_1 \\
&= \int_{\Omega_T} K\big(1 - L_M(||\nabla h_1||_{L^2})\big)T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h_1\cdot\nabla(h+h_1) \\
&\quad + \int_{\Omega_T}\big(\delta\phi + K T_s(h)\mathcal{X}_0^\epsilon(h_1)\big)\nabla h\cdot\nabla h_D \\
&\quad + \int_{\Omega_T}\big(\delta\phi + K T_s(h)\mathcal{X}_0^\epsilon(h_1) + K T_f(h-h_1)\big)\nabla h_1\cdot\nabla h_{1,D} \\
&\quad + \int_{\Omega_T} K T_s(h) L_M(||\nabla h_1||_{L^2})\mathcal{X}_0^\epsilon(h_1)\nabla h_1\cdot\nabla h_D + \int_{\Omega_T} K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\cdot\nabla h_{1,D} \\
&\quad - \int_{\Omega_T}\big(Q_s T_s(h)(h-h_D) + \big(Q_f T_f(h-h_1) + Q_s T_s(h)\big)(h_1-h_{1,D})\big) \\
&\quad - \int_0^T \phi\big(\langle\partial_t h_D, h-h_D\rangle_{V',V} + \langle\partial_t h_{1,D}, h_1-h_{1,D}\rangle_{V',V}\big).
\end{aligned} \tag{39}$$

Writing (39) as $\sum_{i=1}^4 J_i = \sum_{i=5}^{11} J_i$, we now estimate all the integral terms 'J_i'. We set $u = h - h_D$ and $v = h_1 - h_{1,D}$. First, we note that

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$$|J_1| = \frac{\phi}{2}\int_\Omega \big(u^2(T,x) - u_0^2(x)\big)\,dx + \frac{\phi}{2}\int_\Omega \big(v^2(T,x) - v_0^2(x)\big)\,dx,$$

$$|J_2| = \int_{\Omega_T} \delta\phi|\nabla h|^2\,dxdt + \int_{\Omega_T} \delta\phi|\nabla h_1|^2\,dxdt,$$

$$|J_3| \geq \int_{\Omega_T} K_- T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla(h+h_1)|^2\,dxdt,$$

$$|J_4| \geq \int_{\Omega_T} K_-\Big(\big(1 - L_M(||\nabla h_1||_{L^2})\big)T_s(h)\mathcal{X}_0^\epsilon(h_1) + T_f(h-h_1)\Big)|\nabla h_1|^2\,dxdt.$$

Next, applying the Cauchy–Schwarz and Young inequalities, we obtain the following set of estimates for any $\varepsilon_1 > 0$:

$$|J_5| \leq \int_{\Omega_T} \big(1 - L_M\big(||\nabla h_1||_{L^2}\big)\big)T_s(h)\mathcal{X}_0^\epsilon(h_1)\Big(\frac{K_+^2}{4K_-}|\nabla h_1|^2 + K_-|\nabla(h+h_1)|^2\Big),$$

$$|J_6| \leq \int_{\Omega_T} \frac{\delta\phi}{4}\,|\nabla h|^2 + \frac{\varepsilon_1 K_+}{4}\int_{\Omega_T} T_s(h)\mathcal{X}_0^\epsilon(h_1)\,|\nabla h|^2 + \int_{\Omega_T} \big(\delta\phi + \frac{K_+}{\varepsilon_1}T_s(h)\mathcal{X}_0^\epsilon(h_1)\big)|\nabla h_D|^2,$$

$$|J_7| \leq \int_{\Omega_T} \frac{\delta\phi}{4}\,|\nabla h_1|^2 + \int_{\Omega_T} \frac{\varepsilon_1 K_+}{4}T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla h_1|^2 + \int_{\Omega_T} \frac{K_+}{2}T_f(h-h_1)|\nabla h_1|^2 + \int_{\Omega_T} \Big(\delta\phi + \frac{K_+}{2}T_f(h-h_1)\Big)|\nabla h_{1,D}|^2 + \frac{1}{\varepsilon_1}\int_{\Omega_T} T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla h_{1,D}|^2,$$

$$|J_8| \leq \int_{\Omega_T} \frac{\varepsilon_1 K_+}{4}T_s(h)\big)\mathcal{X}_0^\epsilon(h_1)|\nabla h_1|^2 + \int_{\Omega_T} \frac{K_+}{\varepsilon_1}T_s(h)L_M^2\big(||\nabla h_1||_{L^2}\big)\mathcal{X}_0^\epsilon(h_1)|\nabla h_D|^2,$$

and

$$|J_9| \leq \int_{\Omega_T} \frac{\varepsilon_1 K_+}{4}T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla h|^2 + \int_{\Omega_T} \frac{K_+}{\varepsilon_1}T_s(h)|\nabla h_{1,D}|^2,$$

$$|J_{10}| \leq \int_{\Omega_T} T_s(h)|Q_s\,u| + \int_{\Omega_T} T_f(h-h_1)|Q_f\,v|\,dxdt + \int_{\Omega_T} T_s(h)|Q_s\,v| \leq \frac{3\|Q_s\|^2_{L^2(0,T;H)} + 2\|Q_f\|^2_{L^2(0,T;H)}}{2\phi}h_2^2 + \frac{\phi}{2}\int_{\Omega_T}|u|^2 + \frac{\phi}{2}\int_{\Omega_T}|v|^2,$$

$$|J_{11}| \leq \frac{1}{4}\int_{\Omega_T} \delta\phi\,|\nabla(h-h_D)|^2 + \frac{1}{4}\int_{\Omega_T} \delta\phi\,|\nabla(h_1-h_{1,D})|^2 + \frac{\phi}{\delta}\|\partial_t h_D\|^2_{L^2(0,T;V')} + \frac{\phi}{\delta}\|\partial_t h_{1,D}\|^2_{L^2(0,T;V')} \leq \frac{\delta\phi}{4}\int_{\Omega_T}\big(|\nabla h|^2 + |\nabla h_1|^2\big) + \frac{\delta\phi}{4}\int_{\Omega_T}\big(|\nabla h_D|^2 + |\nabla h_{1,D}|^2\big) + \frac{\phi}{\delta}\big(\|\partial_t h_D\|^2_{L^2(0,T;V')} + \|\partial_t h_{1,D}\|^2_{L^2(0,T;V')}\big).$$

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Summing up all these estimates, we obtain

$$\begin{aligned}
&\phi \int_\Omega u^2(T,x) + \phi \int_\Omega v^2(T,x) \\
&\quad + \int_{\Omega_T} \underbrace{\big(\delta\phi - \varepsilon_1 K_+ T_s(h)\mathcal{X}_0^\epsilon(h_1)\big)}\big(|\nabla h|^2 + |\nabla h_1|^2\big) \\
&\quad + \int_{\Omega_T} 2|\nabla h_1|^2 \underbrace{\Big(\big(K_- - \frac{K_+^2}{4K_-}\big)\big(1 - L_M(||\nabla h_1||)\big)\mathcal{X}_0^\epsilon(h_1)T_s(h) + \big(K_- - \frac{K_+}{2}\big)T_f(h-h_1)\Big)} \\
&\quad + 2\int_{\Omega_T} \big(K_- T_s(h)\mathcal{X}_0^\epsilon(h_1) L_M\big(||\nabla h_1||_{L^2}\big)\big)|\nabla(h+h_1)|^2 \\
&\le \phi \int_{\Omega_T} |u|^2 + \phi \int_{\Omega_T} |v|^2 + C,
\end{aligned}$$

where $C = C(u_0, v_0, h_D, h_{1,D}, h_2, Q_s, Q_f)$. We now aim applying the Gronwall lemma in the latter relation. We thus choose $\varepsilon_1 > 0$ such as terms over the curly bracket are respectively positive and nonnegative, namely:

$$\begin{aligned}
&K_+ T_s(h)\mathcal{X}_0^\epsilon(h_1)\varepsilon_1 < \delta\phi, \\
&1 - L_M(x) = 1 - \min(1, M/x) \ge 0 \text{ and } K_+ \le 2K_-.
\end{aligned}$$

The first condition is fulfilled if we choose for instance ε_1 such that $\varepsilon_1 < \delta\phi/(K_+ h_2)$. The second one follows the assumption on permeability in Theorem 1.

Now we apply the Gronwall lemma and we deduce that there exists a real number B, that does not depend on ϵ nor on M, such that

$$||h||_{L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))} \le B \quad \text{and} \quad ||h_1||_{L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))} \le B.$$

In particular, $||\nabla h_1||_{L^2(0,T;H)} \le B$ and this estimate does not depend on the choice of the real number M that defines function L_M. Hence if we choose $M = B$, any weak solution of the system

$$\begin{aligned}
&\phi\partial_t h - \nabla\cdot(\delta\phi\nabla h) - \nabla\cdot\big(K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\big) \\
&\qquad - \nabla\cdot\big(K T_s(h)\mathcal{X}_0^\epsilon(h_1) L_B(||\nabla h_1||_{L^2})\nabla h_1\big) = -Q_s T_s(h), \\
&\phi\partial_t h_1 - \nabla\cdot(\delta\phi\nabla h_1) - \nabla\cdot\big(K\big(T_f(h-h_1) + T_s(h)\mathcal{X}_0^\epsilon(h_1)\big)\nabla h_1\big) \\
&\qquad - \nabla\cdot\big(K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\big) = -Q_f T_f(h-h_1) - Q_s T_s(h)
\end{aligned}$$

in Ω_T, with the initial and boundary conditions

$$h = h_D \text{ and } h_1 = h_{1,D} \text{ on } \Gamma,\ h(0,x) = h_0 \text{ and } h_1(0,x) = h_{1,0}(x) \text{ a.e. in } \Omega,$$

satisfies $L_B\big(||\nabla h_1||_{L^2}\big) = 1$. Then the term $L_B\big(||\nabla h_1||_{L^2}\big) = 1$ may be dropped.

4.3. Step 3: maximum principles

We are going to prove that for almost every $x \in \Omega$ and for all $t \in (0, T)$,

$$0 \leq h_1(t,x) \leq h(t,x) \leq h_2.$$

• First show that $h(t,x) \leq h_2$ a.e. $x \in \Omega$ and $\forall t \in (0,T)$. We set

$$h_m = \big(h - h_2\big)^+ = \sup(0, h - h_2) \in L^2(0,T;V).$$

It satisfies $\nabla h_m = \chi_{\{h>h_2\}}\nabla h$ and $h_m(t,x) \neq 0$ iff $h(t,x) > h_2$, where χ denotes the characteristic function. Let $\tau \in (0,T)$. Taking $w(t,x) = h_m(t,x)\chi_{(0,\tau)}(t)$ in (21) yields:

$$\int_0^T \phi\langle \partial_t h, h_m\chi_{(0,\tau)}\rangle_{V',V} + \int_0^\tau\int_\Omega \delta\phi\nabla h\cdot\nabla h_m + \int_0^\tau\int_\Omega KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\cdot\nabla h_m$$
$$+ \int_0^\tau\int_\Omega KT_s(h)L_M\big(||\nabla h_1||_{L^2}\big)\mathcal{X}_0^\epsilon(h_1)\nabla h_1\cdot\nabla h_m + \int_0^\tau\int_\Omega Q_sT_s(h)h_m = 0,$$

that is

$$\begin{aligned}\int_0^\tau \phi\langle \partial_t h, h_m\rangle_{V',V} &+ \int_0^\tau\int_\Omega \delta\phi\chi_{\{h>h_2\}}|\nabla h|^2\\ &+ \int_0^\tau\int_\Omega KT_s(h)\mathcal{X}_0^\epsilon(h_1)\chi_{\{h>h_2\}}|\nabla h|^2\\ &+ \int_0^\tau\int_\Omega KT_s(h)L_M\big(||\nabla h_1||_{L^2}\big)\mathcal{X}_0^\epsilon(h_1)\nabla h_1\cdot\nabla h_m(x,t)\\ &+ \int_0^\tau\int_\Omega Q_sT_s(h)h_m(x,t) = 0. \end{aligned}\tag{40}$$

In order to evaluate the first term in the lefthand side of (40), we apply Lemma 1 with function f defined by $f(\lambda) = \lambda - h_2$, $\lambda \in \mathbb{R}$. We write

$$\int_0^\tau \phi\langle \partial_t h, h_m\rangle_{V',V}dt = \frac{\phi}{2}\int_\Omega \Big(h_m^2(\tau,x) - h_m^2(0,x)\Big)\,dx = \frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx,$$

since $h_m(0,\cdot) = \big(h_0(\cdot) - h_2(\cdot)\big)^+ = 0$. Since $T_s(h)\chi_{\{h>h_2\}} = 0$ by definition of T_s, the three last terms in the lefthand side of (40) are null. Hence (40) becomes:

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx \leq -\int_0^\tau\int_\Omega \delta\phi\chi_{\{h>h_2\}}|\nabla h|^2\,dxdt \leq 0.$$

Then $h_m = 0$ a.e. in Ω_T. Including the leakage term q_{Ls} defined by (13) does not impact the result because of the factor $\chi_0(h_2 - h)$ in its definition.

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• Now we claim that $h_1(t,x) \le h(t,x)$ a.e. $x \in \Omega$ and $\forall t \in (0,T)$. We now set

$$h_m = \big(h_1 - h\big)^+ \in L^2(0,T;V).$$

Let $\tau \in (0,T)$. We recall that $h_m(0,\cdot) = 0$ a.e. in Ω thanks to the maximum principle satisfied by the initial data h_0 and $h_{1,0}$. Moreover, $\nabla(h_1 - h)\cdot\nabla h_m = \chi_{\{h_1-h>0\}}|\nabla(h_1-h)|^2$. Thus, taking $w(t,x) = h_m(x,t)\chi_{(0,\tau)}(t)$ in $(22)-(21)$ gives:

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega h_m^2(\tau,x) + \int_0^\tau\int_\Omega \delta\phi\chi_{\{h_1-h>0\}}|\nabla(h_1-h)|^2 + \int_0^\tau\int_\Omega K\big(\mathcal{X}_0^\epsilon(h_1)T_s(h)dt \\
&\quad + T_f(h-h_1)\big)\nabla h_1\cdot\nabla h_m - \int_0^\tau\int_\Omega KT_s(h)\mathcal{X}_0^\epsilon(h_1)L_M\big(\|\nabla h_1\|_{L^2}\big)\nabla h_1\cdot\nabla h_m \\
&\quad + \int_0^\tau\int_\Omega Q_fT_f(h-h_1)h_m = 0. \qquad (41)
\end{aligned}$$

Since $T_f(h-h_1)\chi_{\{h_1-h>0\}} = 0$ by definition of T_f and since we now have $M = B$ such that $L_B\big(\|\nabla h_1\|_{L^2}\big) = 1$, we infer from (41) that

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx = -\int_0^\tau\int_\Omega \delta\phi\chi_{\{h_1-h>0\}}|\nabla(h_1-h)|^2\,dxdt \le 0.$$

Thus $h_1(\tau,\cdot) \le h(\tau,\cdot)$ a.e. in Ω and for any $\tau \in (0,T)$. Presence of the leakage terms defined in (13) does not change the picture. Indeed term q_{Ls} disappears in the computation $(22)-(21)$ and $q_{Lf}h_m = 0$ because of the term $\chi_0(h-h_1)$ in the definition of q_{Lf}.

• Finally we show $0 \le h_1(t,x)$ a.e. $x \in \Omega$ and $\forall t \in (0,T)$. We now set

$$h_m = \big(-h_1\big)^+ \in L^2(0,T;V).$$

Let $\tau \in (0,T)$. For this part of the proof, we re-include the leakage terms q_{Lf} and q_{Ls} in the model because they appear in the assertion (20) which is used here. Taking $w(t,x) = -h_m(x,t)\chi_{(0,\tau)}(t)$ in (22) leads to:

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega h_m^2(\tau,x) + \int_0^\tau\int_\Omega \delta\phi\chi_{\{h_1<0\}}|\nabla h_1|^2 - \int_0^\tau\int_\Omega K\big(T_s(h)\mathcal{X}_0^\epsilon(h_1) \\
&\quad + T_f(h-h_1)\big)\nabla h_1\cdot\nabla h_m - \int_0^\tau\int_\Omega KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\cdot\nabla h_m \\
&\quad - \int_0^\tau\int_\Omega \big(Q_fT_f(h-h_1) + Q_sT_s(h) - q_{Lf} - q_{Ls}\big)h_m = 0. \qquad (42)
\end{aligned}$$

We note that if $\nabla h_m \ne 0$ then $\mathcal{X}_0^\epsilon(h_1) = 0$ because $h_1 \le 0$. We have moreover

$$\begin{aligned}
&-\int_0^\tau\int_\Omega K\big(T_s(h)\mathcal{X}_0^\epsilon(h_1) + T_f(h-h_1)\big)\nabla h_1\cdot\nabla h_m\,dxdt \\
&\quad \ge \int_0^\tau\int_\Omega K_-T_f(h-h_1)\chi_{\{h_1<0\}}|\nabla h_1|^2\,dxdt,
\end{aligned}$$

and, due to assumptions (19) and (20),

$$-\int_0^\tau\int_\Omega \big(Q_f T_f(h-h_1)+Q_s T_s(h)-q_{Lf}-q_{Ls}\big)h_m\,dxdt\ge 0.$$

Eq. (42) thus gives:

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)dt\,dx\le\int_0^\tau\int_\Omega \big(\delta\phi+K_-T_f(h-h_1)\big)\chi_{\{h_1<0\}}|\nabla h_1|^2\,dxdt\le 0.$$

We conclude that $h_m(\tau,\cdot)=0$, that is $h_1(\tau,\cdot)\ge 0$, a.e. in Ω for any $\tau\in(0,T)$.

4.4. Step 4: existence for the initial system

In the latter subsections, we have proved the existence of a weak solution $\big(h^\epsilon,h_1^\epsilon\big)\in\big(L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))\big)^2$ of the regularized problem

$$\phi\partial_t h^\epsilon-\nabla\cdot\big(\delta\phi\nabla h^\epsilon\big)-\nabla\cdot\big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla(h^\epsilon+h_1^\epsilon)\big)=-Q_sT_s(h^\epsilon),\tag{43}$$

$$\begin{aligned}\phi\partial_t h_1^\epsilon-\nabla\cdot\big(\delta\phi\nabla h_1^\epsilon\big)-\nabla\cdot\big(K\big(T_f(h^\epsilon-h_1^\epsilon)+T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\big)\nabla h_1^\epsilon\big)&\\ -\nabla\cdot\big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla h^\epsilon\big)=-Q_fT_f(h^\epsilon-h_1^\epsilon)-Q_sT_s(h^\epsilon),&\end{aligned}\tag{44}$$

with the initial and boundary conditions

$$h^\epsilon=h_D,\ h_1^\epsilon=h_{1,D}\text{ in }\Gamma\times(0,T),\quad h^\epsilon(0,x)=h_0,\ h_1^\epsilon(0,x)=h_{1,0}(x)\text{ a.e. in }\Omega.$$

Furthermore this solution satisfies the following maximum principle:

$$\forall t\in(0,T),\ \text{a.e. }x\in\Omega,\quad 0\le h_1^\epsilon(t,x)\le h^\epsilon(t,x)\le h_2,$$

and the following uniform estimates (with respect to ϵ):

$$\text{(UE)}\quad\begin{cases}\|h^\epsilon\|_{L^2(0,T;H^1(\Omega))}\le C, & \|h_1^\epsilon\|_{L^2(0,T;H^1(\Omega))}\le C,\\ \|\partial_t h^\epsilon\|_{L^2(0,T;V')}\le C, & \|\partial_t h_1^\epsilon\|_{L^2(0,T;V')}\le C.\end{cases}$$

We now proceed to the last step in the proof of Theorem 1, namely we let $\epsilon\to 0$. We infer from the above estimates that $(h^\epsilon-h_D)_\epsilon$ and $(h_1^\epsilon-h_{1,D})_\epsilon$ are uniformly bounded in $W(0,T)$. We deduce thanks to the compactness result of Aubin that $(h^\epsilon-h_D)_\epsilon$ and $(h_1^\epsilon-h_{1,D})_\epsilon$ are sequentially compact in $L^2(0,T;H)$. Up to the extraction of a subsequence, not relabeled for convenience, we claim that there exist functions h and h_1 such that $(h-h_D,h_1-h_{1,D})\in W(0,T)^2$ and

$$\begin{cases}h^\epsilon\longrightarrow h & \text{in }L^2(0,T;H)\text{ and a.e. in }\Omega\times(0,T),\\ h^\epsilon\rightharpoonup h & \text{weakly in }L^2(0,T;H^1(\Omega)),\\ \partial_t h^\epsilon\rightharpoonup\partial_t h & \text{weakly in }L^2(0,T;V'),\\ h_1^\epsilon\longrightarrow h_1 & \text{in }L^2(0,T;H)\text{ and a.e. in }\Omega\times(0,T),\\ h_1^\epsilon\rightharpoonup h_1 & \text{weakly in }L^2(0,T;H^1(\Omega)),\\ \partial_t h_1^\epsilon\rightharpoonup\partial_t h_1 & \text{weakly in }L^2(0,T;V').\end{cases}$$

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Letting $\epsilon \to 0$ in the weak formulation of (43)–(44) and using the Lebesgue Theorem (thanks to the uniform estimates (UE)), we get at once (15)–(16). The boundary and initial condition (17)–(18) holds true since the map $i \in W(0,T) \mapsto i(0) \in H$ is continuous. Furthermore (h, h_1) satisfies a maximum principle which is consistent with physical reality:

$$0 \le h_1(x,t) \le h(x,t) \le h_2, \ \forall t \in (0,T), \text{ a.e. } x \in \Omega.$$

The proof of Theorem 1 is complete.

References

[1] P. Ackerer, A. Younes, Efficient approximations for the simulation of density driven flow in porous media, Adv. Water Res. 31 (2008) 15–27.
[2] Y. Amirat, K. Hamdache, A. Ziani, Mathematical analysis for compressible miscible displacement models in porous media, Math. Models Methods Appl. Sci. 6 (1996) 729–747.
[3] J. Bear, A.H.D. Cheng, S. Sorek, D. Ouazar, I. Herrera, Seawater Intrusion in Coastal Aquifers: Concepts, Methods and Practices, Kluwer Academic Pub., 1999.
[4] J. Bear, A. Verruijt, Modelling Groundwater Flow and Pollution, D. Reidel Publishing Company, Dordecht, Holland, 1987.
[5] J. Bear, Dynamics of Fluids in Porous Media, Elsevier, 1972.
[6] C. Choquet, Parabolic and degenerate parabolic models for pressure-driven transport problems, Math. Models Methods Appl. Sci. 20 (2010) 543–566.
[7] C. Choquet, M.M. Diédhiou, C. Rosier, Derivation of a sharp–diffuse interface model in a free aquifer. Numerical simulations, preprint.
[8] H.L. Essaid, A multilayered sharp interface model of coupled freshwater and saltwater flow in coastal systems: model development and application, Water Res. Res. 26 (1990) 1431–1454.
[9] G. Gagneux, M. Madaune-Tort, Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière, Math. Appl., vol. 22, Springer, 1996.
[10] J.L. Lions, E. Magenes, Problèmes aux limites non homogènes, vol. 1, Dunod, 1968.
[11] K. Najib, C. Rosier, On the global existence for a degenerate elliptic–parabolic seawater intrusion problem, Math. Comput. Simulation 81 (1) (2011) 2282–2295.
[12] C. Rosier, L. Rosier, Well-posedness of a degenerate parabolic equation issuing from two-dimensional perfect fluid dynamics, Appl. Anal. 75 (3–4) (2000) 441–465.
[13] J. Simon, Compact sets in the space $L^p(0,T,B)$, Ann. Mat. Pura Appl. 146 (4) (1987) 65–96.
[14] M.H. Tber, M.E. Talibi, A finite element method for hydraulic conductivity identification in a seawater intrusion problem, Comput. Geosci. 33 (2007) 860–874.
[15] E. Zeidler, Nonlinear Functional Analysis and Its Applications, Part 1, Springer Verlag, 1986.

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Please cite this article in press as: C. Choquet et al., Mathematical analysis of a sharp–diffuse interfaces model for seawater intrusion, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.005