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Homogenization of a class of quasilinear elliptic equations with non-standard growth in high-contrast media

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We study the asymptotic behaviour of solutions to a quasilinear equation with high-contrast coefficients. The energy formulation of the problem leads to work with variable exponent Lebesgue spaces $L^{p_\varepsilon(\cdot)}$ in a domain Ω with a complex microstructure depending on a small parameter ε. Assuming only that the functions p_ε converge uniformly to a limit function p_0 and that p_0 satisfy certain logarithmic uniform continuity conditions, we rigorously derive the corresponding homogenized problem which is completely described in terms of local energy characteristics of the original domain. In the framework of our method we do not have to specify the geometrical structure Ω. We illustrate our result with periodical examples, extending, in particular, the classical extension results to variable exponent Sobolev spaces.

1. Introduction

A key feature of this paper is the study of variable exponent Lebesgue spaces. In what follows, we briefly give some motivations and references. In 1931, W. Orlicz [15] was the first to define variable exponent Lebesgue spaces. Very recently, V. V. Zhikov [19] proposed the study of variational problems with non-standard growth and coercivity conditions. At the same time, progress in physics made the study of fluid properties of electrorheological fluids an important issue, used, for instance, in robotics and space technology. As emphasized by W. Winslow in 1949, the viscosity of such fluids in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid which are parallel to the field. They can raise the viscosity by many orders of magnitude. Thus, the mechanical properties of electrorheological fluids, and especially the Winslow effect, can be modelled using variable exponent Lebesgue and Sobolev spaces, $L^{p}(\cdot)$ and $W^{1,p(\cdot)}$.

Roughly speaking, the energy of electrorheological fluids is calculated by minimizing the Dirichlet energy integral $\int |\nabla u(x)|^{p(x)}\,\mathrm{d}x$, where p describes the characteristics of the material as a function of the electric field. For some mathematical results on the problem we refer the reader to [2, 17]. More recently, a new application of variable exponent Lebesgue spaces to image restoration was proposed by Chen *et al.* [9]. They minimize the energy $\int |\nabla u(x)|^{p(x)} + \lambda|u(x) - I(x)|^2\,\mathrm{d}x$, where the input I is the true image corrupted by some noise. For classical image restoration, the power $p = 2$ corresponds to isotropic smoothing, whereas $p = 1$ gives total variation smoothing. In [9] the exponent varies between these two extremes to control the defaults of both procedures: the isotropic smoothing destroys all small details from the image, while total variation smoothing creates edges where there were none in the original image.

We study the asymptotic behaviour of solutions to a quasilinear equation of the form

$$-\operatorname{div}(K_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\nabla u^\varepsilon) + |u^\varepsilon|^{\sigma(x)-2}u^\varepsilon = g^\varepsilon(x), \quad x \in \Omega, \tag{1.1}$$

with a high-contrast coefficient $K_\varepsilon(x)$. Under the assumption that the functions $p_\varepsilon(x)$ converge uniformly to a limit function $p_0(x)$ and that p_0 satisfies certain logarithmic uniform continuity conditions, it is shown that u^ε converges, as $\varepsilon \to 0$, to the solution of a homogenized equation whose coefficients are calculated in terms of local energy characteristics of the domain Ω^ε. The equations that we consider here arise, for example, from compressible flows in porous media, and non-Newtonian flow through thin fissures. The homogenization problem is closely related to the so-called double-porosity models widely discussed in the mathematical literature (see, for instance, [13]). The linear double-porosity model was first studied in [6]. Nonlinear models were treated in [10, 16]. Then a general non-periodic model and a random model were considered in [7] and [8], respectively. Instead of the above-mentioned geometrical assumptions, we follow the approach introduced in [14] and impose conditions on the so-called *local energetic characteristics* associated with the boundary-value problem (1.1). These characteristics include a penalization term. We turn back to usual geometrical assumptions in the last section of the paper by illustrating our result with periodical examples. A key step is here the construction of an appropriate extension operator from the fracture part to the whole domain Ω. This construction extends the classical result of [1] to variable exponent Sobolev spaces.

2. Statement of the problem and the main result

Let $\Omega = \Omega_f^\varepsilon \cup \overline{\Omega_m^\varepsilon}$ be a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz boundary $\partial\Omega$. In what follows, ε is a small positive parameter characterizing the microscopic length-scale. Here $\{\Omega_m^\varepsilon\}_{(\varepsilon>0)}$ is a family of open subsets in Ω. We assume that the set Ω_m^ε is distributed in an asymptotically regular way in Ω, i.e. for any ball $\mathcal{B}(y, r)$ of radius r centred at $y \in \Omega$ and sufficiently small $\varepsilon > 0$, $\varepsilon \leqslant \varepsilon_0(r)$, the set Ω_m^ε satisfies $\mathcal{B}(y, r) \cap \Omega_m^\varepsilon \neq \varnothing$ and $\mathcal{B}(y, r) \cap \Omega_f^\varepsilon \neq \varnothing$. We will assume, for the sake of simplicity, that $\Omega_m^\varepsilon \cap \partial\Omega = \varnothing$.

Remark 2.1. In the framework of the method presented in the paper we do not specify the geometrical structure of the set Ω_m^ε. Generally speaking, it may consist

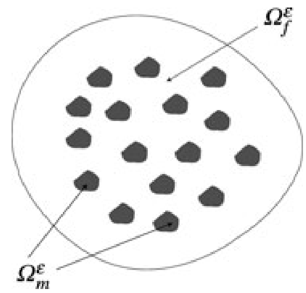

Figure 1. The sets Ω_m^ε and Ω_f^ε.

of N_ε, $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$, small isolated components such that their diameters go to zero as $\varepsilon \to 0$ (see figure 1) or it may be defined as fibres becoming more and more dense as $\varepsilon \to 0$ such that the diameters of the fibres go to zero as $\varepsilon \to 0$.

We consider growth functions in the class $\mathfrak{P}_0^\varepsilon$ described below. First we recall that a function $p = p(x)$ defined in the domain $\bar{\Omega}$ satisfies the log-Hölder continuity property if, for any $x \in \Omega$, $y \in \Omega$,

$$|p(x) - p(y)| \leqslant \omega(|x-y|) \quad \text{with} \quad \overline{\lim_{\tau\to 0}}\, \omega(\tau) \ln\left(\frac{1}{\tau}\right) \leqslant C,$$

where C is a constant. This property was introduced by Zhikov to avoid Lavrentiev phenomena [20]. A sequence of functions $\{p_\varepsilon\}_{(\varepsilon>0)}$ is said to belong to the class $\mathfrak{P}_0^\varepsilon$ if it possesses the following properties.

(A1) For any $\varepsilon > 0$, there exist two real numbers p^- and p^+ such that the function p_ε is bounded in the following sense:

$$1 < p^- \leqslant p_\varepsilon^- \equiv \min_{x\in\bar{\Omega}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\bar{\Omega}} p_\varepsilon(x) \equiv p_\varepsilon^+ \leqslant p^+ < +\infty \quad \text{in } \bar{\Omega}. \tag{2.1}$$

(A2) For any $\varepsilon > 0$, p_ε satisfies the log-Hölder continuity property with the corresponding function ω_{p_ε}.

(A3) The function p_ε converges uniformly in Ω to a function p_0, i.e.

$$\lim_{\varepsilon\to 0} \|p_\varepsilon - p_0\|_{C(\bar{\Omega})} = 0, \tag{2.2}$$

where the limit function p_0 is assumed to satisfy the log–Hölder continuity property.

Note that the constant sequence $\{p_0\}_{(\varepsilon>0)}$ belongs to the family $\mathfrak{P}_0^\varepsilon$.

Let $\sigma \in C(\bar{\Omega})$ be such that

(A4) there exist two real numbers σ^- and σ^+ such that the function σ is bounded in the following sense:

$$0 < \sigma^- \equiv \min_{x\in\bar{\Omega}} \sigma(x) \leqslant \sigma(x) \leqslant \max_{x\in\bar{\Omega}} \sigma(x) \equiv \sigma^+ < \min_{x\in\bar{\Omega}} \frac{p_0(x)n}{n - p_0(x)} \quad \text{in } \bar{\Omega}; \tag{2.3}$$

(A5) the function σ satisfies the log-Hölder continuity property.

In what follows, we refer to [5] (see also the references therein) for the properties of Sobolev spaces with variable exponents. Here $L^{p_\varepsilon(\cdot)}(\Omega)$ denotes the space of measurable functions ϕ in Ω such that

$$\Upsilon_{p_\varepsilon(\cdot),\Omega}(\phi) \overset{\text{def}}{=} \int_\Omega |\phi(x)|^{p_\varepsilon(x)}\,\mathrm{d}x < +\infty. \tag{2.4}$$

This space endowed with the norm $\|\phi\|_{L^{p_\varepsilon(\cdot)}(\Omega)} = \inf\{\lambda > 0 : \Upsilon_{p_\varepsilon(\cdot),\Omega}(\phi/\lambda) \leqslant 1\}$ is a Banach space. Following [5], for any $\varepsilon > 0$ we define the Sobolev space with variable exponent p_ε, $W^{1,p_\varepsilon(\cdot)}(\Omega)$ by

$$W^{1,p_\varepsilon(\cdot)}(\Omega) = \{\phi \in L^{p_\varepsilon(\cdot)}(\Omega) : |\nabla\phi| \in L^{p_\varepsilon(\cdot)}(\Omega)\}.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p(\cdot)}(\Omega)$. We recall the well-known embedding result for Sobolev spaces with variable exponents. Namely, if p and q are continuous functions in $\overline{\Omega}$ and

$$1 < q(x) \leqslant \sup_\Omega q(x) < \inf_\Omega p^\star(x) \quad \text{with } p^\star(x) \overset{\text{def}}{=} \begin{cases} \dfrac{p(x)n}{n - p(x)} & \text{if } p(x) < n, \\ +\infty & \text{if } p(x) \geqslant n, \end{cases} \tag{2.5}$$

then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

Let us now define the variational problem under consideration. To this end, we consider the functional $J^\varepsilon : W^{1,p_\varepsilon(\cdot)}(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

$$J^\varepsilon[u] \overset{\text{def}}{=} \begin{cases} \displaystyle\int_\Omega F_\varepsilon(x, u, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.6}$$

where

$$F_\varepsilon(x, u, \nabla u) \overset{\text{def}}{=} \varkappa_\varepsilon(x)|\nabla u|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|u|^{\sigma(x)} - g^\varepsilon(x)u \quad \text{with } \varkappa_\varepsilon(x) \overset{\text{def}}{=} \frac{K_\varepsilon(x)}{p_\varepsilon(x)}. \tag{2.7}$$

Here the function g^ε is defined by

$$g^\varepsilon(x) \overset{\text{def}}{=} \mathbf{1}_f^\varepsilon(x)g(x), \quad g \in C(\Omega). \tag{2.8}$$

We denote by $\mathbf{1}_k^\varepsilon$ the characteristic function of the set Ω_k^ε, $k = f, m$. Function K_ε is a measurable function in Ω such that

(K1) there exists a real number k_0 such that $0 < k_0 \leqslant K_\varepsilon(x) \leqslant k_0^{-1}$ in Ω_f^ε;

(K2) for any $\varepsilon > 0$ there exists a real number k_ε such that $\sup_{x\in\Omega_m^\varepsilon} K_\varepsilon(x) = k_\varepsilon > 0$ and $k_\varepsilon \to 0$ as $\varepsilon \to 0$.

We consider the following variational problem:

$$J^\varepsilon[u^\varepsilon] \to \min, \quad u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega). \tag{2.9}$$

It is known from [5] that there exists a unique solution $u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ for each $\varepsilon > 0$ of the variational problem (2.9).

We aim to study the asymptotic behaviour of the family $\{u^\varepsilon\}$ as $\varepsilon \to 0$, bearing in mind that the geometry of $\Omega = \Omega_f^\varepsilon \cup \bar{\Omega}_m^\varepsilon$ depends on ε. So we have to specify this geometry. Most of the papers dealing with homogenization assume that Ω is a periodic repetition of a standard cell. This classical periodicity assumption is substituted here by an abstract one covering a variety of concrete behaviours, including periodicity and almost periodicity. We thus make the following assumptions:

(C1) the local concentration of the set Ω_f^ε has a positive continuous limit: that is, the indicator of Ω_f^ε converges weakly in $L^2(\Omega)$ to a continuous positive limit. This implies that there exists a continuous positive function $\rho = \rho(x)$ such that

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} h^{-n}\,\mathrm{meas}(K_h^x \cap \Omega_f^\varepsilon) = \rho(x)$$

for any open cube K_h^x centred at $x \in \Omega$ with lengths equal to $h > 0$;

(C2) for any $\{p_\varepsilon\}_{(\varepsilon>0)} \subset \mathfrak{P}_0^\varepsilon$, there is a constant $C_{p_\varepsilon} \geqslant 0$ such that, if the function $p_\varepsilon^\star$ is defined by $p_\varepsilon^\star(x) = p_\varepsilon(x) - C_{p_\varepsilon}$ in Ω, then

(i) the sequence $\{p_\varepsilon^\star\}_{(\varepsilon>0)}$ belongs to $\mathfrak{P}_0^\varepsilon$, that is $\lim_{\varepsilon\to 0} C_{p_\varepsilon} = 0$;

(ii) there exists a family of extension operators

$$\boldsymbol{P}^\varepsilon : W^{1,p_\varepsilon^\star(\cdot)}(\Omega_f^\varepsilon) \to W^{1,p_\varepsilon^\star(\cdot)}(\Omega)$$

such that, for any $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)$,

$$\boldsymbol{P}^\varepsilon v^\varepsilon = v^\varepsilon \in \Omega_f^\varepsilon \quad \text{and} \quad \|\boldsymbol{P}^\varepsilon v^\varepsilon\|_{W^{1,p_\varepsilon^\star(\cdot)}(\Omega)} \leqslant \Phi(\|v^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)}),$$

where $\Phi = \Phi(t)$ is a strictly monotone continuous function in $\mathbb{R}^+$ such that $\Phi(0) = 0$ and $\Phi(t) \to +\infty$ as $t \to +\infty$.

Remark 2.2. Condition (C2) in the case when $p_\varepsilon = p \in \mathbb{R}$ is well-known in the mathematical literature (see, for example, [1, 3, 11, 14, 16]).

We also impose several conditions on the local characteristic of the set Ω_f^ε and Ω_m^ε associated to the functional (2.6). Let K_h^z be an open cube centred at $z \in \Omega$ with lengths equal to h, $0 < \varepsilon \ll h \ll 1$. We introduce the following functionals.

- The functional $c_{p_\varepsilon(\cdot)}^{\varepsilon,h}$ associated to the energy in Ω_f^ε is defined in $\Omega \times \mathbb{R}^n$ by

$$c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z;\vec{a}) \stackrel{\text{def}}{=} \inf_{v^\varepsilon} \int_{K_h^z \cap \Omega_f^\varepsilon} (\varkappa_\varepsilon(x)|\nabla v^\varepsilon(x)|^{p_\varepsilon(x)} + h^{-p_\varepsilon(x)-\gamma}|v^\varepsilon(x) - (x - z, \vec{a})|^{p_\varepsilon(x)})\,\mathrm{d}x, \quad (2.10)$$

for $z \in \Omega$, $\vec{a} \in \mathbb{R}^n$, where γ is a given positive real number, and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z \cap \Omega_f^\varepsilon)$. The scalar product in $\mathbb{R}^n$ is denoted here by $(\cdot,\cdot)$.

- The functional $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}$ associated to the energy exchange between the sets Ω^ε_f and Ω^ε_m is defined in $\Omega\times\mathbb{R}^n$ by

$$b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z;\beta) \stackrel{\text{def}}{=} \inf_{w^\varepsilon}\int_{K^z_h}\biggl(\varkappa_\varepsilon(x)|\nabla w^\varepsilon|^{p_\varepsilon(x)} + \frac{\mathbf{1}^\varepsilon_m(x)}{\sigma(x)}|w^\varepsilon|^{\sigma(x)} + h^{-p_\varepsilon(x)-\gamma}\mathbf{1}^\varepsilon_f(x)|w^\varepsilon-\beta|^{p_\varepsilon(x)}\biggr)\,\mathrm{d}x, \tag{2.11}$$

for $z\in\Omega$, $\beta\in\mathbb{R}$, the infimum being taken over $w^\varepsilon\in W^{1,p_\varepsilon(\cdot)}(K^z_h)$.

We assume that the local characteristics of Ω are such that

(C3) for any $x\in\Omega$ and any $\vec{a}\in\mathbb{R}^n$, there is a continuous function $\mathsf{A}(x,\vec{a})$ and a real number $\gamma=\gamma_0$, $0<\gamma_0<\mathsf{p}^-$, such that, for any $\{p_\varepsilon\}_{(\varepsilon>0)}\subset\mathfrak{P}^\varepsilon_0$,

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\,h^{-n}c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(x,\vec{a}) = \lim_{h\to0}\underline{\lim_{\varepsilon\to0}}\,h^{-n}c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(x,\vec{a}) = \mathsf{A}(x,\vec{a}); \tag{2.12}$$

(C4) for any $x\in\Omega$ and any $\beta\in\mathbb{R}$, there is a continuous function $\mathsf{b}(x,\beta)$ and a real number $\gamma=\gamma_1$, $0<\gamma_1<\mathsf{p}^-$ such that, for any $\{p_\varepsilon\}_{(\varepsilon>0)}\subset\mathfrak{P}^\varepsilon_0$,

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\,h^{-n}b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(x,\beta) = \lim_{h\to0}\underline{\lim_{\varepsilon\to0}}\,h^{-n}b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(x,\beta) = \mathsf{b}(x,\beta). \tag{2.13}$$

Remark 2.3. It is crucial in conditions (C3) and (C4) that the limit functions $\mathsf{A}(x,\vec{a})$ and $\mathsf{b}(x,\beta)$ do not depend on the particular choice of the sequence $\{p_\varepsilon\}_{(\varepsilon>0)}\subset\mathfrak{P}^\varepsilon_0$. We prove in the last section of the present paper that these assumptions are fulfilled for periodic and locally periodic media.

Remark 2.4. Contrary to the standard growth setting as considered in [4,16], the local characteristic $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z;\beta)$ is not homogeneous with respect to the parameter β. This induces the appearance of a nonlinear function $\mathsf{b}(x,u)$ in the homogenized functional (see theorem 2.5, below).

The main result of the paper is the following theorem.

Theorem 2.5. *Let u^ε be a solution of (2.9). Assume that conditions (A1)–(A5), (K1)–(K2) and (C1)–(C.4) are satisfied. Then u^ε (the solution of the variational problem (2.9)) converges strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon_f)$ to u, which is the solution of the following variational problem:*

$$J_{\text{hom}}[u]\to\min,\quad u\in W^{1,p_0(\cdot)}_0(\Omega), \tag{2.14}$$

the homogenized functional $J_{\text{hom}}: W^{1,p_0(\cdot)}_0(\Omega)\to\mathbb{R}\cup\{+\infty\}$ being defined by

$$J_{\text{hom}}[u] \stackrel{\text{def}}{=} \begin{cases}\displaystyle\int_\Omega \mathsf{F}_0(x,u,\nabla u)\,\mathrm{d}x & \text{if } u\in W^{1,p_0(\cdot)}_0(\Omega),\\ +\infty & \text{otherwise,}\end{cases} \tag{2.15}$$

where

$$\mathsf{F}_0(x,u,\nabla u) \stackrel{\text{def}}{=} \mathsf{A}(x,\nabla u) + \frac{\rho(x)}{\sigma(x)}|u|^{\sigma(x)} + \mathsf{b}(x,u) - g(x)\rho(x)u. \tag{2.16}$$

Moreover, for any smooth function ζ in Ω, we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_m^\varepsilon}\left(\frac{1}{p_\varepsilon(x)}|u^\varepsilon|^{\sigma(x)-2}(u(x)u^\varepsilon-|u^\varepsilon|^2)+\frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)}\right)\zeta(x)\,\mathrm{d}x = \int_\Omega \mathsf{b}(x,u)\zeta(x)\,\mathrm{d}x. \quad (2.17)$$

The paper is organized as follows. In the next section we prove that the homogenized problem is well-posed. The convergence process is rigorously studied in § 5 using auxiliary approximating results previously developed in § 4. Finally, in § 6, we check that our assumptions are fulfilled by periodic (or quasi-periodic) geometries and we explicitly compute the limit functional.

Notational convention. In what follows $C, C_1, C_2, \dots$ are generic constants independent of ε. When we deal with the $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}$ functional we assume that $\gamma=\gamma_0$ is given by (C3). Likewise, when we deal with the $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}$ functional we assume that $\gamma=\gamma_1$ is given by (C4).

3. Properties of the homogenized problem (2.14)

In this section we state the basic properties of the homogenized problem (2.14) and check its well-posedness. First we study the functions $\mathsf{A}(x,\vec{a})$ and $\mathsf{b}(x,\beta)$ defined by conditions (C3) and (C4), respectively. Then, using their properties, we show the continuity of the homogenized functional J_{hom} in the space $W^{1,p_0(\cdot)}(\Omega)$. Finally, we prove that the homogenized problem (2.14) has a unique solution $u\in W^{1,p_0(\cdot)}(\Omega)$.

In what follows we make use of Hölder's inequality for Sobolev spaces with variable exponents. Let $\phi\in L^{\mathsf{p}(\cdot)}(\Omega)$, $\psi\in L^{\mathsf{q}(\cdot)}(\Omega)$ with $1/\mathsf{p}+1/\mathsf{q}=1$, $1<\mathsf{p}^-\leqslant \mathsf{p}(x)\leqslant \mathsf{p}^+<+\infty$ and $1<\mathsf{q}^-\leqslant\mathsf{q}(x)\leqslant\mathsf{q}^+<+\infty$. Then

$$\int_\Omega|\phi\psi|\,\mathrm{d}x\leqslant 2\|\phi\|_{L^{\mathsf{p}(\cdot)}(\Omega)}\|\psi\|_{L^{\mathsf{q}(\cdot)}(\Omega)}. \quad (3.1)$$

We also recall the following result from the theory of Sobolev spaces with nonstandard growth. Let the function p satisfy the log-Hölder continuity property and $1<\mathsf{p}^-\leqslant\mathsf{p}(x)<+\infty$. Then

$$\left.\begin{aligned}\min(\|\phi\|^{\mathsf{p}^-}_{L^{\mathsf{p}(\cdot)}(\Omega)},\|\phi\|^{\mathsf{p}^+}_{L^{\mathsf{p}(\cdot)}(\Omega)})\leqslant \Upsilon_{\mathsf{p}(\cdot),\Omega}(\phi)\leqslant\max(\|\phi\|^{\mathsf{p}^-}_{L^{\mathsf{p}(\cdot)}(\Omega)},\|\phi\|^{\mathsf{p}^+}_{L^{\mathsf{p}(\cdot)}(\Omega)}),\\ \min(\Upsilon^{1/\mathsf{p}^-}_{\mathsf{p}(\cdot),\Omega}(\phi),\Upsilon^{1/\mathsf{p}^+}_{\mathsf{p}(\cdot),\Omega}(\phi))\leqslant\|\phi\|_{L^{\mathsf{p}(\cdot)}(\Omega)}\leqslant\max(\Upsilon^{1/\mathsf{p}^-}_{\mathsf{p}(\cdot),\Omega}(\phi),\Upsilon^{1/\mathsf{p}^+}_{\mathsf{p}(\cdot),\Omega}(\phi)).\end{aligned}\right\} \quad (3.2)$$

Properties of the function $\mathsf{A}(x,\vec{a})$ are given in the following lemma.

LEMMA 3.1. *Under the assumptions of theorem 2.5 the function A has the following properties:*

(i) *it is convex with respect to the variable $\vec{a}$, i.e.*

$$\mathsf{A}(x,\vec{a}_\tau)\leqslant\tau\mathsf{A}(x,\vec{a}_1)+(1-\tau)\mathsf{A}(x,\vec{a}_2) \quad (3.3)$$

for any $x\in\Omega$, $\vec{a}_1\in\mathbb{R}^n$, $\vec{a}_2\in\mathbb{R}^n$, $\tau\in[0,1]$, where $\vec{a}_\tau=\tau\vec{a}_1+(1-\tau)\vec{a}_2$;

(ii) *it admits the bound:*

$$|A(x,\vec{a})| \leqslant C|\vec{a}|^{p_0(x)} \quad \textit{for any } x\in\Omega \textit{ and } \vec{a}\in\mathbb{R}^n; \tag{3.4}$$

(iii) *it is locally Lipschitz in the following sense:*

$$|A(x,\vec{a}_1)-A(x,\vec{a}_2)| \leqslant C(1+|\vec{a}_1|+|\vec{a}_2|)^{p_0(x)-1}|\vec{a}_1-\vec{a}_2| \tag{3.5}$$

for any $x\in\Omega$, $\vec{a}_1\in\mathbb{R}^n$, $\vec{a}_2\in\mathbb{R}^n$.

Proof of lemma 3.1. First, we prove lemma 3.1(i). Let v_1^ε, v_2^ε and $v_{1,2}^\varepsilon$ be minimizers of the functional in (2.10) with $\vec{a}=\vec{a}_1$, $\vec{a}=\vec{a}_2$ and $\vec{a}_\tau=\tau\vec{a}_1+(1-\tau)\vec{a}_2$, respectively. Let $\gamma=\gamma_0$ be given by (C3). By the definition of $v_{1,2}^\varepsilon$, for any $z\in\Omega$, we have

$$\begin{aligned} c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}_\tau) &= \int_{K_h^z\cap\Omega_f^\varepsilon} (\varkappa_\varepsilon(x)|\nabla v_{1,2}^\varepsilon|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)}|v_{1,2}^\varepsilon-(x-z,\vec{a}_\tau)|^{p_\varepsilon(x)})\,\mathrm{d}x \\ &\leqslant \int_{K_h^z\cap\Omega_f^\varepsilon} (\varkappa_\varepsilon(x)|\nabla v_\tau^\varepsilon|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)}|v_\tau^\varepsilon-(x-z,\vec{a}_\tau)|^{p_\varepsilon(x)})\,\mathrm{d}x, \end{aligned} \tag{3.6}$$

where $v_\tau^\varepsilon=\tau v_1^\varepsilon+(1-\tau)v_2^\varepsilon$. It follows from (3.6) that

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}_\tau) \leqslant \tau c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}_1)+(1-\tau)c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}_2) \quad \text{for all } z\in\Omega. \tag{3.7}$$

Lemma 3.1(i) immediately follows from (3.7) and condition (C3).

We turn to lemma 3.1(ii). Let $z\in\Omega$ and let v^ε be the minimizer of the functional in (2.10). Taking $w_a(x)=(x-z,\vec{a})$ as a test function in the integral in (2.10) we obtain

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}) \leqslant \int_{K_h^z\cap\Omega_f^\varepsilon} \varkappa_\varepsilon(x)|\nabla w_a|^{p_\varepsilon(x)}\,\mathrm{d}x = \int_{K_h^z\cap\Omega_f^\varepsilon} \varkappa_\varepsilon(x)|\vec{a}|^{p_\varepsilon(x)}\,\mathrm{d}x.$$

This inequality, condition (K1) and (2.1) immediately imply that

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}) \leqslant \frac{k_0^{-1}}{p^-}\int_{K_h^z\cap\Omega_f^\varepsilon} |\vec{a}|^{p_\varepsilon(x)}\,\mathrm{d}x.$$

We then write

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}) \leqslant \frac{k_0^{-1}}{p^-}\int_{K_h^z\cap\Omega_f^\varepsilon} |\vec{a}|^{p_0(x)}\,\mathrm{d}x + \frac{k_0^{-1}}{p^-}\int_{K_h^z\cap\Omega_f^\varepsilon} (|\vec{a}|^{p_\varepsilon(x)}-|\vec{a}|^{p_0(x)})\,\mathrm{d}x.$$

Using assumption (A3) we obtain

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}) \leqslant \frac{k_0^{-1}}{p^-}\int_{K_h^z\cap\Omega_f^\varepsilon} |\vec{a}|^{p_0(x)}\,\mathrm{d}x + o(1) \quad \text{as } \varepsilon\to 0.$$

We infer from this inequality and assumption (A2) that, for sufficiently small ε and any $z\in\Omega$,

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\vec{a}) \leqslant Ch^n|\vec{a}|^{p_0(z)} + o(h^n) \quad \text{as } h\to 0. \tag{3.8}$$

Statement (ii) of Lemma 3.1 immediately follows from (3.8) and condition (C3).

It remains to prove (iii). Let τ be defined by

$$\tau \stackrel{\text{def}}{=} \frac{|\vec{a}_1 - \vec{a}_2|}{1 + |\vec{a}_1| + |\vec{a}_2|}. \tag{3.9}$$

The result is obvious if $\vec{a}_1 = \vec{a}_2$. We thus assume that $\tau \neq 0$. Let $z \in \Omega$. Consider the functional $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_1)$. It can be represented as follows:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_1) = c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, (1-\tau)\vec{a}_2 + \tau(\vec{a}_2 + \tau^{-1}(\vec{a}_1 - \vec{a}_2))).$$

It therefore follows from the convexity result (3.7) that

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_1) \leqslant (1-\tau)c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_2) + \tau c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_2 + \tau^{-1}(\vec{a}_1 - \vec{a}_2)). \tag{3.10}$$

We use (3.8) to estimate the second term of the right-hand side of (3.10). Bearing in mind (3.9), for sufficiently small ε we have

$$\begin{aligned}
&\tau c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_2 + \tau^{-1}(\vec{a}_1 - \vec{a}_2)) \\
&\qquad \leqslant Ch^n \frac{|\vec{a}_1 - \vec{a}_2|}{1 + |\vec{a}_1| + |\vec{a}_2|} |\vec{a}_2 + \tau^{-1}(\vec{a}_1 - \vec{a}_2)|^{p_0(z)} \\
&\qquad \leqslant C_1 h^n (1 + |\vec{a}_1| + |\vec{a}_2|)^{p_0(z)-1} |\vec{a}_1 - \vec{a}_2| + o(h^n) \quad \text{as } h \to 0.
\end{aligned} \tag{3.11}$$

Because of (3.8), the term $\tau c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_2)$ is also of order $o(h^n)$ as $h \to 0$. Then, from (3.9)–(3.11), for sufficiently small ε, we obtain

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_1) - c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_2) \leqslant C_2 h^n (1 + |\vec{a}_1| + |\vec{a}_2|)^{p_0(z)-1} |\vec{a}_1 - \vec{a}_2| + o(h^n) \quad \text{as } h \to 0. \tag{3.12}$$

In the same way, one checks that, for sufficiently small ε,

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_1) - c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}_2) \geqslant -C_2 h^n (1 + |\vec{a}_1| + |\vec{a}_2|)^{p_0(z)-1} |\vec{a}_1 - \vec{a}_2| + o(h^n) \quad \text{as } h \to 0. \tag{3.13}$$

Statement (iii) of lemma 3.1 follows from (3.12), (3.13) and condition (C3). Hence, lemma 3.1 is proved. □

In a similar way we obtain the properties of the function $\mathsf{b}(x, \beta)$. Namely, the following result holds.

Lemma 3.2. *Under the assumptions of theorem 2.5, function* b *has the following properties:*

(i) *it is convex with respect to the variable* β, *i.e.*

$$\mathsf{b}(x, \beta_\tau) \leqslant \tau \mathsf{b}(x, \beta_1) + (1-\tau)\mathsf{b}(x, \beta_2) \tag{3.14}$$

for any $x \in \Omega$, $(\beta_1, \beta_2) \in \mathbb{R}^2$, $\tau \in [0,1]$, *where* $\beta_\tau = \tau\beta_1 + (1-\tau)\beta_2$;

(ii) *it satisfies the bound*

$$|\mathsf{b}(x, \beta)| \leqslant C|\beta|^{\sigma(x)} \tag{3.15}$$

for any $x \in \Omega$ *and any* $\beta \in \mathbb{R}$;

(iii) *it is locally Lipschitz in the following sense:*

$$|b(x,\beta_1) - b(x,\beta_2)| \leqslant C(1+|\beta_1|+|\beta_2|)^{\sigma(x)-1}|\beta_1-\beta_2| \tag{3.16}$$

for any $x \in \Omega$, $(\beta_1,\beta_2) \in \mathbb{R}^2$.

We now state the continuity of the homogenized functional J_{hom} in the space $W^{1,p_0(\cdot)}(\Omega)$. Namely, we have the following result.

LEMMA 3.3. *Under the assumptions of theorem 2.5, for any* $(u,v) \in (W^{1,p_0(\cdot)}(\Omega))^2$, *the functional* J_{hom} *satisfies*

$$|J_{\mathrm{hom}}[u] - J_{\mathrm{hom}}[v]| \leqslant L\|u-v\|_{W^{1,p_0(\cdot)}(\Omega)}, \tag{3.17}$$

where

$$L = L(\operatorname{meas}\Omega, p_0^{\pm}, \sigma^{\pm}, \|u\|_{W^{1,p_0(\cdot)}(\Omega)}, \|v\|_{W^{1,p_0(\cdot)}(\Omega)}).$$

Proof of lemma 3.3. Let $(u,v) \in (W^{1,p_0(\cdot)}(\Omega))^2$. From the definition of the homogenized functional J_{hom} and regularity properties of functions p_0, σ, ρ, g, we obtain

$$|J_{\mathrm{hom}}[u] - J_{\mathrm{hom}}[v]| \leqslant C\int_\Omega (|A(x,\nabla u) - A(x,\nabla v)| + ||u|^{\sigma(x)} - |v|^{\sigma(x)}| + |b(x,u) - b(x,v)| + |u-v|)\,\mathrm{d}x. \tag{3.18}$$

We have to estimate the right-hand side of (3.18). For the first term, using (3.5), we write

$$\int_\Omega |A(x,\nabla u) - A(x,\nabla v)|\,\mathrm{d}x \leqslant C\int_\Omega (1+|\nabla u|+|\nabla v|)^{p_0(x)-1}|\nabla u - \nabla v|\,\mathrm{d}x. \tag{3.19}$$

To estimate the integral on the right-hand side of (3.19) we apply Hölder's inequality (3.1) and inequalities (3.2). We obtain

$$\int_\Omega (1+|\nabla u|+|\nabla v|)^{p_0(x)-1}|\nabla u-\nabla v|\,\mathrm{d}x \leqslant CL_1\|\nabla u - \nabla v\|_{L^{p_0(\cdot)}(\Omega)}, \tag{3.20}$$

where

$$L_1 = \max(\Upsilon^{1/q_0^-}_{p_0(\cdot),\Omega}(1+|\nabla u|+|\nabla v|), \Upsilon^{1/q_0^+}_{p_0(\cdot),\Omega}(1+|\nabla u|+|\nabla v|))$$

and $1/p_0 + 1/q_0 = 1$ with $1 < q_0^- \leqslant q_0(x) \leqslant q_0^+$. In a similar way we estimate the second, third and fourth terms on the right-hand side of (3.18). We obtain the desired inequality (3.17). Lemma 3.3 is proved. □

We end this section with the existence result for the variational problem (2.14).

LEMMA 3.4. *Under the assumptions of theorem 2.5 there exists a unique solution* $u \in W^{1,p_0(\cdot)}(\Omega)$ *of the variational problem (2.14).*

Proof of lemma 3.4. The existence of the minimizer to the functional (2.15) is a consequence of the proof of theorem 2.5 presented in § 5. The uniqueness of the solution of the homogenized problem (2.14) immediately follows from the strict convexity of the homogenized functional J_{hom}. Lemma 3.4 is proved. □

4. Auxiliary results

In this section we construct a convenient approximation for the solution of the variational problem (2.6) in the subdomains Ω_m^ε and Ω_f^ε. Of course, writing 'convenient', we have in mind 'convenient for the passage to the limit $\varepsilon \to 0$'.

We use the following notation. Let $\{x^\alpha\}$ be a periodic grid in Ω with a period $h' = h - h^{1+\gamma/\mathsf{p}^+}$, $\varepsilon \ll h \ll 1$, $0 < \gamma < \mathsf{p}^-$. Let us cover the domain Ω by cubes K_h^α of length h centred at x^α. With this covering we associate a partition of unity $\{\varphi_\alpha\}$:

$$0 \leqslant \varphi_\alpha(x) \leqslant 1, \qquad \varphi_\alpha(x) = 0 \text{ for } x \notin K_h^\alpha,$$
$$\varphi_\alpha(x) = 1 \quad \text{for } x \in K_h^\alpha \setminus \bigcup_{\beta\neq\alpha} K_h^\beta,$$
$$\sum_\alpha \varphi_\alpha(x) = 1, \qquad |\nabla\varphi_\alpha(x)| \leqslant Ch^{-1-\gamma/\mathsf{p}^+} \text{ for } x \in \Omega.$$

We also denote by $K_{h'}^\alpha$ the cube of length h' centred at the point x^α and we set $\Pi_h^\alpha = K_h^\alpha \setminus K_{h'}^\alpha$.

We begin with the following result of approximation in Ω_m^ε.

Lemma 4.1. *Assume that the conditions of theorem 2.5 are satisfied. Then, for each $h > 0$, there exist a set $\mathcal{B}^{\varepsilon,h} \subset \Omega_f^\varepsilon$ and a function $Y^{\varepsilon,h} \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ such that*

(i) $0 \leqslant Y^{\varepsilon,h}(x) \leqslant 1$ *in* Ω *and* $Y^{\varepsilon,h}(x) = 1$ *in* $\Omega_f^\varepsilon \setminus \mathcal{B}^{\varepsilon,h}$*;*

(ii) $\overline{\lim}_{\varepsilon\to 0} \operatorname{meas} \mathcal{B}^{\varepsilon,h} = O(h^{\gamma/(\mathsf{p}^+ + 1)})$ *as* $h \to 0$*;*

(iii) *for any function* $w \in C_0^1(\Omega)$*, we have, as* $h \to 0$*,*

$$\overline{\lim_{\varepsilon\to 0}} \int_\Omega \left(\varkappa_\varepsilon(x) |w\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)} |wY^{\varepsilon,h}|^{\sigma(x)} \right) \mathrm{d}x \leqslant \int_\Omega \left(\mathsf{b}(x,w) + \frac{\rho(x)}{\sigma(x)} |w|^{\sigma(x)} \right) \mathrm{d}x + o(1). \quad (4.1)$$

Proof of lemma 4.1. Let $w_\alpha^{\varepsilon,h}$ be a minimizer of the functional in (2.11) with $z = x^\alpha$ and $\beta = \beta_\alpha = w(x^\alpha)$. It follows from condition (C4) that

$$\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha} \left(\varkappa_\varepsilon(x) |\nabla w_\alpha^{\varepsilon,h}|^{p_\varepsilon(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)} |w_\alpha^{\varepsilon,h}|^{\sigma(x)} \right) \mathrm{d}x = O(h^n) \qquad \text{as } h \to 0, \quad (4.2)$$

$$\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-p_\varepsilon(x)} |w_\alpha^{\varepsilon,h} - \beta_\alpha|^{p_\varepsilon(x)} \, \mathrm{d}x = O(h^{n+\gamma}) \quad \text{as } h \to 0. \quad (4.3)$$

Furthermore, due to conditions (A1) and (C4),

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Pi_h^\alpha} \left(\varkappa_\varepsilon(x) |\nabla w_\alpha^{\varepsilon,h}|^{p_\varepsilon(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)} |w_\alpha^{\varepsilon,h}|^{\sigma(x)} \right) \mathrm{d}x = o(h^n) \qquad \text{as } h \to 0, \quad (4.4)$$

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Pi_h^\alpha \cap \Omega_f^\varepsilon} h^{-p_\varepsilon(x)} |w_\alpha^{\varepsilon,h} - \beta_\alpha|^{p_\varepsilon(x)} \, \mathrm{d}x = o(h^{n+\gamma}) \quad \text{as } h \to 0. \quad (4.5)$$

Moreover, the minimizer $w_\alpha^{\varepsilon,h}$ of functional (2.11) is bounded. Namely,

$$|w_\alpha^{\varepsilon,h}| \leqslant |\beta_\alpha| \quad \text{in } K_h^\alpha. \tag{4.6}$$

Now, for any cube K_h^α we introduce the set

$$\mathcal{B}_\alpha^{\varepsilon,h} \stackrel{\text{def}}{=} \{x \in K_h^\alpha : |w_\alpha^{\varepsilon,h}(x) - \beta_\alpha \mathbf{1}_f^\varepsilon(x)| \geqslant h^{p^-/p^+ + \gamma/(p^++1)}\} \tag{4.7}$$

and the function

$$\hat{w}_\alpha^{\varepsilon,h}(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{in } K_h^\alpha \cap (\Omega_f^\varepsilon \setminus \mathcal{B}_\alpha^{\varepsilon,h}), \\ \dfrac{w_\alpha^{\varepsilon,h} - h^{p^-/p^++\gamma/(p^++1)}}{\beta_\alpha - 2h^{p^-/p^++\gamma/(p^++1)}} & \text{in } \mathcal{B}_\alpha^{\varepsilon,h}, \\ 0 & \text{in } K_h^\alpha \cap (\Omega_m^\varepsilon \setminus \mathcal{B}_\alpha^{\varepsilon,h}). \end{cases} \tag{4.8}$$

Note that in the non-trivial case $\beta_\alpha \neq 0$, we can choose sufficiently small h to ensure that $\beta_\alpha - 2h^{p^-/p^++\gamma/(p^++1)} \neq 0$. Then we set

$$\hat{\mathcal{B}}_\alpha^{\varepsilon,h} \stackrel{\text{def}}{=} \mathcal{B}_\alpha^{\varepsilon,h} \cap \Omega_f^\varepsilon. \tag{4.9}$$

It follows from condition (C4) and (4.7) that

$$\overline{\lim_{\varepsilon\to 0}} \operatorname{meas} \hat{\mathcal{B}}_\alpha^{\varepsilon,h} = O(h^{n+\gamma/(p^++1)}). \tag{4.10}$$

Indeed, we have the following relation:

$$\begin{aligned} O(h^n) &= \int_{K_h^z} \bigg(\varkappa_\varepsilon(x) |\nabla w_\alpha^{\varepsilon,h}|^{p_\varepsilon(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)} |w_\alpha^{\varepsilon,h}|^{\sigma(x)} \\ &\qquad + h^{-p_\varepsilon(x)-\gamma} \mathbf{1}_f^\varepsilon(x) |w_\alpha^{\varepsilon,h} - \beta|^{p_\varepsilon(x)} \bigg) \, \mathrm{d}x \\ &\geqslant h^{-p^--\gamma} \int_{\hat{\mathcal{B}}_\alpha^{\varepsilon,h}} |w_\alpha^{\varepsilon,h} - \beta|^{p_\varepsilon(x)} \, \mathrm{d}x \\ &\geqslant h^{-p^--\gamma} \int_{\hat{\mathcal{B}}_\alpha^{\varepsilon,h}} (h^{p^-/p^++\gamma/(p^++1)})^{p_\varepsilon(x)} \, \mathrm{d}x \\ &\geqslant h^{-p^--\gamma} h^{p^-+\gamma p^+/(p^++1)} \operatorname{meas} \hat{\mathcal{B}}_\alpha^{\varepsilon,h} \\ &= h^{-(\gamma/(p^++1))} \operatorname{meas} \hat{\mathcal{B}}_\alpha^{\varepsilon,h}. \end{aligned}$$

It also follows from definition (4.8) that

$$0 \leqslant \hat{w}_\alpha^{\varepsilon,h}(x) \leqslant 1 \quad \text{in } K_h^\alpha. \tag{4.11}$$

Moreover, we note that condition (C4) implies

$$\begin{aligned} \overline{\lim_{\varepsilon\to 0}} \int_{K_h^z} \bigg(&\varkappa_\varepsilon(x) |\beta_\alpha \nabla \hat{w}_\alpha^{\varepsilon,h}|^{p_\varepsilon(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)} |\beta_\alpha \hat{w}_\alpha^{\varepsilon,h}|^{\sigma(x)} \\ &+ h^{-p_\varepsilon(x)-\gamma} \mathbf{1}_f^\varepsilon(x) |\beta_\alpha \hat{w}_\alpha^{\varepsilon,h} - \beta|^{p_\varepsilon(x)} \bigg) \, \mathrm{d}x \leqslant h^n b(x^\alpha, \beta_\alpha) + o(h^n) \end{aligned} \tag{4.12}$$

as $h \to 0$. Now we are in a position to define the desired set $\mathcal{B}^{\varepsilon,h}$ and the function $Y^{\varepsilon,h}$. We set

$$\mathcal{B}^{\varepsilon,h} \stackrel{\text{def}}{=} \bigcup_{\alpha} \hat{\mathcal{B}}_{\alpha}^{\varepsilon,h} \quad \text{and} \quad Y^{\varepsilon,h}(x) \stackrel{\text{def}}{=} \sum_{\alpha} \hat{w}_{\alpha}^{\varepsilon,h}(x)\varphi_{\alpha}(x). \tag{4.13}$$

Assertion (i) of the lemma immediately follows from (4.11) and definition (4.13). Assertion (ii) follows from estimate (4.10) and definition (4.13). Finally, using conditions (C1), (C4), estimates (4.2)–(4.5), the definition of the function $\hat{w}_{\alpha}^{\varepsilon,h}$ and estimate (4.12), we prove assertion (iii). Lemma 4.1 is proved. □

The second step of the approximation process is the following lemma.

LEMMA 4.2. *Let the conditions of theorem 2.5 be satisfied and let $\mathcal{B}^{\varepsilon,h}$ be the set defined in lemma 4.1. Let $w \in C_0^1(\Omega)$. Then there are a set $\mathcal{D}^{\varepsilon,h} \subset \Omega$ and a function $V^{\varepsilon,h} = V^{\varepsilon,h}(\cdot, w) \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ such that*

(i) $\mathcal{B}^{\varepsilon,h} \subset \mathcal{D}^{\varepsilon,h}$ *and* $\overline{\lim}_{\varepsilon\to 0} \operatorname{meas} \mathcal{D}^{\varepsilon,h} = o(1)$ *as* $h \to 0$,

(ii) $\max_{x\in\Omega} |V^{\varepsilon,h}(x) - w(x)| \leqslant Ch$,

(iii) *the following relations hold true:*

$$\overline{\lim_{\varepsilon\to 0}} \int_{\mathcal{D}^{\varepsilon,h}\cup\Omega_m^\varepsilon} \varkappa_\varepsilon(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x = o(1) \quad \textit{as } h \to 0, \tag{4.14}$$

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Omega_f^\varepsilon} \varkappa_\varepsilon(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x \leqslant \int_{\Omega} A(x, \nabla w)\,\mathrm{d}x + o(1) \quad \textit{as } h \to 0. \tag{4.15}$$

Proof of lemma 4.2. Let $\{\tilde{p}_\varepsilon\}_{(\varepsilon>0)}$ be a sequence of $\mathfrak{P}_0^\varepsilon$ such that, for any $x \in \Omega$, $\tilde{p}_\varepsilon(x) - C_{\tilde{p}_\varepsilon} \geqslant p_\varepsilon(x)$, where $C_{\tilde{p}_\varepsilon}$ is the constant defined in condition (C2). Due to condition (C2), there exists a family of extension operators $\boldsymbol{P}^\varepsilon : W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon) \to W^{1,\tilde{p}_\varepsilon(\cdot)}(\Omega)$ such that $\boldsymbol{P}^\varepsilon v^\varepsilon = v^\varepsilon$ in Ω_f^ε and $\|\boldsymbol{P}^\varepsilon v^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \leqslant C\|v^\varepsilon\|_{W^{1,\tilde{p}_\varepsilon(\cdot)}(\Omega_f^\varepsilon)}$ for any $v^\varepsilon \in W^{1,\tilde{p}_\varepsilon(\cdot)}(\Omega_f^\varepsilon)$.

Let $v_\alpha^{\varepsilon,h}$ be a minimizer of the functional

$$\begin{aligned} &c_{\tilde{p}_\varepsilon(\cdot)}^{\varepsilon,h}(x^\alpha, \vec{a}) \\ &\quad = \inf_{v^\varepsilon} \int_{K_h^\alpha\cap\Omega_f^\varepsilon} \left(\frac{K_\varepsilon(x)}{\tilde{p}_\varepsilon(x)} |\nabla v^\varepsilon|^{\tilde{p}_\varepsilon(x)} + h^{-\tilde{p}_\varepsilon(x)-\gamma}|v^\varepsilon - (x - x^\alpha, \vec{a}_\alpha)|^{\tilde{p}_\varepsilon(x)} \right) \mathrm{d}x, \end{aligned} \tag{4.16}$$

where $\gamma > 0$, the infimum is taken over $v^\varepsilon \in W^{1,\tilde{p}_\varepsilon(\cdot)}(K_h^\alpha \cap \Omega_f^\varepsilon)$ and $\vec{a}_\alpha = \nabla w(x^\alpha)$. It follows from condition (C3) that

$$\int_{K_h^\alpha\cap\Omega_f^\varepsilon} \frac{K_\varepsilon(x)}{\tilde{p}_\varepsilon(x)} |\nabla v_\alpha^{\varepsilon,h}|^{\tilde{p}_\varepsilon(x)}\,\mathrm{d}x = O(h^n) \qquad \text{as } h \to 0, \tag{4.17}$$

$$\int_{K_h^\alpha\cap\Omega_f^\varepsilon} h^{-\tilde{p}_\varepsilon(x)}|v_\alpha^{\varepsilon,h} - (x - x^\alpha, \vec{a}_\alpha)|^{\tilde{p}_\varepsilon(x)}\,\mathrm{d}x = O(h^{n+\gamma}) \quad \text{as } h \to 0, \tag{4.18}$$

$$\int_{\Pi_h^\alpha\cap\Omega_f^\varepsilon}\frac{K_\varepsilon(x)}{\tilde p_\varepsilon(x)}|\nabla v_\alpha^{\varepsilon,h}|^{\tilde p_\varepsilon(x)}\,\mathrm{d}x=o(h^n)\qquad\text{as } h\to 0,\tag{4.19}$$

$$\int_{\Pi_h^\alpha\cap\Omega_f^\varepsilon}h^{-\tilde p_\varepsilon(x)}|v_\alpha^{\varepsilon,h}-(x-x^\alpha,\vec a_\alpha)|^{\tilde p_\varepsilon(x)}\,\mathrm{d}x=o(h^{n+\gamma})\qquad\text{as } h\to 0.\tag{4.20}$$

Since $v_\alpha^{\varepsilon,h}$ is a minimizer of (4.16), then we have

$$\max_{K_h^\alpha\cap\Omega_f^\varepsilon}|v_\alpha^{\varepsilon,h}(x)|\leqslant h.\tag{4.21}$$

Let us now introduce the function $W^{\varepsilon,h}=W^{\varepsilon,h}(x)$ defined by

$$W^{\varepsilon,h}(x)=w(x)+\sum_\alpha(v_\alpha^{\varepsilon,h}(x)-(x-x^\alpha,\vec a_\alpha))\varphi_\alpha(x)\quad\text{for all } x\in\Omega.\tag{4.22}$$

The function $W^{\varepsilon,h}$ is constructed such that it belongs to $W^{1,\tilde p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)$ and it satisfies the bound:

$$\max_{x\in\Omega_f^\varepsilon}|W^{\varepsilon,h}(x)-w(x)|\leqslant Ch.\tag{4.23}$$

In addition, using estimates (4.17)–(4.20) and bearing in mind that $w\in C_0^1(\Omega)$, we obtain

$$\overline{\lim_{\varepsilon\to 0}}\int_{\Omega_f^\varepsilon}\frac{K_\varepsilon(x)}{\tilde p_\varepsilon(x)}|\nabla W^{\varepsilon,h}|^{\tilde p_\varepsilon(x)}\,\mathrm{d}x\leqslant\int_\Omega A(x,\nabla w)\,\mathrm{d}x+o(1)\quad\text{as } h\to 0.\tag{4.24}$$

Setting

$$U^{\varepsilon,h}=W^{\varepsilon,h}-w,\tag{4.25}$$

we infer from (4.23) and (4.24) that

$$\max_{x\in\Omega_f^\varepsilon}|U^{\varepsilon,h}(x)|\leqslant Ch\quad\text{and}\quad\|U^{\varepsilon,h}\|_{W^{1,\tilde p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)}\leqslant C.\tag{4.26}$$

Now, using the extension operator defined in the first lines of the present proof, we claim that there exists a function $\boldsymbol{U}^{\varepsilon,h}\in W^{1,p_\varepsilon(\cdot)}(\Omega)$ such that

$$\boldsymbol{U}^{\varepsilon,h}(x)=U^{\varepsilon,h}(x)\text{ in }\Omega_f^\varepsilon,\qquad\max_{x\in\Omega}|\boldsymbol{U}^{\varepsilon,h}(x)|\leqslant Ch\quad\text{and}\quad\|\boldsymbol{U}^{\varepsilon,h}\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)}\leqslant C.\tag{4.27}$$

We also recall that the set $\mathcal{B}^{\varepsilon,h}$ defined in lemma 4.1 satisfies

$$\overline{\lim_{\varepsilon\to 0}}\operatorname{meas}\mathcal{B}^{\varepsilon,h}=O(h^{\gamma/(p^++1)})\quad\text{as } h\to 0.$$

Thus, following the ideas of the proof of [14, lemma 4.4, ch. 4], we assert that there exist a set $\mathcal{D}^{\varepsilon,h}$ and a function $\hat{\boldsymbol{U}}^{\varepsilon,h}$ such that $\hat{\boldsymbol{U}}^{\varepsilon,h}=\boldsymbol{U}^{\varepsilon,h}$ in $\Omega\setminus\mathcal{D}^{\varepsilon,h}$, satisfying the following properties:

$$\max_{x\in\Omega}|\hat{\boldsymbol{U}}^{\varepsilon,h}(x)|\leqslant Ch,\tag{4.28}$$

$$\|\hat{\boldsymbol{U}}^{\varepsilon,h}\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)}\leqslant C,\tag{4.29}$$

$$\mathcal{B}^{\varepsilon,h}\subset\mathcal{D}^{\varepsilon,h}\quad\text{and}\quad\overline{\lim_{\varepsilon\to 0}}\operatorname{meas}\mathcal{D}^{\varepsilon,h}=o(1)\quad\text{as } h\to 0,\tag{4.30}$$

$$\|\hat{\boldsymbol{U}}^{\varepsilon,h}\|_{W^{1,p_\varepsilon(\cdot)}(\mathcal{D}^{\varepsilon,h})}\to 0\quad\text{as } h\to 0.\tag{4.31}$$

Finally, we define the function $V^{\varepsilon,h}$ by

$$V^{\varepsilon,h} = w + \hat{\boldsymbol{U}}^{\varepsilon,h} \quad \text{in } \Omega. \tag{4.32}$$

Let us show that the function $V^{\varepsilon,h}$ and the set $\mathcal{D}^{\varepsilon,h}$ satisfy all the assertions of the lemma. Assertion (i) immediately follows from (4.30), while assertion (ii) is a consequence of the definition of the function $V^{\varepsilon,h}$ and (4.28). It remains to prove assertion (iii). Relation (4.14) immediately follows from the definitions of the functions K_ε and $V^{\varepsilon,h}$ and equations (4.30) and (4.31). Let us prove inequality (4.15). We write

$$\begin{aligned}
\int_{\Omega_f^\varepsilon} &\varkappa_\varepsilon(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x \\
&= \int_{\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x + \int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x &(4.33)\\
&= \int_{\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla W^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x + \int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x. &(4.34)
\end{aligned}$$

The second term on the right-hand side of (4.34) is $o(1)$ as $h \to 0$ because of (4.30), (4.31) and definition (4.32). Consider the first term on the right-hand side of (4.34). It can be estimated as follows:

$$\int_{\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla W^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x \leqslant \int_{\Omega_f^\varepsilon} \frac{K_\varepsilon(x)}{\tilde{p}_\varepsilon(x)}|\nabla W^{\varepsilon,h}|^{\tilde{p}_\varepsilon(x)}\,\mathrm{d}x + I^{\varepsilon,h},$$

where

$$I^{\varepsilon,h} = \int_{\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h}} K_\varepsilon(x)\bigg(\frac{1}{p_\varepsilon(x)}|\nabla W^{\varepsilon,h}|^{p_\varepsilon(x)} - \frac{1}{\tilde{p}_\varepsilon(x)}|\nabla W^{\varepsilon,h}|^{\tilde{p}_\varepsilon(x)}\bigg)\,\mathrm{d}x. \tag{4.35}$$

Because $\{p_\varepsilon\}_{(\varepsilon>0)} \subset \mathfrak{P}_0^\varepsilon$ and $\{\tilde{p}_\varepsilon\}_{(\varepsilon>0)} \subset \mathfrak{P}_0^\varepsilon$, we note that $\tilde{p}_\varepsilon - p_\varepsilon$ converges uniformly to zero in $C(\bar{\Omega})$. We therefore have

$$I^{\varepsilon,h} \leqslant k_0^{-1} \int_{(\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h})\cap\Theta_V^{\varepsilon,h}} \bigg(\frac{1}{p_\varepsilon(x)}|\nabla W^{\varepsilon,h}|^{p_\varepsilon(x)} - \frac{1}{\tilde{p}_\varepsilon(x)}|\nabla W^{\varepsilon,h}|^{\tilde{p}_\varepsilon(x)}\bigg)\,\mathrm{d}x \to 0 \quad \text{as } \varepsilon \to 0, \tag{4.36}$$

where

$$\Theta_V^{\varepsilon,h} = \bigg\{x \in \Omega : |\nabla W^{\varepsilon,h}(x)| < \bigg(\frac{\tilde{p}_\varepsilon(x)}{p_\varepsilon(x)}\bigg)^{1/(\tilde{p}_\varepsilon(x)-p_\varepsilon(x))}\bigg\}.$$

Now inequality (4.15) follows from (4.24), (4.34)–(4.36) and (4.14). Lemma 4.2 is proved. □

We now use the following notation. Let $\{p_\varepsilon^\star\}_{(\varepsilon>0)} \subset \mathfrak{P}_0^\varepsilon$ be the sequence of functions defined in condition (C2). We consider the sequence $\{\pi_\varepsilon^\star\}_{(\varepsilon>0)} \subset \mathfrak{P}_0^\varepsilon$ defined by

$$\pi_\varepsilon^\star(x) = \min\{p_\varepsilon^\star(x), p_0(x)\}, \quad x \in \Omega. \tag{4.37}$$

It is clear that $\{\pi^\star_\varepsilon\}_{(\varepsilon>0)} \subset \mathfrak{P}^\varepsilon_0$ and, moreover, that

$$\pi^\star_\varepsilon(x) \leqslant p^\star_\varepsilon(x) \leqslant p_\varepsilon(x) \quad \text{and} \quad \pi^\star_\varepsilon(x) \leqslant p_0(x) \quad \text{in } \Omega.$$

The next lemma gives an auxiliary result which will be used in the proof of the lower bound (see § 5.2 below).

LEMMA 4.3. *Let the conditions of theorem 2.5 be satisfied. Assume that a sequence* $\{u^\varepsilon\}_{(\varepsilon>0)} \subset W^{1,p^\star_\varepsilon(\cdot)}_0(\Omega)$ *converges to a function* $u \in C^1_0(\Omega)$ *in* $L^{p_0(\cdot)}(\Omega^\varepsilon_f)$ *and, moreover, that*

$$\int_\Omega \left(\mathbf{1}^\varepsilon_f(x)\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} \right) \mathrm{d}x \leqslant C. \tag{4.38}$$

Then there exist a set $\mathcal{G}^\varepsilon \subset \Omega$ *with* $\Omega^\varepsilon_m \subset \mathcal{G}^\varepsilon$, *a function* $\hat{u}^\varepsilon$ *and a subsequence* $\varepsilon_k \to 0$ *(still denoted by* ε *for convenience) such that*

(i) $\lim_{\varepsilon\to 0} \operatorname{meas} \mathcal{G}^\varepsilon_f = 0$, *where* $\mathcal{G}^\varepsilon_f = \mathcal{G}^\varepsilon \cap \Omega^\varepsilon_f$,

(ii) $\hat{u}^\varepsilon = u^\varepsilon$ *in* $\Omega^\varepsilon_f \setminus \mathcal{G}^\varepsilon_f$ *and, moreover,*

$$\lim_{\varepsilon\to 0} \|\hat{u}^\varepsilon\|_{W^{1,\pi^\star_\varepsilon(\cdot)}(\mathcal{G}^\varepsilon_f)} = 0, \tag{4.39}$$

(iii) *the following inequality holds true:*

$$\lim_{\varepsilon\to 0} \int_{\mathcal{G}^\varepsilon} \left(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi^\star_\varepsilon(x)} + \frac{\mathbf{1}^\varepsilon_m(x)}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} \right) \mathrm{d}x \geqslant \int_\Omega b(x,u)\,\mathrm{d}x. \tag{4.40}$$

Proof of lemma 4.3. Let u^ε_f be the restriction of the function u^ε to the domain Ω^ε_f. Due to condition (C2), the function $\boldsymbol{P}^\varepsilon u^\varepsilon_f = U^\varepsilon_f \in W^{1,p^\star_\varepsilon(\cdot)}(\Omega)$ satisfies in particular

$$\|U^\varepsilon_f\|_{W^{1,\pi^\star_\varepsilon(\cdot)}(\Omega)} \leqslant C, \tag{4.41}$$

because the function $\pi^\star_\varepsilon$ defined in (4.37) is such that $\pi^\star_\varepsilon \leqslant p^\star_\varepsilon$ in Ω.

Since $\{u^\varepsilon\}_{(\varepsilon>0)}$ strongly converges to the function u in the space $L^{p_0(\cdot)}(\Omega^\varepsilon_f)$, there exists a set G^ε such that $\operatorname{meas} G^\varepsilon \to 0$ as $\varepsilon \to 0$ and the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ converges to u uniformly in the domain $\Omega^\varepsilon_f \setminus G^\varepsilon$ and

$$\sup_{\Omega^\varepsilon_f \setminus G^\varepsilon} |U^\varepsilon_f - u| = \varrho_\varepsilon \quad \text{with } \varrho_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{4.42}$$

We set

$$G^\varepsilon_f = G^\varepsilon \cap \Omega^\varepsilon_f.$$

Following the ideas of the proof of [14, lemma 4.4, ch. 4], we show that there exist a set $\hat{G}^\varepsilon_f$ and a function $\hat{u}^\varepsilon \in W^{1,\pi^\star_\varepsilon(\cdot)}_0(\Omega)$ such that $G^\varepsilon_f \subset \hat{G}^\varepsilon_f$, $\hat{u}^\varepsilon = u^\varepsilon$ in $\Omega^\varepsilon_f \setminus \hat{G}^\varepsilon_f$,

$$\lim_{\varepsilon\to 0} \operatorname{meas} \hat{G}^\varepsilon_f = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \|\hat{u}^\varepsilon\|_{W^{1,\pi^\star_\varepsilon(\cdot)}(\hat{G}^\varepsilon_f)} = 0. \tag{4.43}$$

Now we set

$$\mathcal{G}^\varepsilon = \Omega^\varepsilon_m \cup \hat{G}^\varepsilon_f \quad \text{and} \quad \mathcal{G}^\varepsilon_f = \mathcal{G}^\varepsilon \cap \Omega^\varepsilon_f. \tag{4.44}$$

The sets $\mathcal{G}^\varepsilon$, $\mathcal{G}^\varepsilon_f$ and the function $\hat{u}^\varepsilon$ satisfy lemma 4.3(i) and (ii). Assertion (iii) remains to be proved. To this end, let us introduce the function

$$\hat{U}^\varepsilon(x) = \begin{cases} \hat{u}^\varepsilon(x) & \text{if } |\hat{u}^\varepsilon(x) - u(x)| \leqslant \varrho_\varepsilon, \\ u(x) + \varrho_\varepsilon & \text{if } \hat{u}^\varepsilon(x) > u(x) + \varrho_\varepsilon, \\ u(x) - \varrho_\varepsilon & \text{if } \hat{u}^\varepsilon(x) < u(x) - \varrho_\varepsilon. \end{cases} \tag{4.45}$$

The function $\hat{U}^\varepsilon$ belongs to $W^{1,\pi^\star_\varepsilon(\cdot)}(\Omega)$ and

$$|\hat{U}^\varepsilon(x) - u(x)| \leqslant \varrho_\varepsilon \quad \text{with } \varrho_\varepsilon \to 0 \text{ as } \varepsilon \to 0,\ x \in \Omega. \tag{4.46}$$

We set

$$\boldsymbol{U}^\varepsilon = u^\varepsilon - \hat{U}^\varepsilon \tag{4.47}$$

and consider the functional

$$I^\varepsilon[u^\varepsilon] \overset{\text{def}}{=} \int_{\mathcal{G}^\varepsilon} \biggl(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi^\star_\varepsilon(x)} + \frac{\mathbf{1}^\varepsilon_m(x)}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} \biggr)\,\mathrm{d}x. \tag{4.48}$$

Since $\mathcal{G}^\varepsilon = \Omega^\varepsilon_m \cup \mathcal{G}^\varepsilon_f$ and $\boldsymbol{U}^\varepsilon(x) = 0$ in $\Omega^\varepsilon_f \setminus \mathcal{G}^\varepsilon_f$, we have

$$\begin{aligned} I^\varepsilon[u^\varepsilon] &= \biggl(\int_{\Omega^\varepsilon_f} \varkappa_\varepsilon(x)|\nabla \boldsymbol{U}^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x + \int_{\Omega^\varepsilon_m} \biggl(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi^\star_\varepsilon(x)} + \frac{\mathbf{1}^\varepsilon_m(x)}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} \biggr)\,\mathrm{d}x \biggr) \\ &\quad + \biggl(\int_{\mathcal{G}^\varepsilon_f} \varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x - \int_{\mathcal{G}^\varepsilon_f} \varkappa_\varepsilon(x)|\nabla \boldsymbol{U}^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x \biggr) \\ &= \biggl(\int_{\Omega^\varepsilon_f} \varkappa_\varepsilon(x)|\nabla \boldsymbol{U}^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x + \int_{\Omega^\varepsilon_m} \biggl(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi^\star_\varepsilon(x)} + \frac{\mathbf{1}^\varepsilon_m(x)}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} \biggr)\,\mathrm{d}x \biggr) \\ &\quad + \biggl(\int_{\mathcal{G}^\varepsilon_f} \varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x - \int_{\mathcal{G}^\varepsilon_f} \varkappa_\varepsilon(x)|\nabla (u^\varepsilon - \hat{U}^\varepsilon)|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x \biggr) \\ &\overset{\text{def}}{=} \boldsymbol{i}^\varepsilon_1 + \boldsymbol{i}^\varepsilon_2. \end{aligned} \tag{4.49}$$

Consider the first term on the right-hand side of (4.49). First, for any $\xi > 0$ we define the set $\Omega_\zeta \subset \Omega$ by $\Omega_\zeta = \{x \in \Omega : |u(x)| > 2\zeta\}$. Let us cover Ω_ζ with cubes $K^{\alpha,\xi}_h$ of length h centred at x^α. Because of (4.46), for ε and sufficiently small h we have $|\hat{U}^\varepsilon| > \zeta$ in $K^{\alpha,\xi}_h$. Following the lines of the proof of [4, (6.29)], for $x \in \Omega^\varepsilon_f \cap K^{\alpha,\xi}_h$ we have

$$\begin{aligned} (1 + A_1 h^{\mathsf{p}^+/(\mathsf{p}^+-1)})&\varkappa_\varepsilon(x)|\nabla \boldsymbol{U}^\varepsilon|^{\pi^\star_\varepsilon(x)} \\ &\geqslant \varkappa_\varepsilon(x)|\hat{U}^\varepsilon|^{\pi^\star_\varepsilon(x)} \biggl| \nabla \biggl(\frac{u^\varepsilon}{\hat{U}^\varepsilon} \biggr) \biggr|^{\pi^\star_\varepsilon(x)} \\ &\qquad - k_0^{-1} A_2 \biggl(1 + \frac{1}{h^{\mathsf{p}^+}} \biggr) |\boldsymbol{U}^\varepsilon(x)| \frac{|\nabla \hat{U}^\varepsilon|^{\pi^\star_\varepsilon(x)}}{|\hat{U}^\varepsilon|^{\pi^\star_\varepsilon(x)}}, \end{aligned} \tag{4.50}$$

where A_1 and A_2 are positive constants independent of ε, δ and M. In a similar way, for $x \in \Omega_m^\varepsilon \cap K_h^{\alpha,\xi}$ we have

$$\begin{aligned}(1 + A_1 h^{p^+/(p^+-1)})\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)} &\geqslant \varkappa_\varepsilon(x)|\hat{U}^\varepsilon|^{\pi_\varepsilon^\star(x)}\left|\nabla\left(\frac{u^\varepsilon}{\hat{U}^\varepsilon}\right)\right|^{\pi_\varepsilon^\star(x)} \\ &\quad - k_\varepsilon A_2\left(1 + \frac{1}{h^{p^+}}\right)|u^\varepsilon|^{\pi_\varepsilon^\star(x)}\frac{|\nabla \hat{U}^\varepsilon|^{\pi_\varepsilon^\star(x)}}{|\hat{U}^\varepsilon|^{\pi_\varepsilon^\star(x)}}, \qquad (4.51)\end{aligned}$$

where k_ε is defined in condition (K2). Now we make use of the strong convergence of the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ to the function u in the space $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ and of the definition of the function $\hat{U}^\varepsilon$. Let

$$w_\alpha^\varepsilon \stackrel{\text{def}}{=} u(x^\alpha)\frac{u^\varepsilon}{\hat{U}^\varepsilon}.$$

Then, for any $K_h^{\alpha,\xi}$, we infer from (4.50) and (4.51) that

$$\begin{aligned}\varliminf_{\varepsilon\to 0}\Bigg(\int_{K_h^{\alpha,\xi}\cap\Omega_f^\varepsilon} &\varkappa_\varepsilon(x)|\nabla U^\varepsilon|^{\pi_\varepsilon^\star(x)}\,\mathrm{d}x \\ &+ \int_{K_h^{\alpha,\xi}\cap\Omega_m^\varepsilon}\left(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)}|u^\varepsilon|^{\sigma(x)}\right)\mathrm{d}x\Bigg) \\ &\qquad \geqslant \varliminf_{\varepsilon\to 0}\int_{K_h^{\alpha,\xi}}\left(\varkappa_\varepsilon(x)|\nabla w_\alpha^\varepsilon|^{\pi_\varepsilon^\star(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)}|w_\alpha^\varepsilon|^{\sigma(x)}\right)\mathrm{d}x \\ &\qquad + o(h^n) \quad \text{as } h \to 0. \qquad (4.52)\end{aligned}$$

Condition (C4) implies that

$$\begin{aligned}\varliminf_{\varepsilon\to 0}\int_{K_h^{\alpha,\xi}}&\left(\varkappa_\varepsilon(x)|\nabla w_\alpha^\varepsilon|^{\pi_\varepsilon^\star(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)}|w_\alpha^\varepsilon|^{\sigma(x)}\right)\mathrm{d}x \\ &\geqslant h^n b(x, u(x^\alpha)) + o(h^n) \quad \text{as } h \to 0. \qquad (4.53)\end{aligned}$$

Now it follows from (4.52) and (4.53) that

$$\varliminf_{\varepsilon\to 0}\boldsymbol{i}_1^\varepsilon \geqslant \int_{\Omega_\zeta} b(x, u)\,\mathrm{d}x. \qquad (4.54)$$

Taking into account the definition of Ω_ζ and passing to the limit as $\zeta \to 0$ in (4.54), we obtain

$$\varliminf_{\varepsilon\to 0}\boldsymbol{i}_1^\varepsilon \geqslant \int_{\Omega} b(x, u)\,\mathrm{d}x. \qquad (4.55)$$

Let us now estimate from below the term $\boldsymbol{i}_2^\varepsilon$ in (4.49). We argue as follows. Using the definition (4.45) of the function $\hat{U}^\varepsilon$, we obtain

$$\begin{aligned}|\boldsymbol{i}_2^\varepsilon| &\leqslant C\left|\int_{\mathcal{G}_f^\varepsilon}(|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)} - |\nabla(u^\varepsilon - \hat{U}^\varepsilon)|^{\pi_\varepsilon^\star(x)})\,\mathrm{d}x\right| \\ &= C\left|\int_{\mathcal{G}_f^\varepsilon}(|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)} - |\nabla(u^\varepsilon - u)|^{\pi_\varepsilon^\star(x)})\,\mathrm{d}x\right|\end{aligned}$$

$$\leqslant C\int_{\mathcal{G}_f^\varepsilon}|\nabla u|(|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)-1}+|\nabla u|^{\pi_\varepsilon^\star(x)-1})\,\mathrm{d}x.$$

Since u is a smooth function in Ω we conclude that

$$|\boldsymbol{i}_2^\varepsilon|\leqslant C\int_{\mathcal{G}_f^\varepsilon}(1+|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)-1})\,\mathrm{d}x. \tag{4.56}$$

It is now easy to see that (4.38), the estimate for the measure of the set $\mathcal{G}_f^\varepsilon$ (see (4.43) and (4.44)) and Hölder's inequality yields

$$\overline{\lim_{\varepsilon\to 0}}\,|\boldsymbol{i}_2^\varepsilon|=0. \tag{4.57}$$

Now assertion (iii) of the lemma immediately follows from (4.55) and (4.57). Lemma 4.3 is proved. □

5. Proof of theorem 2.5

We begin this section by obtaining *a priori* estimates for the minimizer u^ε of problem (2.9). Since $J^\varepsilon[u^\varepsilon]\leqslant J^\varepsilon[0]\equiv 0$, by virtue of the regularity properties of $\varkappa_\varepsilon$, σ, g^ε, (3.2) and Young's inequality, we can show that

$$\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)}\leqslant C. \tag{5.1}$$

It follows from (C2) that there is a function $\mathsf{u}^\varepsilon=\boldsymbol{P}^\varepsilon u^\varepsilon$ such that $\mathsf{u}^\varepsilon=u^\varepsilon$ in Ω_f^ε and

$$\|\mathsf{u}^\varepsilon\|_{W^{1,p_\varepsilon^\star(\cdot)}(\Omega)}\leqslant C. \tag{5.2}$$

Moreover, the definition of the function p_ε which converges uniformly to p_0 implies that there exists a parameter ς that does not depend on ε and such that

$$\|\mathsf{u}^\varepsilon\|_{W^{1,p_0(\cdot)-\varsigma}(\Omega)}\leqslant C \tag{5.3}$$

and the family $\{\mathsf{u}^\varepsilon\}_{(\varepsilon>0)}$ is a compact set in the space $L^{p_0(\cdot)}(\Omega)$. We can then extract a subsequence (still denoted by $\{\mathsf{u}^\varepsilon\}$) which converges to a function $u\in L^{p_0(\cdot)}(\Omega)$. In particular,

$$u^\varepsilon\to u\quad\text{in } L^{p_0(\cdot)}(\Omega_f^\varepsilon). \tag{5.4}$$

Let us show that u is the solution of the homogenized problem (2.14). The proof will be done in three steps. In § 5.1 we prove that

$$\overline{\lim}\,J^\varepsilon[u^\varepsilon]\leqslant J_{\mathrm{hom}}[w]\quad\text{for any } w\in W^{1,p_0(\cdot)}(\Omega).$$

Section 5.2 is devoted to the proof of the inequality

$$\underline{\lim}\,J^\varepsilon[u^\varepsilon]\geqslant J_{\mathrm{hom}}[u].$$

It follows that u is the minimizer of functional J_{hom} in $W_0^{1,p_0(\cdot)}(\Omega)$. We conclude by studying the convergence in the matrix part Ω_m^ε. Indeed, we prove the weak convergence of

$$\mathbf{1}_m^\varepsilon\left(\frac{1}{p_\varepsilon}|u^\varepsilon|^{\sigma(\cdot)-2}(uu^\varepsilon-|u^\varepsilon|^2)+\frac{1}{\sigma}|u^\varepsilon|^{\sigma(\cdot)}\right)$$

to $\mathsf{b}(\cdot,u)$ in § 5.3.

5.1. Step 1: upper bound

The aim of this section is to prove that

$$\overline{\lim}\, J^{\varepsilon}[u^{\varepsilon}] \leqslant J_{\mathrm{hom}}[w]$$

for any $w \in W^{1,p_0(\cdot)}(\Omega)$. We first state the result for an arbitrary function $w \in C_0^1(\Omega)$. Let $Y^{\varepsilon,h}$, $V^{\varepsilon,h}$ and $\mathcal{D}^{\varepsilon,h}$ be the corresponding functions and set defined in lemmas 4.1 and 4.2. We define the function $T^{\varepsilon,h} \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ by

$$T^{\varepsilon,h}(x) \overset{\mathrm{def}}{=} Y^{\varepsilon,h}(x)V^{\varepsilon,h}(x), \quad x \in \Omega. \tag{5.5}$$

We first prove that

$$\overline{\lim_{h\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, J^{\varepsilon}[T^{\varepsilon,h}] \leqslant J_{\mathrm{hom}}[w], \tag{5.6}$$

where we recall for the convenience of the reader that

$$J^{\varepsilon}[T^{\varepsilon,h}] = \int_{\Omega} \left(\varkappa_{\varepsilon}(x)|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|T^{\varepsilon,h}|^{\sigma(x)} - g^{\varepsilon}(x)T^{\varepsilon,h} \right) \mathrm{d}x, \tag{5.7}$$

$$J_{\mathrm{hom}}[w] = \int_{\Omega} \left(\mathsf{A}(x,\nabla w) + \frac{\rho(x)}{\sigma(x)}|w|^{\sigma(x)} + \mathsf{b}(x,w) - g(x)\rho(x)w \right) \mathrm{d}x. \tag{5.8}$$

Consider the third term in (5.7). It follows from condition (C1), the definition (2.8) of the function g^{ε}, assertions (i) and (ii) of lemma 4.1 and assertion (ii) of lemma 4.2 that

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \int_{\Omega} g^{\varepsilon}(x)T^{\varepsilon,h}(x)\,\mathrm{d}x = \int_{\Omega} g(x)\rho(x)w(x)\,\mathrm{d}x. \tag{5.9}$$

Consider the second term in (5.7). We write it as follows:

$$\int_{\Omega} \frac{1}{\sigma}|T^{\varepsilon,h}|^{\sigma} = \int_{\Omega} \frac{1}{\sigma}|Y^{\varepsilon,h}|^{\sigma}|w|^{\sigma}\,\mathrm{d}x + \int_{\Omega} \frac{1}{\sigma}|Y^{\varepsilon,h}|^{\sigma}(|V^{\varepsilon,h}|^{\sigma} - |w|^{\sigma})\,\mathrm{d}x.$$

It follows from assertion lemma 4.1(i) and lemma 4.2(i) that

$$\overline{\lim_{h\to 0}}\, \overline{\lim_{\varepsilon\to 0}} \int_{\Omega} \frac{1}{\sigma(x)}|Y^{\varepsilon,h}|^{\sigma(x)}||V^{\varepsilon,h}|^{\sigma(x)} - |w|^{\sigma(x)}|\,\mathrm{d}x = 0.$$

We thus write

$$\int_{\Omega} \frac{1}{\sigma}|T^{\varepsilon,h}|^{\sigma} = \int_{\Omega} \frac{1}{\sigma}|Y^{\varepsilon,h}|^{\sigma}|w|^{\sigma}\,\mathrm{d}x + \boldsymbol{j}_1^{\varepsilon,h} \quad \text{with } \overline{\lim_{h\to 0}}\, \overline{\lim_{\varepsilon\to 0}} |\boldsymbol{j}_1^{\varepsilon,h}| = 0. \tag{5.10}$$

Using $\Omega = (\Omega_f^{\varepsilon} \setminus \mathcal{D}^{\varepsilon,h}) \cup (\mathcal{D}^{\varepsilon,h} \cap \Omega_f^{\varepsilon}) \cup \Omega_m^{\varepsilon}$, we rewrite the first term in (5.7) as follows:

$$\begin{aligned}\int_{\Omega} \varkappa_{\varepsilon}(x)|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x &= \int_{\Omega_f^{\varepsilon}\setminus\mathcal{D}^{\varepsilon,h}} \varkappa_{\varepsilon}(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x + \int_{\mathcal{D}^{\varepsilon,h}\cap\Omega_f^{\varepsilon}} \varkappa_{\varepsilon}(x)|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x \\ &\quad + \int_{\Omega_m^{\varepsilon}} \varkappa_{\varepsilon}(x)|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x. \end{aligned} \tag{5.11}$$

Because of inequality (4.15), the first term on the right-hand side of (5.11) is such that

$$\overline{\lim_{h\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\int_{\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla V^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x \leqslant \int_\Omega A(x,\nabla w)\,\mathrm{d}x. \tag{5.12}$$

Consider the second term on the right-hand side of (5.11). Using

$$|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)} = |V^{\varepsilon,h}\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)} + (|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)} - |V^{\varepsilon,h}\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}),$$

we rewrite it as follows:

$$\begin{aligned}\int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x &= \int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}(|V^{\varepsilon,h}|^{p_\varepsilon(x)} - |w(x)|^{p_\varepsilon(x)})\,\mathrm{d}x \\ &\quad + \int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)(|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)} - |V^{\varepsilon,h}\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)})\,\mathrm{d}x \\ &\quad + \int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|w(x)\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x. \end{aligned} \tag{5.13}$$

We now study the first term on the right-hand side of (5.13). It follows from lemma 4.1(iii), lemma 4.2(ii), Hölder's inequality and (3.2) that

$$\overline{\lim_{h\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)|\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}||V^{\varepsilon,h}|^{p_\varepsilon(x)} - |w(x)|^{p_\varepsilon(x)}|\,\mathrm{d}x = 0. \tag{5.14}$$

In a similar way, for the second term on the right-hand side of (5.13), from lemmas 4.1, and 4.2(ii), Hölder's inequality and (3.2) we obtain

$$\overline{\lim_{h\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\int_{\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)||\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)} - |V^{\varepsilon,h}\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}|\,\mathrm{d}x = 0. \tag{5.15}$$

Using the same decomposition of $|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)}$ as in (5.13), we write the third term on the right-hand side of (5.11) as follows:

$$\begin{aligned}\int_{\Omega_m^\varepsilon} \varkappa_\varepsilon(x)|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x &= \int_{\Omega_m^\varepsilon} \varkappa_\varepsilon(x)|\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}(|V^{\varepsilon,h}|^{p_\varepsilon(x)} - |w(x)|^{p_\varepsilon(x)})\,\mathrm{d}x \\ &\quad + \int_{\Omega_m^\varepsilon} \varkappa_\varepsilon(x)(|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)} - |V^{\varepsilon,h}\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)})\,\mathrm{d}x \\ &\quad + \int_{\Omega_m^\varepsilon} \varkappa_\varepsilon(x)|w(x)\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x. \end{aligned} \tag{5.16}$$

Using similar arguments to the ones used in the proof of bounds (5.14) and (5.15), we obtain

$$\overline{\lim_{h\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\int_{\Omega_m^\varepsilon} \varkappa_\varepsilon(x)|\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}||V^{\varepsilon,h}|^{p_\varepsilon(x)} - |w(x)|^{p_\varepsilon(x)}|\,\mathrm{d}x = 0, \tag{5.17}$$

$$\overline{\lim_{h\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\int_{\Omega_m^\varepsilon} \varkappa_\varepsilon(x)||\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)} - |V^{\varepsilon,h}\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}|\,\mathrm{d}x = 0. \tag{5.18}$$

Finally, it follows from (5.11)–(5.18) that the first term in (5.7) satisfies

$$\int_\Omega \varkappa_\varepsilon(x)|\nabla T^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x \leqslant \int_\Omega A(x,\nabla w)\,\mathrm{d}x + \int_\Omega \varkappa_\varepsilon(x)|w(x)\nabla Y^{\varepsilon,h}|^{p_\varepsilon(x)}\,\mathrm{d}x + \boldsymbol{j}_2^{\varepsilon,h}, \tag{5.19}$$

where $\overline{\lim}_{h\to 0}\,\overline{\lim}_{\varepsilon\to 0}\,|\boldsymbol{j}_2^{\varepsilon,h}| = 0$.

Now inequality (5.6) immediately follows from (5.9), (5.10), (5.19) and assertion (iii) of lemma 4.1. Since u^ε minimizes the functional J^ε, it follows from (5.6) that

$$\overline{\lim_{\varepsilon\to 0}}\, J^\varepsilon[u^\varepsilon] \leqslant J_{\mathrm{hom}}[w] \tag{5.20}$$

for any smooth function w. By density arguments, (5.20) holds for any function $w \in W_0^{1,p_0(\cdot)}(\Omega)$ as well.

5.2. Step 2: lower bound

The derivation of the lower bound

$$\underline{\lim}\, J^\varepsilon[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u]$$

is done in two main steps. In the first step we introduce an auxiliary functional J_π^ε and obtain the lower bound for this functional. In the second step we obtain the desired result for the initial functional J^ε.

Step 1 (an auxiliary inequality). Let $\{\pi_\varepsilon^\star\}_{(\varepsilon>0)}$ be the sequence of functions defined in (4.37). On the space $W^{1,\pi_\varepsilon^\star(\cdot)}(\Omega^\varepsilon)$ we define the functional

$$J^{\pi_\varepsilon^\star} : W^{1,\pi_\varepsilon^\star(\cdot)}(\Omega^\varepsilon) \to \mathbb{R}\cup\{+\infty\}$$

by setting

$$J^{\pi_\varepsilon^\star}[u] \overset{\mathrm{def}}{=} \begin{cases} \displaystyle\int_\Omega \Big(\varkappa_\varepsilon(x)|\nabla u|^{\pi_\varepsilon^\star(x)} + \frac{1}{\sigma(x)}|u|^{\sigma(x)} - g^\varepsilon(x)u\Big)\,\mathrm{d}x & \text{if } u \in W^{1,\pi_\varepsilon^\star(\cdot)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.21}$$

The functional $J^{\pi_\varepsilon^\star}$ is continuous in $W^{1,\pi_\varepsilon^\star(\cdot)}(\Omega^\varepsilon)$ and the following inequality holds:

$$|J^{\pi_\varepsilon^\star}[u] - J^{\pi_\varepsilon^\star}[v]| \leqslant CL_2\|u-v\|_{W^{1,\pi_\varepsilon^\star(\cdot)}(\Omega)}$$

for any $(u,v) \in (W^{1,p_0(\cdot)}(\Omega))^2$, where

$$L_2 = \max(\Upsilon^{1/q_0^-}_{p_0(\cdot),\Omega}(1+|u|+|\nabla u|+|v|+|\nabla v|), \Upsilon^{1/q_0^+}_{p_0(\cdot),\Omega}(1+|u|+|\nabla u|+|v|+|\nabla v|)),$$

with the exponent $q_0 = q_0(x)$ and the values $q_0^\pm$ being defined after (3.20).

Now let u be an arbitrary $C_0^1(\Omega)$ function and let $\{u^\varepsilon\}_{(\varepsilon>0)}$ be a sequence which converges to the function u strongly in $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ and such that $J^{\pi_\varepsilon^\star}[u^\varepsilon] \leqslant C$. We will show that

$$\underline{\lim}_{\varepsilon\to 0}\, J^{\pi_\varepsilon^\star}[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u]. \tag{5.22}$$

We consider a new set of points $\{x^\alpha\}$ in the domain Ω that form a h-periodic space lattice. Let us cover the domain Ω by the cubes K_h^α with non-intersecting interiors. We introduce the following notation:

$$\Omega_h = \Big\{ \bigcup_\alpha K_h^\alpha,\ K_h^\alpha \subset \Omega \Big\}, \qquad \tilde{\Omega}_h = \Omega \setminus \Omega_h,$$
$$\Omega_h^\varepsilon = \Omega^\varepsilon \cap \Omega_h, \qquad \tilde{\Omega}_h^\varepsilon = \Omega^\varepsilon \cap \tilde{\Omega}_h.$$

By construction,

$$\operatorname{meas} \tilde{\Omega}_h = O(h) \quad \text{as } h \to 0. \tag{5.23}$$

The functional $J^{\pi_\varepsilon^\star}[u^\varepsilon]$ is then decomposed as follows.

$$J^{\pi_\varepsilon^\star}[u^\varepsilon] = \int_{\tilde{\Omega}_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon^\star}(x, u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x + \int_{\Omega_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon^\star}(x, u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x, \tag{5.24}$$

where

$$\mathsf{F}_{\pi_\varepsilon^\star}(x, u, \nabla u) = \varkappa_\varepsilon(x)|\nabla u|^{\pi_\varepsilon^\star(x)} + \frac{1}{\sigma(x)}|u|^{\sigma(x)} - g^\varepsilon(x)u. \tag{5.25}$$

Consider the first term on the right-hand side of (5.24). It follows from the definition of g^ε, from the strong convergence of the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ to $u \in C_0^1(\Omega)$ in the space $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ and from (5.23) that

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \int_{\tilde{\Omega}_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon^\star}(x, u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x \geqslant 0. \tag{5.26}$$

Consider now the second term on the right-hand side of (5.24). We have

$$\int_{\Omega_h^\varepsilon} \mathsf{F}_{\pi_\varepsilon^\star}(x, u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x = \sum_{K_h^\alpha \subset \Omega} \int_{K_h^\alpha \cap \Omega^\varepsilon} \Big(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)} + \frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} - g^\varepsilon(x)u^\varepsilon \Big)\,\mathrm{d}x. \tag{5.27}$$

For any α such that $K_h^\alpha \subset \Omega$, the first term on the right-hand side of (5.27) is

$$\int_{K_h^\alpha \cap \Omega^\varepsilon} \mathsf{F}_{\pi_\varepsilon^\star}(x, u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x = \int_{K_h^\alpha \cap \Omega_f^\varepsilon} \mathsf{F}_{\pi_\varepsilon^\star}(x, u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x + \int_{K_h^\alpha \cap \Omega_m^\varepsilon} \mathsf{F}_{\pi_\varepsilon^\star}(x, u^\varepsilon, \nabla u^\varepsilon)\,\mathrm{d}x. \tag{5.28}$$

We now apply lemma 4.3 to the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ and the function u. So, there exist an open set $\mathcal{G}^\varepsilon \subset \Omega$, such that $\Omega_m^\varepsilon \subset \mathcal{G}^\varepsilon$ and a function $\hat{u}^\varepsilon$ that satisfy lemma 4.3(i)–(iii). We define the function v^ε in Ω by

$$v^\varepsilon = \hat{u}^\varepsilon - u(x^\alpha). \tag{5.29}$$

We aim to go back to the functional $c^{\varepsilon,h}_{\pi^\star_\varepsilon(\cdot)}$. Bearing in mind condition (A3), we note that, as $\varepsilon \to 0$, we have

$$\begin{aligned}
&\int_{K_h^\alpha \cap \Omega_f^\varepsilon} \varkappa_\varepsilon(x)|\nabla v^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x \\
&\qquad = \int_{K_h^\alpha \cap \Omega_f^\varepsilon} \frac{K_\varepsilon(x)}{\pi^\star_\varepsilon(x)}|\nabla v^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x + o(1) \\
&\qquad = \int_{K_h^\alpha \cap \Omega_f^\varepsilon} \left(\frac{K_\varepsilon(x)}{\pi^\star_\varepsilon(x)}|\nabla v^\varepsilon|^{\pi^\star_\varepsilon(x)} + h^{-\gamma-\pi^\star_\varepsilon(x)}|v^\varepsilon - (x - x^\alpha, \vec{a})|^{\pi^\star_\varepsilon(x)}\right)\mathrm{d}x \\
&\qquad\quad - h^{-\gamma}\int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\pi^\star_\varepsilon(x)}|v^\varepsilon - (x - x^\alpha, \vec{a})|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x + o(1),
\end{aligned} \tag{5.30}$$

where the parameter $\vec{a}$ will be specified later. We now study the second term on the right-hand side of (5.30). It follows from the regularity of the function u and from assumptions (A1) and (A3) that, for any $\vec{a} \in \mathbb{R}^n$ and any $\varepsilon > 0$, we have

$$\begin{aligned}
&\int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\pi^\star_\varepsilon(x)}|v^\varepsilon - (x - x^\alpha, \vec{a})|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x \\
&\qquad = \int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\pi^\star_\varepsilon(x)}|(\hat{u}^\varepsilon(x) - u(x)) \\
&\qquad\qquad + (u(x) - u(x^\alpha) - (x - x^\alpha, \nabla u(x^\alpha))) + (x - x^\alpha, \nabla u(x^\alpha) - \vec{a})|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x.
\end{aligned}$$

Obviously, for $h \to 0$ we have

$$\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\pi^\star_\varepsilon(x)}|u(x) - u(x^\alpha) - (x - x^\alpha, \nabla u(x^\alpha))|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x = O(h^{n+p^-}). \tag{5.31}$$

Now it follows from (5.31) that

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\pi^\star_\varepsilon(x)}|v^\varepsilon - (x - x^\alpha, \vec{a})|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x \\
&\qquad \leqslant C\,\overline{\lim_{\varepsilon\to 0}} \left(\int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\pi^\star_\varepsilon(x)}|\hat{u}^\varepsilon(x) - u(x)|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x\right. \\
&\qquad\qquad \left. + \int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\pi^\star_\varepsilon(x)}|(x - x^\alpha, \nabla u(x^\alpha) - \vec{a})|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x\right) + O(h^{n+p^-})
\end{aligned} \tag{5.32}$$

as $h \to 0$. We set $\vec{a} = \vec{a}_\alpha = \nabla u(x^\alpha)$. It follows from the strong convergence of the sequence $\{u^\varepsilon\}$ to u in the space $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ and (5.32) that

$$\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha \cap \Omega_f^\varepsilon} h^{-\gamma-\pi^\star_\varepsilon(x)}|v^\varepsilon - (x - z, \nabla u(x^\alpha))|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x = O(h^{n+p^- -\gamma}) \quad \text{as } h \to 0. \tag{5.33}$$

Now definition (2.10) and relations (5.30)–(5.33) give

$$\underline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha \cap \Omega_f^\varepsilon} \varkappa_\varepsilon(x)|\nabla v^\varepsilon|^{\pi^\star_\varepsilon(x)}\,\mathrm{d}x \geqslant \underline{\lim_{\varepsilon\to 0}}\, c^{\varepsilon,h}_{\pi^\star_\varepsilon(\cdot)}(x^\alpha, \nabla u(x^\alpha)) - O(h^{n+p^- -\gamma}). \tag{5.34}$$

Due to the definition of the function v^ε, relation (5.34) means that

$$\begin{aligned}\varliminf_{\varepsilon\to 0}\sum_\alpha\int_{K_h^\alpha\cap(\Omega_f^\varepsilon\setminus\mathcal{G}_f^\varepsilon)}&\varkappa_\varepsilon(x)|\nabla\hat{u}^\varepsilon|^{\pi_\varepsilon^\star(x)}\,\mathrm{d}x\\ &\geqslant\varliminf_{\varepsilon\to 0}\sum_\alpha c^{\varepsilon,h}_{\pi_\varepsilon^\star(\cdot)}(x^\alpha,\nabla u(x^\alpha))\\ &\qquad-\varlimsup_{\varepsilon\to 0}k_0^{-1}\int_{\mathcal{G}_f^\varepsilon}|\nabla\hat{u}^\varepsilon|^{\pi_\varepsilon^\star(x)}\,\mathrm{d}x+o(1)\quad\text{as } h\to 0.\end{aligned}\tag{5.35}$$

Moreover, due to (4.39), the second term on the right-hand side of (5.35) equals zero. Finally, we go back to the functional $J^{\pi_\varepsilon^\star}$. Since $\hat{u}^\varepsilon=u^\varepsilon$ in $\Omega^\varepsilon\setminus\mathcal{G}^\varepsilon$, we obtain from (5.35) that

$$\begin{aligned}\varliminf_{\varepsilon\to 0}J^{\pi_\varepsilon^\star}[u^\varepsilon]\geqslant\varliminf_{\varepsilon\to 0}\Bigg(&\sum_\alpha c^{\varepsilon,h}_{\pi_\varepsilon^\star(\cdot)}(x^\alpha,\nabla u(x^\alpha))\\ &+\sum_\alpha\int_{K_h^\alpha\cap\Omega_f^\varepsilon}\left(\frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)}-g^\varepsilon(x)u^\varepsilon\right)\mathrm{d}x\\ &+\int_{\mathcal{G}^\varepsilon}\mathsf{F}_{\pi_\varepsilon^\star}(x,u^\varepsilon,\nabla u^\varepsilon)\,\mathrm{d}x\Bigg)+o(1)\end{aligned}\tag{5.36}$$

as $h\to 0$. We pass to the limit in the inequality (5.36) first as $\varepsilon\to 0$ and then as $h\to 0$. Taking into account the strong convergence of the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ to u in the space $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$, the regularity of the function g given in (2.8), the properties of the function p_ε, conditions (C1), (C3) and lemma 4.3, we obtain the desired inequality (5.22).

It remains to pass from the result in $C_0^1(\Omega)$ to the result in $W^{1,p_0(\cdot)}(\Omega)$. Function $\pi_\varepsilon^\star$ satisfies $\pi_\varepsilon^\star\leqslant p_0$ in Ω. Therefore, the family $\{J^{\pi_\varepsilon^\star}\}$ is (uniformly in ε) continuous in the $W^{1,p_0(\cdot)}(\Omega)$ topology. In addition, as proved in lemma 3.2, the functional J_{hom} is continuous in the $W^{1,p_0(\cdot)}(\Omega)$ topology. Then the fact that inequality (5.22) holds for any $u\in C_0^1(\Omega)$ implies that (5.22) holds for any $u\in W_0^{1,p_0(\cdot)}(\Omega)$. This completes the proof of the 'lim inf' inequality for the functional $J^{\pi_\varepsilon^\star}$.

STEP 2 (lower bound for the original functional). Let u be an arbitrary function from $L^{p_0(\cdot)}(\Omega)$ and let $\{u^\varepsilon\}$ be a sequence which converges to the function u strongly in $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ and such that $J^\varepsilon[u^\varepsilon]\leqslant C$. First we note that one can prove the inequality

$$\varliminf_{\varepsilon\to 0}J^\varepsilon[u^\varepsilon]\geqslant J_{\mathrm{hom}}[u]\quad\text{for all } u\in C_0^1(\Omega)\tag{5.37}$$

in the same way as the inequality (5.22). Note that, in contrast with $J^{\pi_\varepsilon^\star}$, the functional J^ε is not continuous in the $W^{1,p_0(\cdot)}$ topology unless we restrict ourself to the case when $p_\varepsilon\leqslant p_0$. Therefore, the fact that (5.37) holds for any C_0^1-function does not imply that it is true for any $u\in W_0^{1,p_0(\cdot)}(\Omega)$. So, let $u\in W^{1,p_0(\cdot)}(\Omega)$. Consider the value

$$I^\varepsilon[u^\varepsilon]=\int_\Omega\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)}(|\nabla u^\varepsilon|^{p_\varepsilon(x)-\pi_\varepsilon^\star(x)}-1)\,\mathrm{d}x.\tag{5.38}$$

We check easily that

$$\max_{0<B<1}(-B^{\pi_\varepsilon^\star(x)}(B^{p_\varepsilon(x)-\pi_\varepsilon^\star(x)}-1)) \geqslant C(\varepsilon) \quad \text{with } C(\varepsilon)\to 0 \text{ as } \varepsilon\to 0 \tag{5.39}$$

and

$$\begin{aligned}\varliminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] &\geqslant \varliminf_{\varepsilon\to 0} I^\varepsilon[u^\varepsilon] + \varliminf_{\varepsilon\to 0} J^{\pi_\varepsilon^\star}[u^\varepsilon]\\ &\geqslant \varliminf_{\varepsilon\to 0}\int_{\{|\nabla u^\varepsilon|<1\}\cap\Omega} \varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{\pi_\varepsilon^\star(x)}(|\nabla u^\varepsilon|^{p_\varepsilon(x)-\pi_\varepsilon^\star(x)}-1)\,\mathrm{d}x + J_{\mathrm{hom}}[u].\end{aligned} \tag{5.40}$$

Combining (5.39) and (5.40), we assert that

$$\varliminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u] \quad \text{for all } u\in W^{1,p_0(\cdot)}(\Omega). \tag{5.41}$$

If u is an arbitrary function from $W_0^{1,p_0(\cdot)}(\Omega)$ and $\{u^\varepsilon\}$ is a sequence converging to the function u strongly in $L^{p_0(\cdot)}(\Omega)$, then inequalities (5.37) and (5.41) mean that

$$\varliminf_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u] \tag{5.42}$$

and the lower bound is obtained.

5.3. Step 3: convergence result (2.17) in the matrix part

It remains to prove the convergence result (2.17). Suppose that the solution u of the homogenized problem is a sufficiently smooth function (if not, we use smooth approximations of u to construct $\tilde u^\varepsilon$). Let $\tilde u^\varepsilon$ be the function defined in (5.5) with $w=u$. Then it follows from (5.20) and (5.42) that

$$\lim_{\varepsilon\to 0} J^\varepsilon[\tilde u^\varepsilon] = \lim_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] = J_{\mathrm{hom}}[u]. \tag{5.43}$$

It follows that $\lim_{\varepsilon\to 0} J^\varepsilon[\tilde u^\varepsilon] - \lim_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon]=0$, that is

$$\begin{aligned}\lim_{\varepsilon\to 0}\Bigg(\int_\Omega\left(\varkappa_\varepsilon(x)|\nabla\tilde u^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|\tilde u^\varepsilon|^{\sigma(x)} + g^\varepsilon\tilde u^\varepsilon\right)\mathrm{d}x\\ - \int_\Omega\left(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} + g^\varepsilon u^\varepsilon\right)\mathrm{d}x\Bigg) = 0.\end{aligned} \tag{5.44}$$

We aim to deduce from the latter relation that

$$\lim_{\varepsilon\to 0}\|\tilde u^\varepsilon - u^\varepsilon\|_{L^{\sigma(\cdot)}(\Omega)} = 0. \tag{5.45}$$

To this end we make use of the following lemma.

Lemma 5.1.

(i) *Let $\mathsf{p}=\mathsf{p}(x)$ be a continuous function such that $1\leqslant \mathsf{p}^-\leqslant \mathsf{p}(x)\leqslant \mathsf{p}^+$ in $\bar\Omega$. Then, for any vectors $\xi_1,\xi_2\in\mathbb{R}^n$, we have*

$$\frac{1}{\mathsf{p}(\cdot)}|\xi_2|^{\mathsf{p}(\cdot)} - \frac{1}{\mathsf{p}(\cdot)}|\xi_1|^{\mathsf{p}(\cdot)} \geqslant (\xi_1|\xi_1|^{\mathsf{p}(\cdot)-2}, \xi_2-\xi_1). \tag{5.46}$$

(ii) *Let* $\mathsf{p} = \mathsf{p}(x)$ *be a continuous function such that* $2 \leqslant \mathsf{p}^- \leqslant \mathsf{p}(x) \leqslant \mathsf{p}^+$ *in* $\bar{\Omega}$. *Then, for any vectors* $\xi_1, \xi_2 \in \mathbb{R}^n$, *there exists a strictly positive constant* $L = L(\mathsf{p}^-)$ *such that*

$$\frac{1}{\mathsf{p}(\cdot)}|\xi_2|^{\mathsf{p}(\cdot)} - \frac{1}{\mathsf{p}(\cdot)}|\xi_1|^{\mathsf{p}(\cdot)} \geqslant (\xi_1|\xi_1|^{\mathsf{p}(\cdot)-2}, \xi_2 - \xi_1) + L|\xi_2 - \xi_1|^{\mathsf{p}(\cdot)}. \tag{5.47}$$

(iii) *Let* $\mathsf{p} = \mathsf{p}(x)$ *be a continuous function such that* $1 \leqslant \mathsf{p}^- \leqslant \mathsf{p}(x) \leqslant \mathsf{p}^+ < 2$ *in* $\bar{\Omega}$. *Then, for any vectors* $\xi_1, \xi_2 \in \mathbb{R}^n$, *there exists a strictly positive constant* $L = L(\mathsf{p}^-)$ *such that*

$$\frac{1}{\mathsf{p}(\cdot)}|\xi_2|^{\mathsf{p}(\cdot)} - \frac{1}{\mathsf{p}(\cdot)}|\xi_1|^{\mathsf{p}(\cdot)} \geqslant (\xi_1|\xi_1|^{\mathsf{p}(\cdot)-2}, \xi_2 - \xi_1) + L\frac{|\xi_2 - \xi_1|^2}{(|\xi_1| + |\xi_2|)^{2-\mathsf{p}(\cdot)}}. \tag{5.48}$$

The proof of lemma 5.1 is given in the appendix.

Now we prove the relation (5.45). In view of lemma 5.1 we should consider four subsets of the domain Ω:

$$\Omega_+^{p_\varepsilon} \stackrel{\text{def}}{=} \Omega \cap \{x : p_\varepsilon(x) \geqslant 2\}, \qquad \Omega_-^{p_\varepsilon} \stackrel{\text{def}}{=} \Omega \setminus \overline{\Omega_+^{p_\varepsilon}},$$

$$\Omega_+^{\sigma} \stackrel{\text{def}}{=} \Omega \cap \{x : \sigma(x) \geqslant 2\}, \qquad \Omega_-^{\sigma} \stackrel{\text{def}}{=} \Omega \setminus \overline{\Omega_+^{\sigma}}.$$

However, for the sake of simplicity we assume here that $p_\varepsilon, \sigma \geqslant 2$ in Ω. Then we apply (5.47) for $\mathsf{p} = p^\varepsilon$, $\xi_1 = \nabla u^\varepsilon$ and $\xi_2 = \nabla \tilde{u}^\varepsilon$ and we obtain

$$L\int_\Omega \varkappa_\varepsilon(x)|\nabla u^\varepsilon - \nabla \tilde{u}^\varepsilon|^{p_\varepsilon(x)} \leqslant \int_\Omega \varkappa_\varepsilon(x)|\nabla \tilde{u}^\varepsilon|^{p_\varepsilon(x)} - \int_\Omega \varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)} - \int_\Omega K_\varepsilon(x)(|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\nabla u^\varepsilon, \nabla \tilde{u}^\varepsilon - \nabla u^\varepsilon). \tag{5.49}$$

In a similar way, with $\mathsf{p} = \sigma$ we obtain the following inequality:

$$L_1\int_\Omega \frac{1}{\sigma(x)}|u^\varepsilon - \tilde{u}^\varepsilon|^{\sigma(x)} \leqslant \int_\Omega \frac{1}{\sigma(x)}|\tilde{u}^\varepsilon|^{\sigma(x)} - \int_\Omega \frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)} - \int_\Omega |u^\varepsilon|^{\sigma(x)-2}u^\varepsilon(\tilde{u}^\varepsilon - u^\varepsilon) \tag{5.50}$$

Using (5.49) and (5.50) we obtain

$$\begin{aligned} \min\{L, L_1\} &\int_\Omega (\varkappa_\varepsilon(x)|\nabla u^\varepsilon - \nabla \tilde{u}^\varepsilon|^{p_\varepsilon(x)} + |u^\varepsilon - \tilde{u}^\varepsilon|^{\sigma(x)})\,\mathrm{d}x \\ &\leqslant \int_\Omega \left(\varkappa_\varepsilon(x)|\nabla \tilde{u}^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|\tilde{u}^\varepsilon|^{\sigma(x)}\right)\mathrm{d}x \\ &\quad - \int_\Omega \left(\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)}\right)\mathrm{d}x \\ &\quad - \int_\Omega (K_\varepsilon(x)(|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\nabla u^\varepsilon, \nabla \tilde{u}^\varepsilon - \nabla u^\varepsilon) + |u^\varepsilon|^{\sigma(x)-2}u^\varepsilon(\tilde{u}^\varepsilon - u^\varepsilon))\,\mathrm{d}x. \end{aligned} \tag{5.51}$$

Consider the third term on the right-hand side of (5.51). Since $u^\varepsilon = \tilde u^\varepsilon = 0$ on $\partial\Omega$, then we have

$$\int_\Omega (K_\varepsilon(x)(|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\nabla u^\varepsilon, \nabla\tilde u^\varepsilon - \nabla u^\varepsilon) + |u^\varepsilon|^{\sigma(x)-2}u^\varepsilon(\tilde u^\varepsilon - u^\varepsilon))\,\mathrm{d}x$$
$$= \int_\Omega (-\operatorname{div}(K_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\nabla u^\varepsilon) + |u^\varepsilon|^{\sigma(x)-2}u^\varepsilon)(\tilde u^\varepsilon - u^\varepsilon)\,\mathrm{d}x. \tag{5.52}$$

Since u^ε is the solution of variational problem (2.9), the first term on the right-hand side of (5.52) is such that

$$\int_\Omega (-\operatorname{div}(K_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\nabla u^\varepsilon)+|u^\varepsilon|^{\sigma(x)-2}u^\varepsilon)(\tilde u^\varepsilon - u^\varepsilon)\,\mathrm{d}x = \int_\Omega g^\varepsilon(x)(\tilde u^\varepsilon - u^\varepsilon)\,\mathrm{d}x. \tag{5.53}$$

Finally, from (5.44) and (5.51)–(5.53) we obtain the desired relation (5.45).

Consider now the functional $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta)$ defined in (2.11). It is clear that the minimizer $w^{\varepsilon,h}_z$ of the functional (2.11) satisfies the Neumann boundary-value problem for the following equation:

$$-\operatorname{div}(K_\varepsilon(x)\nabla w^{\varepsilon,h}_z|\nabla w^{\varepsilon,h}_z|^{p_\varepsilon(x)-2}) + \mathbf{1}^\varepsilon_m(x)w^{\varepsilon,h}_z|w^{\varepsilon,h}_z|^{\sigma(x)-2}$$
$$+ p_\varepsilon(x)h^{-p_\varepsilon(x)-\gamma}\mathbf{1}^\varepsilon_f(x)(w^\varepsilon - \beta)|w^\varepsilon - \beta|^{p_\varepsilon(x)-2} = 0 \quad \text{in } K^z_h. \tag{5.54}$$

Let us multiply equation (5.54) by $(w^\varepsilon - \beta)$ and integrate over the cube K^z_h. We obtain

$$\int_{K^z_h} (K_\varepsilon(x)|\nabla w^{\varepsilon,h}_z|^{p_\varepsilon(x)} + \mathbf{1}^\varepsilon_m(x)|w^{\varepsilon,h}_z|^{\sigma(x)}$$
$$+ p_\varepsilon(x)h^{-p_\varepsilon(x)-\gamma}\mathbf{1}^\varepsilon_f(x)|w^\varepsilon - \beta|^{p_\varepsilon(x)})\,\mathrm{d}x$$
$$= \beta\int_{K^z_h} \mathbf{1}^\varepsilon_m(x)w^{\varepsilon,h}_z|w^{\varepsilon,h}_z|^{\sigma(x)-2}\,\mathrm{d}x. \tag{5.55}$$

Now we represent the left-hand side of (5.55) in terms of the local energy characteristic $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta)$. Using condition (A3), we write

$$\int_{K^z_h} (K_\varepsilon(x)|\nabla w^{\varepsilon,h}_z|^{p_\varepsilon(x)} + \mathbf{1}^\varepsilon_m(x)|w^{\varepsilon,h}_z|^{\sigma(x)} + p_\varepsilon(x)h^{-p_\varepsilon(x)-\gamma}\mathbf{1}^\varepsilon_f(x)|w^\varepsilon - \beta|^{p_\varepsilon(x)})\,\mathrm{d}x$$
$$= \int_{K^z_h} p_\varepsilon(x)\biggl(\varkappa_\varepsilon(x)|\nabla w^{\varepsilon,h}_z|^{p_\varepsilon(x)} + \frac{\mathbf{1}^\varepsilon_m(x)}{\sigma(x)}|w^{\varepsilon,h}_z|^{\sigma(x)}$$
$$+ h^{-p_\varepsilon(x)-\gamma}\mathbf{1}^\varepsilon_f(x)|w^\varepsilon - \beta|^{p_\varepsilon(x)}\biggr)\,\mathrm{d}x$$
$$+ \int_{K^z_h} p_\varepsilon(x)\biggl(\frac{1}{p_\varepsilon(x)}|w^{\varepsilon,h}_z|^{\sigma(x)} - \frac{1}{\sigma(x)}|w^{\varepsilon,h}_z|^{\sigma(x)}\biggr)\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x$$
$$= p_0(z)b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta) + \int_{K^z_h} p_\varepsilon(x)|w^{\varepsilon,h}_z|^{\sigma(x)}\biggl(\frac{1}{p_\varepsilon(x)} - \frac{1}{\sigma(x)}\biggr)\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x + o(h^n) \tag{5.56}$$

as $h \to 0$. Therefore, it follows from (5.55) and (5.56) that

$$p_0(z) b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta) = \int_{K^z_h} |w^{\varepsilon,h}_z|^{\sigma(x)-2}(\beta w^{\varepsilon,h}_z - |w^{\varepsilon,h}_z|^2)\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x + \int_{K^z_h} \frac{p_\varepsilon(x)}{\sigma(x)} |w^{\varepsilon,h}_z|^{\sigma(x)}\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x + o(h^n) \quad \text{as } h \to 0$$

or

$$b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta) = \int_{K^z_h} \frac{1}{p_\varepsilon(x)} |w^{\varepsilon,h}_z|^{\sigma(x)-2}(\beta w^{\varepsilon,h}_z - |w^{\varepsilon,h}_z|^2)\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x + \int_{K^z_h} \frac{1}{\sigma(x)} |w^{\varepsilon,h}_z|^{\sigma(x)}\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x + o(h^n) \quad \text{as } h \to 0. \tag{5.57}$$

Then it follows from condition (C4) of theorem 2.5 that

$$\mathsf{b}(x,\beta) = \lim_{\varepsilon\to 0} h^{-n}(\varepsilon)\biggl(\int_{K^z_h} \frac{1}{p_\varepsilon(x)} |w^{\varepsilon,h}_z|^{\sigma(x)-2}(\beta w^{\varepsilon,h}_z - |w^{\varepsilon,h}_z|^2)\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x + \int_{K^z_h} \frac{1}{\sigma(x)} |w^{\varepsilon,h}_z|^{\sigma(x)}\mathbf{1}^\varepsilon_m(x)\,\mathrm{d}x\biggr). \tag{5.58}$$

Now let ζ be a smooth function in Ω. Consider the quantity:

$$\mathcal{J}^\varepsilon[\tilde{u}^\varepsilon] \overset{\mathrm{def}}{=} \int_{\Omega^\varepsilon_m} \biggl(\frac{1}{p_\varepsilon(x)} |\tilde{u}^\varepsilon|^{\sigma(x)-2}(u\tilde{u}^\varepsilon - |\tilde{u}^\varepsilon|^2) + \frac{1}{\sigma(x)} |\tilde{u}^\varepsilon|^{\sigma(x)}\biggr)\zeta(x)\,\mathrm{d}x.$$

It follows from (5.58) that

$$\lim_{\varepsilon\to 0} \mathcal{J}^\varepsilon[\tilde{u}^\varepsilon] = \int_\Omega \mathsf{b}(x,u)\zeta(x)\,\mathrm{d}x. \tag{5.59}$$

Now the desired relation (2.17) follows from (5.45) and (5.59). This completes the proof of theorem 2.5.

6. Periodic example

Theorem 2.5 provides sufficient conditions for the existence of the homogenized functional (2.15) and for the convergence of minimizers of the variational problem (2.9) to the minimizer of the homogenized variational problem (2.14) under conditions (A1)–(A5), (K1), (K2) and (C1)–(C4). It is important to show that the 'intersection' of these conditions is not empty. The aim of this section is to prove that all the conditions of the above-mentioned theorem are satisfied for periodic media, and to compute the coefficients of the homogenized functional (2.15) either in an explicit form or, as is usually the case, by the solution of a corresponding cell problem.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz boundary. We assume that, in the standard periodic cell $\mathcal{Y} = (0,1)^n$, there is an obstacle $\mathcal{M} \subset \mathcal{Y}$ with Lipschitz boundary $\partial\mathcal{M}$ (see figure 2). We assume that this geometry is repeated periodically in the whole $\mathbb{R}^n$. The geometric structure within the domain Ω is then

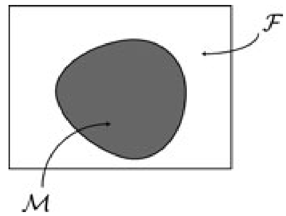

Figure 2. The reference cell $\mathcal{Y}$.

obtained by intersecting the ε-multiple of this geometry with Ω, ε being a small positive parameter. Let $\{x^{i,\varepsilon}\}$ be an ε-periodic grid in Ω. Then we define Ω_m^ε as the union of sets $\mathcal{M}_i^\varepsilon \subset K_\varepsilon^i$, $i = 1, 2, \ldots, N_\varepsilon$, obtained from $\varepsilon\mathcal{M}$ by translations of vectors

$$\varepsilon \sum_{j=1}^{n} k_j \boldsymbol{e}_j,$$

that is,

$$\Omega_m^\varepsilon = \bigcup_{i}^{N_\varepsilon} \mathcal{M}_i^\varepsilon \quad \text{and} \quad \Omega_f^\varepsilon = \Omega \setminus \overline{\Omega_m^\varepsilon}, \tag{6.1}$$

where K_ε^i is the cube centred at the point $x^{i,\varepsilon}$ and of length ε, $k_j \in \mathbb{Z}, \{\boldsymbol{e}_j\}_{j=1}^n$ is the canonical basis of $\mathbb{R}^n$ and $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

Let $p_0 = p_0(x)$ be a *log-Hölder continuous* function such that

$$2 < \mathsf{p}^- \equiv \min_{x \in \bar{\Omega}} p_0(x) \leqslant p_0(x) \leqslant \max_{x \in \bar{\Omega}} p_0(x) \equiv \mathsf{p}^+ < +\infty \quad \text{in } \bar{\Omega}. \tag{6.2}$$

Let $\{p_\varepsilon\}_{(\varepsilon>0)} \subset \mathfrak{P}_0^\varepsilon$ be a sequence defined by

$$p_\varepsilon(x) \stackrel{\text{def}}{=} p_0(x) + \boldsymbol{d}_\varepsilon(x), \tag{6.3}$$

where the function $\boldsymbol{d}_\varepsilon$ is such that $\boldsymbol{d}_\varepsilon = o(1)$ as $\varepsilon \to 0$. The asymptotic behaviour of $\boldsymbol{d}_\varepsilon$ will be specified in convergence theorems below. On the space $W^{1,p_\varepsilon(\cdot)}(\Omega)$ we define the functional $J^\varepsilon : W^{1,p_\varepsilon(\cdot)}(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

$$J^\varepsilon[u] \stackrel{\text{def}}{=} \begin{cases} \displaystyle\int_\Omega \left(\frac{K_\varepsilon(x)}{p_\varepsilon(x)} |\nabla u|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)} |u|^{\sigma(x)} - g^\varepsilon(x) u \right) \mathrm{d}x & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{6.4}$$

where

$$K^\varepsilon(x) = \begin{cases} k_f & \text{in } \Omega_f^\varepsilon, \\ k_m \varepsilon^{p_0(x)} & \text{in } \Omega_m^\varepsilon, \end{cases} \tag{6.5}$$

the function σ satisfies condition (A4) with $\sigma^- > 2$ and (A5), and the function g^ε is defined in (2.8). Here k_f, k_m are strictly positive constants independent of ε.

Consider the following variational problem:

$$J^\varepsilon[u^\varepsilon] \to \min, \quad u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega). \tag{6.6}$$

We aim to study the asymptotic behaviour of the solution of (6.6): u^ε.

To formulate the main result of this section we introduce some notation. We denote by $u^a = u^a(x,y)$ the unique solution in $W^{1,p_0(\cdot)}_{\#}(\mathcal{F})$ of the following cell problem:

$$\left.\begin{aligned} \operatorname{div}_y(k_f|\nabla_y u^a|^{p_0(x)-2}\nabla_y u^a) = 0 \text{ in } \mathcal{F},\\ (k_f|\nabla_y u^a|^{p_0(x)-2}\nabla_y u^a - \vec{a},\vec{\nu}) = 0 \text{ on } \partial\mathcal{M}, \quad y \to u^a(y)\mathcal{Y}\text{-periodic,} \end{aligned}\right\} \tag{6.7}$$

where $\mathcal{F} = \mathcal{Y}\setminus\overline{\mathcal{M}}$, $\vec{\nu}$ is the outward normal vector to $\partial\mathcal{M}$, and $\vec{a} \in \mathbb{R}^n$. We denote by $w^\beta = w^\beta(x,y)$ the unique solution in $W^{1,p_0(\cdot)}_{\#}(\mathcal{M})$ of the following cell problem:

$$\left.\begin{aligned} -\operatorname{div}_y(k_m\boldsymbol{d}(x)|\nabla_y w^\beta|^{p_0(x)-2}\nabla_y w^\beta) + |w^\beta|^{p_0(x)-2}w^\beta = 0 \text{ in } \mathcal{M},\\ w^\beta(y) = \beta \text{ on } \partial\mathcal{M}, \quad y \to w^\beta(y)\mathcal{Y}\text{-periodic.} \end{aligned}\right\} \tag{6.8}$$

Note that, in the cell problems (6.7) and (6.8), x is a parameter. Regularity results for u^a and w^β are thus easily deduced from [12] and [18]. We also introduce the homogenized functional $J_{\text{hom}} : W^{1,p_0(\cdot)}(\Omega) \to \mathbb{R}\cup\{+\infty\}$:

$$J_{\text{hom}}[u] \stackrel{\text{def}}{=} \begin{cases} \displaystyle\int_\Omega \left(A(x,\nabla u) + \frac{\rho^\star}{\sigma(x)}|u|^{\sigma(x)} + b(x,u) - g(x)\rho^\star u\right)\mathrm{d}x \\ \qquad \text{if } u \in W^{1,p_0(\cdot)}_0(\Omega), \\ +\infty \quad \text{otherwise.} \end{cases} \tag{6.9}$$

The following results hold.

Theorem 6.1. *Let u^ε be a solution of (6.6). Assume that*

$$\lim_{\varepsilon\to 0}\varepsilon^{-\boldsymbol{d}_\varepsilon(\cdot)} = \boldsymbol{d}(\cdot) \tag{6.10}$$

uniformly in Ω. Then u^ε converges strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon_f)$ to u the solution of the variational problem

$$J_{\text{hom}}[u] \to \min, \quad u \in W^{1,p_0(\cdot)}_0(\Omega),$$

where

$$\rho^\star \stackrel{\text{def}}{=} \operatorname{meas}\mathcal{F}, \tag{6.11}$$

$$A(x,\vec{a}) \stackrel{\text{def}}{=} \frac{1}{p_0(x)}\int_{\mathcal{F}}|\nabla_y u^a(x,y) - \vec{a}|^{p_0(x)}\,\mathrm{d}y, \tag{6.12}$$

$$b(x,\beta) \stackrel{\text{def}}{=} \beta\int_{\mathcal{M}} w^\beta(x,y)|w^\beta(x,y)|^{p_0(x)-2}\,\mathrm{d}y. \tag{6.13}$$

Theorem 6.2. *Let u^ε be a solution of (6.6). Assume that, for any $x \in \Omega$,*

$$\lim_{\varepsilon\to 0}\varepsilon^{-\boldsymbol{d}_\varepsilon(x)} = +\infty. \tag{6.14}$$

Then u^ε converges strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon_f)$ to u, the solution of the variational problem

$$J_{\text{hom}}[u] \to \min, \quad u \in W^{1,p_0(\cdot)}_0(\Omega),$$

where $\rho^\star$ and the function $\mathsf{A}(x,\vec{a})$ are given in (6.11), (6.12) and

$$\mathsf{b}(x,\beta) \overset{\text{def}}{=} \frac{\operatorname{meas}\mathcal{M}}{\sigma(x)}|\beta|^{\sigma(x)}. \tag{6.15}$$

THEOREM 6.3. *Let u^ε be a solution of (6.6). Assume that, for any $x \in \Omega$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\boldsymbol{d}_\varepsilon(x)} = 0. \tag{6.16}$$

Then u^ε converges strongly in $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ to u the solution of the variational problem

$$J_{\text{hom}}[u] \to \min, \quad u \in W_0^{1,p_0(\cdot)}(\Omega),$$

where $\rho^\star$ and the function $\mathsf{A}(x,\vec{a})$ are given in (6.11), (6.12) and

$$\mathsf{b}(x,\beta) = 0. \tag{6.17}$$

REMARK 6.4. Note that if

$$K_\varepsilon(x) = \mathbf{1}_f^\varepsilon k_f + \mathbf{1}_m^\varepsilon k_m \varepsilon^{p_\varepsilon(x)},$$

then theorem 6.1 holds true with $\boldsymbol{d}(x) \equiv 1$.

The proofs of theorems 6.1–6.3 are similar (with evident modifications) and we restrict ourselves to the proof of theorem 6.1.

6.1. Proof of theorem 6.1

The proof of theorem 6.1 is made in four steps. In the first step we prove that condition (C1) is satisfied and compute the function $\rho(x)$. In the second step we prove condition (C2), i.e. the extension condition. Then we prove (C3) and compute $\mathsf{A}(x,\vec{a})$. Finally, in the fourth step we compute the function $\mathsf{b}(x,\beta)$.

6.1.1. *Condition (C1): the function $\rho(x)$*

Let K_h^z be an open cube centred at $z \in \Omega$ with length equal to h with $0 < \varepsilon \ll h < 1$. It is easy to check that

$$\operatorname{meas}(K_h^z \cap \Omega_f^\varepsilon) = \frac{h^n}{\varepsilon^n}\operatorname{meas}(\varepsilon\mathcal{F}) + o(h^n) \quad \text{as } h \to 0. \tag{6.18}$$

Then condition (C1) is satisfied and the function $\rho(x) = \rho^\star \in \mathbb{R}$ is given by (6.11).

6.1.2. *Extension condition (C2)*

Let u^ε be an arbitrary function from the space $W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)$. Taking its restriction to any cell $\mathcal{F}_i^\varepsilon = K_\varepsilon^i \setminus \overline{\mathcal{M}_i^\varepsilon}$, we reduce our problem to the proof of the *strong connectedness condition* (C2) for the cube K_ε^i. We map the cell $\mathcal{F}_i^\varepsilon$ on the standard domain $\mathcal{F} = \mathcal{Y} \setminus \overline{\mathcal{M}}$ by considering the function $U^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\mathcal{F})$ defined by

$$U^\varepsilon(\xi) = u^\varepsilon(\varepsilon\xi + x^{i,\varepsilon}).$$

We have

$$\int_{\mathcal{F}_i^\varepsilon} |\nabla u^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x = \varepsilon^n \int_{\mathcal{F}} \varepsilon^{P_\varepsilon(\xi)}|\nabla U^\varepsilon|^{P_\varepsilon(\xi)}\,\mathrm{d}\xi, \tag{6.19}$$

where $P_\varepsilon(\xi) = p_\varepsilon(\varepsilon\xi + x^{i,\varepsilon})$. We denote by $U^\varepsilon_{\mathcal{F}}$ the mean value of the function U^ε in $\mathcal{F}$. Due to assumptions (6.2) and (6.3), there exists an extension $\boldsymbol{E}^\varepsilon$ of the function $E^\varepsilon(\cdot) = U^\varepsilon(\cdot) - U^\varepsilon_{\mathcal{F}}$ in the ball $\mathcal{M}$ such that

$$\|\boldsymbol{E}^\varepsilon\|_{W^{1,P^\star_\varepsilon(\cdot)}(\mathcal{Y})} \leqslant C_1 \|E^\varepsilon\|_{W^{1,P_\varepsilon(\cdot)}(\mathcal{F})},$$

where $P^\star_\varepsilon(\xi) = p^\star_\varepsilon(\varepsilon\xi + x^{i,\varepsilon})$ and the sequence of functions $\{p^\star_\varepsilon\}_{(\varepsilon>0)}$ satisfies assertions (i) and (ii) of condition (C2). Namely, the extension being constructed as usual by reflexion (see, for instance, [1]), we can easily guess that $p^\star_\varepsilon = p_\varepsilon$ in $\mathcal{F}^\varepsilon_i$ and $C_{p_\varepsilon} \leqslant \sup_{\cup K^i_\varepsilon} |\omega_{p_\varepsilon}|$. Let us now extend the function U^ε in $\mathcal{M}$ by

$$\boldsymbol{U}^\varepsilon(\xi) = \boldsymbol{E}^\varepsilon(\xi) + U^\varepsilon_{\mathcal{F}}.$$

Applying the Poincaré inequality to the function E^ε in the domain $\mathcal{F}$, we obtain

$$\begin{aligned}\|\nabla \boldsymbol{U}^\varepsilon\|_{L^{P^\star_\varepsilon(\cdot)}(\mathcal{Y})} &= \|\nabla \boldsymbol{E}^\varepsilon\|_{L^{P^\star_\varepsilon(\cdot)}(\mathcal{Y})} \leqslant \|\boldsymbol{E}^\varepsilon\|_{W^{1,P^\star_\varepsilon(\cdot)}(\mathcal{F})} \leqslant C_1 \|E^\varepsilon\|_{W^{1,P_\varepsilon(\cdot)}(\mathcal{F})} \\ &\leqslant C\|\nabla U^\varepsilon\|_{L^{P_\varepsilon(\cdot)}(\mathcal{F})} \leqslant C \max(\Upsilon^{1/\mathsf{p}^-}_{P_\varepsilon(\cdot),\mathcal{F}}(|\nabla U^\varepsilon|), \Upsilon^{1/\mathsf{p}^+}_{P_\varepsilon(\cdot),\mathcal{F}}(|\nabla U^\varepsilon|)),\end{aligned} \tag{6.20}$$

where C is a constant which does not depend on $\varepsilon, U^\varepsilon$. Now, for any i, we introduce the function $\boldsymbol{u}^\varepsilon_i$ defined by

$$\boldsymbol{u}^\varepsilon_i(x) = \boldsymbol{U}^\varepsilon\left(\frac{x - x^{i,\varepsilon}}{\varepsilon}\right). \tag{6.21}$$

This is an extension of the function u^ε in the ball $\mathcal{M}$ and $\boldsymbol{u}^\varepsilon_i \in W^{1,p_\varepsilon(\cdot)}(K^i_\varepsilon)$. Moreover, inequality (6.19) remains true. With (6.20), this means that conditions (C2) are satisfied.

6.1.3. *Condition (C3): the function $\mathsf{A}(x, \vec{a})$*

Let $z \in \Omega$. We recall that the functional $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a})$ that appeared in condition (C3) has the form:

$$c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a}) \stackrel{\text{def}}{=} \inf_{v^\varepsilon} \int_{K^z_h \cap \Omega^\varepsilon_f} (\varkappa_\varepsilon(x)|\nabla v^\varepsilon|^{p_\varepsilon(x)} + h^{-p_\varepsilon(x)-\gamma}|v^\varepsilon - (x - z, \vec{a})|^{p_\varepsilon(x)})\,\mathrm{d}x, \tag{6.22}$$

where $\varkappa_\varepsilon(x) = k_f/p_\varepsilon(x)$ in Ω^ε_f, $\gamma > 0$, $\vec{a} \in \mathbb{R}^n$, and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K^z_h \cap \Omega^\varepsilon_f)$.

The idea of the proof of condition (C3) for the functional $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a})$ is as follows. Firstly, using the solution of (6.7), we will approximate the minimizer of $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a})$ and show that the residue, i.e. the difference between the minimizer and the approximation, gives a small contribution (as $\varepsilon, h \to 0$) in the functional $c^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z, \vec{a})$. The function $\mathsf{A}(x, \vec{a})$ is then calculated in terms of the approximating function.

Let $\mathsf{U}^a(z, \cdot)$ be a $\mathcal{Y}$-periodic extension of the function $\mathsf{u}^a(z, \cdot)$ solution of the cell problem (6.7), on $\mathbb{R}^n \setminus \overline{\mathfrak{M}}$, where $\mathfrak{M}$ is the union of sets $\mathcal{M}_j$ obtained from $\mathcal{M}$ by translations of vectors

$$\sum_{j=1}^n l_j \boldsymbol{e}_j, \quad l_j \in \mathbb{Z}.$$

The regularity properties of the function U^a are given by the following lemma.

LEMMA 6.5. *The function $\mathcal{U}^a$ possesses the following properties:*

$$\mathcal{U}^a(z,\cdot) \in L^q(\mathcal{F}) \quad \text{and} \quad \nabla \mathcal{U}^a(z,\cdot) \in L^{p_0(z)+\delta}(\mathcal{F}), \tag{6.23}$$

where $\delta > 0$ and

$$q = \begin{cases} \dfrac{p_0(z)n}{n-p_0(z)} & \text{if } p_0(z) < n, \\ \text{any number} & \text{if } p_0(z) \geqslant n. \end{cases}$$

Denote by $\{x^{i,\varepsilon}\}$ an ε-periodic grid in the cube K_h^z and denote by K_ε^i the cube centred at the point $x^{i,\varepsilon}$ and of length ε. We cover K_h^z by cubes K_ε^i and introduce a function W^ε defined by

$$W^\varepsilon(x) = (x-z,\vec{a}) - \varepsilon \mathcal{U}^a\Big(z,\frac{x}{\varepsilon}\Big) \quad \text{in } \Big(\bigcup K_\varepsilon^i\Big) \cap K_h^z. \tag{6.24}$$

Now let $v_{\min}^\varepsilon$ be the minimizer of (6.22). We represent this function as follows:

$$v_{\min}^\varepsilon(x) = W^\varepsilon(x) + \zeta^\varepsilon(x), \quad x \in K_h^z \cap \Omega_f^\varepsilon. \tag{6.25}$$

We will prove that ζ^ε gives a vanishing contribution (as $\varepsilon \to 0$ and $h \to 0$) in $c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z,\vec{a})$ and, therefore, the functional (6.22) may be approximated in terms of the function W^ε.

The property (6.23) of the function $\mathcal{U}^a$ and the uniform convergence of p_ε to p_0 imply that, for ε small enough, $W^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z \cap \Omega_f^\varepsilon)$. Thus, by definition of $c_{p_\varepsilon(\cdot)}^{\varepsilon,h}$, we have

$$c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z,\vec{a}) \leqslant \mathcal{W}^{\varepsilon,h}(z,\vec{a}), \tag{6.26}$$

where

$$\mathcal{W}^{\varepsilon,h}(z,\vec{a}) = \int_{K_h^z \cap \Omega_f^\varepsilon} (\varkappa_\varepsilon(x)|\nabla W^\varepsilon|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)}|W^\varepsilon - (x-z,\vec{a})|^{p_\varepsilon(x)})\,\mathrm{d}x. \tag{6.27}$$

Due to the definition of the function W^ε, for sufficiently small ε, we have

$$\mathcal{W}^{\varepsilon,h}(z,\vec{a}) = h^n \int_{\mathcal{F}} \frac{k_f}{p_0(z)}|\nabla u^a(z,y) - \vec{a}|^{p_0(z)}\,\mathrm{d}y + o(h^n) \quad \text{as } h \to 0. \tag{6.28}$$

Let us estimate $c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z,\vec{a})$ from below. In view of lemma 5.1(ii), for any vectors $\xi_1, \xi_2 \in \mathbb{R}^n$, there exists δ which does not depend on ε and such that

$$|\xi_1+\xi_2|^{p_\varepsilon(\cdot)} \geqslant |\xi_1|^{p_\varepsilon(\cdot)} + \delta|\xi_2|^{p_\varepsilon(\cdot)} + p_\varepsilon(\cdot)|\xi_1|^{p_\varepsilon(\cdot)-2}(\xi_1,\xi_2).$$

Then we obtain

$$\begin{aligned} &c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z,\vec{a}) \\ &\quad \geqslant \mathcal{W}^{\varepsilon,h}(z,\vec{a}) + \delta \mathcal{Z}^{\varepsilon,h}(z,\vec{a}) + \int_{K_h^z \cap \Omega_f^\varepsilon} k_f |\nabla W^\varepsilon|^{p_\varepsilon(x)-2}(\nabla W^\varepsilon, \nabla \zeta^\varepsilon)\,\mathrm{d}x \\ &\qquad + h^{-\gamma}\int_{K_h^z \cap \Omega_f^\varepsilon} p_\varepsilon(x) h^{-p_\varepsilon(x)}|W^\varepsilon - (x-z,\vec{a})|^{p_\varepsilon(x)-2}(W^\varepsilon - (x-z,\vec{a}))\zeta^\varepsilon\,\mathrm{d}x \\ &\quad = \mathcal{W}^{\varepsilon,h}(z,\vec{a}) + \delta \mathcal{Z}^{\varepsilon,h}(z,\vec{a}) + \boldsymbol{j}_1^{\varepsilon,h} + \boldsymbol{j}_2^{\varepsilon,h}, \end{aligned} \tag{6.29}$$

where

$$Z^{\varepsilon,h}(z,\vec{a}) = \int_{K_h^z \cap \Omega_f^\varepsilon} (\varkappa_\varepsilon(x)|\nabla\zeta^\varepsilon|^{p_\varepsilon(x)} + h^{-\gamma-p_\varepsilon(x)}|\zeta^\varepsilon|^{p_\varepsilon(x)})\,\mathrm{d}x. \tag{6.30}$$

Finally, from (6.26) and (6.29) we obtain the following bound for the residue ζ^ε:

$$0 \leqslant Z^{\varepsilon,h}(z,\vec{a}) \leqslant C(|\boldsymbol{j}_1^{\varepsilon,h}| + |\boldsymbol{j}_2^{\varepsilon,h}|). \tag{6.31}$$

Now let us estimate the right-hand side of inequality (6.31).

Estimate for $\boldsymbol{j}_1^{\varepsilon,h}$. We rewrite $\boldsymbol{j}_1^{\varepsilon,h}$ as follows:

$$\begin{aligned}\boldsymbol{j}_1^{\varepsilon,h} &= k_f \int_{K_h^z \cap \Omega_f^\varepsilon} (|\nabla W^\varepsilon|^{p_\varepsilon(x)-2} - |\nabla W^\varepsilon|^{p_0(x)-2})(\nabla W^\varepsilon, \nabla\zeta^\varepsilon)\,\mathrm{d}x \\ &\quad + k_f \int_{K_h^z \cap \Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-2}(\nabla W^\varepsilon, \nabla\zeta^\varepsilon)\,\mathrm{d}x \\ &= \boldsymbol{j}_{11}^{\varepsilon,h} + \boldsymbol{j}_{12}^{\varepsilon,h}.\end{aligned} \tag{6.32}$$

First, let us estimate $\boldsymbol{j}_{11}^{\varepsilon,h}$. Applying Hölder's inequality (3.1), we obtain

$$|\boldsymbol{j}_{11}^{\varepsilon,h}| \leqslant C\|\nabla\zeta^\varepsilon\|_{L^{p_\varepsilon(\cdot)}(K_h^z \cap \Omega_f^\varepsilon)} \||\nabla W^\varepsilon|||\nabla W^\varepsilon|^{p_\varepsilon(\cdot)-2} - |\nabla W^\varepsilon|^{p_0(\cdot)-2}|\|_{L^{q_\varepsilon(\cdot)}(K_h^z \cap \Omega_f^\varepsilon)}, \tag{6.33}$$

where $1/p_\varepsilon + 1/q_\varepsilon = 1$. We note that, due to the properties of $\{p_\varepsilon\}$, there exist two real numbers q^- and q^+ such that

$$1 < \mathsf{q}^- \leqslant q_\varepsilon^- \equiv \min_{x\in\bar{\Omega}} q_\varepsilon(x) \leqslant q_\varepsilon(x) \leqslant \max_{x\in\bar{\Omega}} q_\varepsilon(x) \equiv q_\varepsilon^+ \leqslant \mathsf{q}^+.$$

It now follows from (3.2) that

$$\begin{aligned}&\||\nabla W^\varepsilon|||\nabla W^\varepsilon|^{p_\varepsilon(\cdot)-2} - |\nabla W^\varepsilon|^{p_0(\cdot)-2}|\|_{L^{q_\varepsilon(\cdot)}(K_h^z \cap \Omega_f^\varepsilon)} \\ &\quad \leqslant \left(\int_{K_h^z \cap \Omega_f^\varepsilon} |\nabla W^\varepsilon|^{q_\varepsilon(x)} |\nabla W^\varepsilon|^{p_\varepsilon(x)-2} - |\nabla W^\varepsilon|^{p_0(x)-2}|^{q_\varepsilon(x)}\,\mathrm{d}x\right)^{1/\mathsf{q}^+}.\end{aligned} \tag{6.34}$$

Taking into account the properties of the function U^a, we estimate the right-hand side of (6.34) and obtain the following bound:

$$\||\nabla W^\varepsilon|||\nabla W^\varepsilon|^{p_\varepsilon(\cdot)-2} - |\nabla W^\varepsilon|^{p_0(\cdot)-2}|\|_{L^{q_\varepsilon(\cdot)}(K_h^z \cap \Omega_f^\varepsilon)} \leqslant C_1(\varepsilon,h), \tag{6.35}$$

with $C_1(\varepsilon,h) \to 0$ as $\varepsilon \to 0$. Now, turning back to (6.33), we use the following estimates for ζ^ε deduced from (6.25) and (6.26):

$$\int_{K_h^z \cap \Omega_f^\varepsilon} |\zeta^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x \leqslant Ch^{n+\gamma+\mathsf{p}_K^-} \quad \text{and} \quad \int_{K_h^z \cap \Omega_f^\varepsilon} |\nabla\zeta^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x \leqslant Ch^n, \tag{6.36}$$

where

$$2 \leqslant \mathsf{p}_K^- \leqslant p_\varepsilon^- \equiv \min_{x\in\overline{K_h^z}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\overline{K_h^z}} p_\varepsilon(x) \equiv p_\varepsilon^+ \leqslant \mathsf{p}_K^+$$

in $\overline{K_h^z}$ to conclude with (6.33) and (6.35) that

$$|\boldsymbol{j}_{11}^{\varepsilon,h}| \leqslant C_2(\varepsilon,h) \quad \text{with} \quad C_2(\varepsilon,h)\to 0 \quad \text{as } \varepsilon\to 0. \tag{6.37}$$

Now consider the term $\boldsymbol{j}_{12}^{\varepsilon,h}$. Let $K_{\hbar}^z$ be the cube centred at the point $z\in K_h^z$ and of length $\hbar = h - h^{1+\alpha}$, where α is a positive parameter which will be specified later. Let φ_h be a smooth cutoff function defined in K_h^z and such that $\varphi_h(x)=1$ in $K_{\hbar}^z$ and $\varphi_h(x)=0$ for $x\in \Omega\setminus\overline{K_h^z}$. Then

$$\begin{aligned}\boldsymbol{j}_{12}^{\varepsilon,h} &= k_f\int_{K_h^z\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-2}(\nabla W^\varepsilon, \nabla(\varphi_h\zeta^\varepsilon))\,\mathrm{d}x\\ &\quad + k_f\int_{K_h^z\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-2}(\nabla W^\varepsilon, \nabla((1-\varphi_h)\zeta^\varepsilon))\,\mathrm{d}x\\ &= \boldsymbol{i}_1^{\varepsilon,h} + \boldsymbol{i}_2^{\varepsilon,h}.\end{aligned} \tag{6.38}$$

Integrating by parts and using the boundary condition of the function $\boldsymbol{u}^a$ on $\partial\mathcal{M}$, we rewrite $\boldsymbol{i}_1^{\varepsilon,h}$ as follows:

$$\begin{aligned}\boldsymbol{i}_1^{\varepsilon,h} = -\int_{K_h^z\cap\Omega_f^\varepsilon} \operatorname{div}(k_f|\nabla W^\varepsilon|^{p_0(x)-2}\nabla W^\varepsilon)\varphi_h\zeta^\varepsilon\,\mathrm{d}x&\\ + k_f\int_{\partial K_h^z} |\nabla W^\varepsilon|^{p_0(x)-2}\frac{\partial W^\varepsilon}{\partial\vec{\nu}}\varphi_h\zeta^\varepsilon\,\mathrm{d}s_x&.\end{aligned}$$

But in view of the definition of functions W^ε in (6.7) and φ_h, the latter relation proves that

$$\boldsymbol{i}_1^{\varepsilon,h}\equiv 0. \tag{6.39}$$

Consider the second term of the right-hand side of (6.38). The definition of the function φ_h implies the following bound:

$$\begin{aligned}\left|\frac{\boldsymbol{i}_2^{\varepsilon,h}}{k_f}\right| \leqslant \int_{(K_h^z\setminus K_{\hbar}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-1}|\nabla\zeta^\varepsilon|\,\mathrm{d}x&\\ + \int_{(K_h^z\setminus K_{\hbar}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-1}|\nabla\varphi_h||\zeta^\varepsilon|\,\mathrm{d}x&.\end{aligned} \tag{6.40}$$

Using Hölder's inequality (3.1) and (6.23), for the first term of the right-hand side of (6.40) we obtain

$$\begin{aligned}&\int_{(K_h^z\setminus K_{\hbar}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-1}|\nabla\zeta^\varepsilon|\,\mathrm{d}x\\ &\qquad\leqslant 2\|\nabla\zeta^\varepsilon\|_{L^{p_\varepsilon(\cdot)}(K_h^z\cap\Omega_f^\varepsilon)}\||\nabla W^\varepsilon|^{p_0(x)-1}\|_{L^{q_\varepsilon(\cdot)}((K_h^z\setminus K_{\hbar}^z)\cap\Omega_f^\varepsilon)}\\ &\qquad\leqslant Ch^{n/p_0(z)}\biggl(\int_{(K_h^z\setminus K_{\hbar}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p(z,h)}\,\mathrm{d}x\biggr)^{1/q(z,h)},\end{aligned} \tag{6.41}$$

where

$$p(z,h) = p_0(z) + o(1) \quad\text{and}\quad q(z,h) = \frac{p_0(z)}{p_0(z)-1} + o(1) \quad \text{as } h\to 0.$$

Let us estimate the integral of the right-hand side of (6.41). We have

$$\begin{aligned}
&\int_{(K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p(z,h)} \\
&\quad\leqslant \biggl(\int_{(K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(z)+\delta}\biggr)^{1/(q_\delta(z,h))} [\operatorname{meas}((K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon)]^{1/(\eta_\delta(z,h))} \\
&\quad\leqslant Ch^{n/q_\delta(z,h)}\cdot h^{(n+\alpha)/(\eta_\delta(z,h))}, \qquad (6.42)
\end{aligned}$$

where $q_\delta(z,h)=(p_0(z)+\delta)/p(z,h)$ and $1/\eta_\delta(z,h)+1/q_\delta(z,h)=1$. It follows from (6.42) that

$$\biggl(\int_{(K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p(z,h)}\biggr)^{1/q(z,h)} \leqslant Ch^{\theta(n)}, \qquad (6.43)$$

where

$$\theta(n)=n\frac{p_0(z)-1}{p_0(z)}+\alpha\biggl(\frac{p_0(z)-1}{p_0(z)}-\frac{p_0(z)-1}{p_0(z)+\delta}\biggr)+o(1)\quad\text{as } h\to 0. \qquad (6.44)$$

Then, from (6.41), (6.43) and (6.44) we deduce that, for sufficiently small h,

$$\int_{(K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-1}|\nabla\zeta^\varepsilon|\,\mathrm{d}x = o(h^n)\quad\text{as } h\to 0. \qquad (6.45)$$

For the second term on the right-hand side of (6.40), from (6.36) and (6.44) we have

$$\begin{aligned}
&\int_{(K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-1}|\nabla\varphi_h||\zeta^\varepsilon|\,\mathrm{d}x \\
&\qquad\leqslant Ch^{-1-\alpha}\int_{(K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-1}|\zeta^\varepsilon|\,\mathrm{d}x \\
&\qquad\leqslant Ch^{-1-\alpha}\biggl(\int_{(K_h^z\setminus K_{\tilde h}^z)\cap\Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_\varepsilon(x)(p_0(x)-1)/(p_\varepsilon(x)-1)}\,\mathrm{d}x\biggr)^{(p_K^- -1)/p_K^-} \\
&\qquad\quad\times\biggl(\int_{K_h^z\cap\Omega_f^\varepsilon} |\zeta^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x\biggr)^{1/p_K^+} \\
&\qquad\leqslant Ch^{\mu(\gamma,n)}, \qquad (6.46)
\end{aligned}$$

where

$$\mu(\gamma,n)=n-\alpha\biggl(\frac{1}{p_0(z)}+\frac{p_0(z)-1}{p_0(z)+\delta}\biggr)+\frac{\gamma}{p_0(z)}+o(1)\quad\text{as } h\to 0. \qquad (6.47)$$

We choose α such that

$$\alpha<\gamma\frac{p_0(z)+\delta}{p_0^2(z)+p_0(z)+\delta-1}. \qquad (6.48)$$

Then, for sufficiently small h we have that $\mu(\gamma, n) > n$ and, therefore, for the first term on the right-hand side of (6.40) we obtain

$$\int_{(K_h^z \setminus K_{\bar{h}}^z)\cap \Omega_f^\varepsilon} |\nabla W^\varepsilon|^{p_0(x)-1} |\nabla \varphi_h| |\zeta^\varepsilon| \,\mathrm{d}x = o(h^n) \quad \text{as } h \to 0. \tag{6.49}$$

Now, from inequalities (6.38), (6.39), (6.45) and (6.49), for sufficiently small ε, we obtain the bound:

$$|\boldsymbol{j}_{12}^{\varepsilon,h}| = o(h^n) \quad \text{as } h \to 0. \tag{6.50}$$

Finally, alongside (6.37), estimate (6.50) gives:

$$|\boldsymbol{j}_1^{\varepsilon,h}| = o(h^n) \quad \text{as } h \to 0. \tag{6.51}$$

Estimate for $\boldsymbol{j}_2^{\varepsilon,h}$. In a similar way, by using the definition of the function W^ε, we obtain

$$|\boldsymbol{j}_2^{\varepsilon,h}| \leqslant C_3(\varepsilon, h) \quad \text{with } C_3(\varepsilon, h) \to 0 \text{ as } \varepsilon \to 0. \tag{6.52}$$

Estimates (6.51) and (6.52) together with (6.31) imply that

$$Z^{\varepsilon,h}(z, \vec{a}) = o(h^n) \quad \text{as } h \to 0. \tag{6.53}$$

Thus the residue ζ^ε gives a small contribution (as $\varepsilon, h \to 0$) in the functional $c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z, \vec{a})$. With (6.26), (6.29), (6.51) and (6.52) we conclude that, for sufficiently small ε,

$$c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z, \vec{a}) = W^{\varepsilon,h}(z, \vec{a}) + o(h^n) \quad \text{as } h \to 0.$$

Because $W^{\varepsilon,h}(z, \vec{a})$ satisfies (6.28), the latter relation proves that condition (C3) is fulfilled and that

$$A(z, \vec{a}) = \int_{\mathcal{F}} \frac{k_f}{p_0(z)} |\nabla u^a(z, y) - \vec{a}|^{p_0(z)} \,\mathrm{d}y,$$

where u^a is the solution of the cell problem (6.7).

6.1.4. *Condition (C4): the function* $b(x, \beta)$

First, we recall that the functional $b_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z, \beta)$ appearing in condition (C4) has the form:

$$b_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z, \beta) = \inf_{w^\varepsilon} \int_{K_h^z} \left(\varkappa_\varepsilon |\nabla w^\varepsilon|^{p_\varepsilon(x)} + \frac{\mathbf{1}_m^\varepsilon}{\sigma} |w^\varepsilon|^{\sigma(x)} + h^{-p_\varepsilon(x)-\gamma} \mathbf{1}_f^\varepsilon |w^\varepsilon - \beta|^{p_\varepsilon(x)} \right) \mathrm{d}x, \tag{6.54}$$

where the infimum is taken over $w^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z)$.

Let us denote w^ε, the minimizer of (6.54), in the form

$$w^\varepsilon(x) = \tilde{w}^\varepsilon + \xi^\varepsilon(x), \tag{6.55}$$

where

$$\tilde{w}^\varepsilon(x) = \begin{cases} w^\beta\left(\dfrac{x - x^{i,\varepsilon}}{\varepsilon}\right) & \text{in } \Omega_m^\varepsilon \cap K_\varepsilon^i, \\ \beta & \text{in } \Omega_f^\varepsilon \cap K_h^z. \end{cases} \tag{6.56}$$

We will prove that ξ^ε gives a vanishing contribution (as $\varepsilon \to 0$ and $h \to 0$) in $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta)$ and, therefore, that the functional (6.54) may be calculated in terms of the function $\tilde{w}^\varepsilon$ or, more precisely, in terms of the function w^β.

Firstly, it is easy to see that

$$\begin{aligned} b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta) &\leqslant \int_{K^z_h} \left(\varkappa_\varepsilon |\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)} + \frac{\mathbf{1}^\varepsilon_m}{\sigma} |\tilde{w}^\varepsilon|^{\sigma(x)} + h^{-p_\varepsilon(x)-\gamma} \mathbf{1}^\varepsilon_f |\tilde{w}^\varepsilon - \beta|^{p_\varepsilon(x)} \right) \mathrm{d}x \\ &= \int_{K^z_h \cap \Omega^\varepsilon_m} \left(\varkappa_\varepsilon(x) |\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)} |\tilde{w}^\varepsilon|^{\sigma(x)} \right) \mathrm{d}x \\ &\stackrel{\mathrm{def}}{=} \mathsf{M}^{\varepsilon,h}(z,\beta). \end{aligned} \tag{6.57}$$

Let us calculate the integral on the right-hand side of (6.57). Taking (6.2), (6.3) and the regularity properties of $p_0, \sigma, \mathsf{w}^\beta$ into account, for sufficiently small ε, we have

$$\begin{aligned} \mathsf{M}^{\varepsilon,h}(z,\beta) &= \int_{K^z_h \cap \Omega^\varepsilon_m} \left(\frac{k_m \varepsilon^{p_0(x)}}{p_\varepsilon(x)} |\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)} |\tilde{w}^\varepsilon|^{\sigma(x)} \right) \mathrm{d}x \\ &= \int_{K^z_h \cap \Omega^\varepsilon_m} \left(\frac{k_m \varepsilon^{-\boldsymbol{d}_\varepsilon(x)}}{p_\varepsilon(x)} \left| \nabla \mathsf{w}^\beta \left(\frac{x - x^\varepsilon_i}{\varepsilon} \right) \right|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)} |\tilde{w}^\varepsilon|^{\sigma(x)} \right) \mathrm{d}x \\ &= h^n \int_{\mathcal{M}} \left(\frac{k_m \boldsymbol{d}(z)}{p_0(z)} |\nabla \mathsf{w}^\beta|^{p_0(z)} + \frac{1}{\sigma(z)} |\mathsf{w}^\beta|^{\sigma(z)} \right) \mathrm{d}x + o(h^n) \quad \text{as } h \to 0. \end{aligned} \tag{6.58}$$

To rewrite the right-hand side of (6.58) we consider the boundary-value problem (6.8). Multiplying (6.8) by $(\mathsf{w}^\beta - \beta)/p_0(z)$ and integrating over $\mathcal{M}$, we obtain

$$\int_{\mathcal{M}} \frac{k_m \boldsymbol{d}(z)}{p_0(z)} |\nabla \mathsf{w}^\beta|^{p_0(z)} = \frac{1}{p_0(z)} \int_{\mathcal{M}} (\beta \mathsf{w}^\beta |\mathsf{w}^\beta|^{\sigma(z)-2} - |\mathsf{w}^\beta|^{\sigma(z)})\, \mathrm{d}x. \tag{6.59}$$

It then follows from (6.58) and (6.59) that, for sufficiently small ε and as $h \to 0$,

$$\begin{aligned} \mathsf{M}^{\varepsilon,h}(z,\beta) = h^n \int_{\mathcal{M}} \left(\frac{1}{p_0(z)} (\beta \mathsf{w}^\beta |\mathsf{w}^\beta|^{\sigma(z)-2} - |\mathsf{w}^\beta|^{\sigma(z)}) + \frac{1}{\sigma(z)} |\mathsf{w}^\beta|^{\sigma(z)} \right) \mathrm{d}x \\ + o(h^n). \end{aligned} \tag{6.60}$$

Let us estimate $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta)$ from below. As before, we use lemma 5.1(ii). We have

$$\begin{aligned} b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta) \geqslant \mathsf{M}^{\varepsilon,h}(z,\beta) + \delta \mathsf{X}^{\varepsilon,h}(z,\beta) + \int_{K^z_h} K_\varepsilon(x) |\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)-2} (\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon)\, \mathrm{d}x \\ + \int_{K^z_h} \mathbf{1}^\varepsilon_m(x) |\tilde{w}^\varepsilon|^{\sigma(x)-2} \tilde{w}^\varepsilon \xi^\varepsilon\, \mathrm{d}x \\ + \int_{K^z_h} h^{-p_\varepsilon(x)-\gamma} \mathbf{1}^\varepsilon_f(x) |\tilde{w}^\varepsilon - \beta|^{p_\varepsilon(x)-2} (\tilde{w}^\varepsilon - \beta) \xi^\varepsilon\, \mathrm{d}x \end{aligned}$$

$$= \mathsf{M}^{\varepsilon,h}(z,\beta) + \delta \mathsf{X}^{\varepsilon,h}(z,\beta) + \int_{K_h^z} K_\varepsilon(x)|\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon)\,\mathrm{d}x$$

$$+ \int_{K_h^z} \mathbf{1}_m^\varepsilon(x)|\tilde{w}^\varepsilon|^{\sigma(x)-2}\tilde{w}^\varepsilon \xi^\varepsilon\,\mathrm{d}x$$

$$= \mathsf{M}^{\varepsilon,h}(z,\beta) + \delta \mathsf{X}^{\varepsilon,h}(z,\beta) + \mu^{\varepsilon,h}(z,\beta), \tag{6.61}$$

where

$$\mathsf{X}^{\varepsilon,h}(z,\beta) \stackrel{\text{def}}{=} \int_{K_h^z} \left(\varkappa_\varepsilon(x)|\nabla \xi^\varepsilon|^{p_\varepsilon(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)}|\xi^\varepsilon|^{\sigma(x)} + h^{-p_\varepsilon(x)-\gamma}\mathbf{1}_f^\varepsilon(x)|\xi^\varepsilon|^{p_\varepsilon(x)} \right)\mathrm{d}x, \tag{6.62}$$

$$\mu^{\varepsilon,h}(z,\beta) \stackrel{\text{def}}{=} \int_{K_h^z\cap\Omega_m^\varepsilon} (K_\varepsilon(x)|\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon) + |\tilde{w}^\varepsilon|^{\sigma(x)-2}\tilde{w}^\varepsilon\xi^\varepsilon)\,\mathrm{d}x. \tag{6.63}$$

It follows from (6.57) and (6.61) that

$$0 \leqslant \mathsf{X}^{\varepsilon,h}(z,\beta) \leqslant \frac{1}{\delta}|\mu^{\varepsilon,h}(z,\beta)|. \tag{6.64}$$

Now let us prove that ξ^ε gives a vanishing contribution in $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta)$. To this end, we have to estimate the right-hand side of (6.64). First, it is easy to see that

$$\int_{K_h^z\cap\Omega_m^\varepsilon} K_\varepsilon(x)|\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon)\,\mathrm{d}x$$

$$= \int_{\bigcup_i \mathcal{M}_i^\varepsilon} k_m \varepsilon^{p_0(x)}|\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon)\,\mathrm{d}x$$

$$= \int_{\bigcup_i \mathcal{M}_i^\varepsilon} k_m \boldsymbol{d}(z) \varepsilon^{p_0(z)}|\nabla \tilde{w}^\varepsilon|^{p_0(z)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon)\,\mathrm{d}x$$

$$+ k_m \int_{\bigcup_i \mathcal{M}_i^\varepsilon} (\varepsilon^{p_0(x)}|\nabla \tilde{w}^\varepsilon|^{p_\varepsilon(x)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon) - \boldsymbol{d}(z)\varepsilon^{p_0(z)}|\nabla \tilde{w}^\varepsilon|^{p_0(z)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon))\,\mathrm{d}x. \tag{6.65}$$

Moreover, it follows from the definition of the functions $\tilde{w}^\varepsilon, p_\varepsilon$ and (6.10) that the second term on the right-hand side of (6.65) goes to zero as $\varepsilon, h \to 0$.

Using (6.10), (6.8), (6.65) and the regularity properties of the functions $\mathsf{w}^\beta, p_0, \sigma$ and then integrating by parts, we obtain

$$\mathsf{X}^{\varepsilon,h}(z,\beta) \leqslant C\left| \int_{K_h^z\cap\Omega_m^\varepsilon} (k_m \boldsymbol{d}(z)\varepsilon^{p_0(z)}|\nabla \tilde{w}^\varepsilon|^{p_0(z)-2}(\nabla \tilde{w}^\varepsilon, \nabla \xi^\varepsilon) + |\tilde{w}^\varepsilon|^{\sigma(z)-2}\tilde{w}^\varepsilon\xi^\varepsilon)\,\mathrm{d}x \right|$$

$$\leqslant C\varepsilon^{p_0(z)} \int_{K_h^z\cap\partial\Omega_m^\varepsilon} |\nabla \tilde{w}^\varepsilon|^{p_0(z)-2}\left|\frac{\partial \tilde{w}^\varepsilon}{\partial \nu}\right||\xi^\varepsilon|\,\mathrm{d}s_x \stackrel{\text{def}}{=} \mathcal{J}_\mu^\varepsilon. \tag{6.66}$$

To estimate the integral on the right-hand side of (6.66) we apply Hölder's inequality. We obtain

$$\int_{K_h^z\cap\partial\Omega_m^\varepsilon} |\nabla\tilde{w}^\varepsilon|^{p_0(z)-2}\left|\frac{\partial\tilde{w}^\varepsilon}{\partial\nu}\right||\xi^\varepsilon|\,\mathrm{d}s_x \leqslant \left(\sum_i\int_{\partial\mathcal{M}_i^\varepsilon}|\xi^\varepsilon|^{p_K^-}\,\mathrm{d}s_x\right)^{1/p_K^-} \times\left(\sum_i\int_{\partial\mathcal{M}_i^\varepsilon}\left[|\nabla\tilde{w}^\varepsilon|^{p_0(z)-2}\left|\frac{\partial\tilde{w}^\varepsilon}{\partial\nu}\right|\right]^{p_K^-/(p_K^- -1)}\mathrm{d}s_x\right)^{(p_K^- -1)/p_K^-}, \quad (6.67)$$

where

$$2\leqslant p_K^-\leqslant p_\varepsilon^-\equiv\min_{x\in\overline{K_h^z}}p_\varepsilon(x)\leqslant p_\varepsilon(x)\leqslant\max_{x\in\overline{K_h^z}}p_\varepsilon(x)\equiv p_\varepsilon^+\leqslant p_K^+ \quad \text{in } \overline{K_h^z}.$$

Consider the second term on the right-hand side of (6.67). We have

$$\begin{aligned}\int_{\partial\mathcal{M}_i^\varepsilon}&\left[|\nabla\tilde{w}^\varepsilon|^{p_0(z)-2}\left|\frac{\partial\tilde{w}^\varepsilon}{\partial\nu}\right|\right]^{p_K^-/(p_K^- -1)}\mathrm{d}s_x\\ &=\varepsilon^{n-1}\cdot(\varepsilon^{-p_0(z)+1})^{p_K^-/(p_K^- -1)}\int_{\partial\mathcal{M}}\left[|\nabla w^\beta|^{p_0(z)-2}\left|\frac{\partial w^\beta}{\partial\nu}\right|\right]^{p_K^-/(p_K^- -1)}\mathrm{d}s_x\\ &\leqslant C\varepsilon^n\cdot\varepsilon^{-1-p_0(z)}. \end{aligned} \quad (6.68)$$

Since $(\varepsilon^{-1-p_0(z)})^{p_K^-/(p_K^- -1)}\leqslant C\varepsilon^{-p_0(z)+1/p_0(z)}$, then it follows from (6.68) that

$$\left(\sum_i\int_{\partial\mathcal{M}_i^\varepsilon}\left[|\nabla\tilde{w}^\varepsilon|^{p_0(z)-2}\left|\frac{\partial\tilde{w}^\varepsilon}{\partial\nu}\right|\right]^{p_K^-/(p_K^- -1)}\mathrm{d}s_x\right)^{(p_K^- -1)/p_K^-} \leqslant Ch^{n(p_K^- -1)/p_K^-}\varepsilon^{-p_0(z)+1/p_0(z)}. \quad (6.69)$$

Inequalities (6.69), (6.66) and (6.67) give

$$\mathcal{J}_\mu^\varepsilon\leqslant Ch^{n(p_K^- -1)/p_K^-}\varepsilon^{1/p_0(z)}\left(\sum_i\int_{\partial\mathcal{M}_i^\varepsilon}|\xi^\varepsilon|^{p_K^-}\,\mathrm{d}s_x\right)^{1/p_K^-}.$$

Using the following bound [14, (7.135) in the case $p_0(z)\equiv 2$],

$$\int_{\partial\mathcal{M}_i^\varepsilon}|\xi^\varepsilon|^{p_K^-}\,\mathrm{d}s_x\leqslant\frac{C}{\varepsilon}\int_{K_i^\varepsilon\setminus\mathcal{M}_i^\varepsilon}|\xi^\varepsilon|^{p_K^-}\,\mathrm{d}x+C\varepsilon^{p_0(z)-1}\int_{K_i^\varepsilon\setminus\mathcal{M}_i^\varepsilon}|\nabla\xi^\varepsilon|^{p_K^-}\,\mathrm{d}x,$$

we obtain

$$\mathcal{J}_\mu^\varepsilon\leqslant Ch^{n(p_K^- -1)/p_K^-}\left(\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\xi^\varepsilon|^{p_K^-}\,\mathrm{d}x+\varepsilon^{p_0(z)}\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\nabla\xi^\varepsilon|^{p_K^-}\,\mathrm{d}x\right)^{1/p_K^-}. \quad (6.70)$$

Let us estimate the right-hand side of inequality (6.70). Consider the first integral in this inequality. It follows from Hölder's inequality (3.1) and inequalities (3.2) that

$$\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\xi^\varepsilon|^{p_K^-}\,\mathrm{d}x \leqslant C\||\xi^\varepsilon|^{p_K^-}\|_{L^{\eta_\varepsilon(\cdot)}(K_h^z\cap\partial\Omega_f^\varepsilon)}$$

$$\leqslant C\biggl(\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\xi^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x\biggr)^{p_K^-/p_K^+}, \tag{6.71}$$

$$\varepsilon^{p_0(z)}\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\nabla\xi^\varepsilon|^{p_K^-}\,\mathrm{d}x \leqslant C\biggl(\varepsilon^{p_0(z)(p_K^+/p_K^-)}\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\nabla\xi^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x\biggr)^{p_K^-/p_K^+}. \tag{6.72}$$

Then it follows from (6.70)–(6.72) and the definition of the functional $X^{\varepsilon,h}(z,\beta)$ that

$$\begin{aligned}\mathcal{J}_\mu^\varepsilon &\leqslant Ch^{n(p_K^- -1)/p_K^-}\biggl(\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\xi^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x \\ &\qquad + \varepsilon^{p_0(z)(p_K^+/p_K^-)}\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\nabla\xi^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x\biggr)^{1/p_K^+} \\ &= Ch^{n(p_K^- -1)/p_K^-}\biggl(\int_{K_h^z\cap\partial\Omega_f^\varepsilon}\frac{h^{-p_\varepsilon(x)-\gamma}}{h^{-p_\varepsilon(x)-\gamma}}|\xi^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x \\ &\qquad + \varepsilon^{p_0(z)(p_K^+/p_K^-)}\int_{K_h^z\cap\partial\Omega_f^\varepsilon}|\nabla\xi^\varepsilon|^{p_\varepsilon(x)}\,\mathrm{d}x\biggr)^{1/p_K^+} \\ &\leqslant Ch^{n(p_K^- -1)/p_K^-}\cdot h^{(p_K^-/p_K^+)+(\gamma/p_K^+)}(X^{\varepsilon,h}(z,\beta))^{1/p_K^+}.\end{aligned} \tag{6.73}$$

Now it follows from (6.66) and (6.73) that

$$X^{\varepsilon,h}(z,\beta) \leqslant Ch^{n(p_K^- -1)/p_K^-}\cdot h^{(p_K^-/p_K^+)+(\gamma/p_K^+)}(X^{\varepsilon,h}(z,\beta))^{1/p_K^+}$$

or

$$X^{\varepsilon,h}(z,\beta) \leqslant Ch^{\varsigma(n)}, \tag{6.74}$$

where

$$\varsigma(n) \overset{\text{def}}{=} n\frac{p_K^- -1}{p_K^-}\cdot\frac{p_K^+}{p_K^+ -1} + \frac{p_K^-}{p_K^+}\cdot\frac{p_K^+}{p_K^+ -1} + \frac{\gamma}{p_K^+}\cdot\frac{p_K^+}{p_K^+ -1}. \tag{6.75}$$

Due to the properties of the function p_0 and (6.75),

$$\varsigma(n) = n + \frac{p_K^-}{p_K^- -1} + \frac{\gamma}{p_K^- -1} + O(h) \quad \text{as } h\to 0.$$

Since $p_K^- \geqslant 2$, then $\varsigma(n) > n$ and we infer from (6.74) that

$$\overline{\lim_{\varepsilon\to 0}}\, X^{\varepsilon,h}(z,\beta) = o(h^n) \quad \text{as } h\to 0. \tag{6.76}$$

Thus, the residue ξ^ε gives a small contribution (as $\varepsilon, h\to 0$) in the functional $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta)$. We conclude that, for sufficiently small ε,

$$b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z,\beta) = M^{\varepsilon,h}(z,\beta) + o(h^n) \quad \text{as } h\to 0.$$

Because $M^{\varepsilon,h}(z,\vec{a})$ satisfies (6.60), the latter relation proves that condition (C4) is fulfilled and that

$$b(x,\beta)=\int_{\mathcal{M}}\left(\frac{1}{p_0(x)}(\beta w^\beta|w^\beta|^{\sigma(x)-2}-|w^\beta|^{\sigma(x)})+\frac{1}{\sigma(x)}|w^\beta|^{\sigma(x)}\right)\mathrm{d}y,$$

where w^β is the solution of (6.8).

Theorem 6.1 is proved.

Remark 6.6. Note that, for the sake of simplicity and brevity, in this section we consider a periodic case only. We make use of the same arguments (with evident modifications) for locally periodic or disperse media. In the proof of the corresponding results we follow the ideas of [7, 14], where locally periodic and disperse media were considered.

Appendix A. Proof of lemma 5.1

It is clear that the dependence of the function p on the variable x plays no role in the proof. Then we take an arbitrary constant $p>1$. Let f be a function defined in $\mathbb{R}$ by

$$f(t)=|\xi_1+t(\xi_2-\xi_1)|^p.$$

Assertion (i) is justified by the convexity of f.

Now we prove assertions (ii) and (iii). The Taylor–MacLaurin formula gives

$$f(1)=f(0)+f'(0)+\int_0^1(1-t)f''(t)\,\mathrm{d}t,$$

that is,

$$\begin{aligned}|\xi_2|^p&=|\xi_1|^p+p|\xi_1|^{p-2}\xi_1\cdot(\xi_2-\xi_1)\\&\quad+\int_0^1(1-t)(p(p-2)|\xi_1+t(\xi_2-\xi_1)|^{p-4}((\xi_1+t(\xi_2-\xi_1))\cdot(\xi_2-\xi_1))^2\\&\qquad+p|\xi_1+t(\xi_2-\xi_1)|^{p-2}|\xi_2-\xi_1|^2)\,\mathrm{d}t,\end{aligned}\tag{A 1}$$

if $|\xi_1+t(\xi_2-\xi_1)|\neq 0$ for $0<t<1$. Of course, if there exists some $t\in(0,1)$ such that $\xi_1+t(\xi_2-\xi_1)=0$, then we have

$$|\xi_2|^p-|\xi_1|^p-p|\xi_1|^{p-2}\xi_1\cdot(\xi_2-\xi_1)=|\xi_2|^p\left(1-\frac{t^p}{(1-t)^p}+\frac{pt^{p-1}}{(1-t)^p}\right)$$

and the result is obvious. We now have to estimate the integral term in (A 1).

We begin by assuming $p\geqslant 2$. We denote

$$I_1=\int_0^1(1-t)p(p-2)|\xi_1+t(\xi_2-\xi_1)|^{p-4}((\xi_1+t(\xi_2-\xi_1))\cdot(\xi_2-\xi_1))^2\,\mathrm{d}t,$$

$$I_2=\int_0^1(1-t)p|\xi_1+t(\xi_2-\xi_1)|^{p-2}|\xi_2-\xi_1|^2\,\mathrm{d}t.$$

Then $I_1 \geqslant 0$ and

$$\begin{aligned}I_2 &= p|\xi_2 - \xi_1|^2 \int_0^1 (1-t)|\xi_1 + t(\xi_2 - \xi_1)|^{p-2}\,\mathrm{d}t \\ &\geqslant p|\xi_2 - \xi_1|^2 L(p)(|\xi_1|^{p-2} + |\xi_2 - \xi_1|^{p-2}) \\ &\geqslant p|\xi_2 - \xi_1|^2 L(p)|\xi_2 - \xi_1|^{p-2},\end{aligned}$$

where the constant $L(p)$ depends solely on p. Assertion (ii) is proved.

Now we assume $1 < p \leqslant 2$. We note that

$$|\xi_1 + t(\xi_2 - \xi_1)|^{p-4}((\xi_1 + t(\xi_2 - \xi_1)) \cdot (\xi_2 - \xi_1))^2 \leqslant |\xi_1 + t(\xi_2 - \xi_1)|^{p-2}|\xi_2 - \xi_1|^2.$$

Then

$$\begin{aligned}&p(p-2)|\xi_1 + t(\xi_2 - \xi_1)|^{p-4}((\xi_1 + t(\xi_2 - \xi_1)) \cdot (\xi_2 - \xi_1))^2 \\ &\qquad\qquad \geqslant p(p-2)|\xi_1 + t(\xi_2 - \xi_1)|^{p-2}|\xi_2 - \xi_1|^2\end{aligned}$$

and

$$f''(t) \geqslant p(p-1)|\xi_1 + t(\xi_2 - \xi_1)|^{p-2}|\xi_2 - \xi_1|^2.$$

Therefore, the following relation holds true:

$$\begin{aligned}\int_0^1 (1-t)f''(t)\,\mathrm{d}t &\geqslant p(p-1)|\xi_2 - \xi_1|^2 \int_0^1 (1-t)|\xi_1 + t(\xi_2 - \xi_1)|^{p-2}\,\mathrm{d}t \\ &\geqslant p(p-1)|\xi_2 - \xi_1|^2(1-c)\int_0^c |\xi_1 + t(\xi_2 - \xi_1)|^{p-2}\,\mathrm{d}t\end{aligned}$$

for any $0 < c < 1$. Since $|\xi_1 + t(\xi_2 - \xi_1)| \leqslant |\xi_1| + |\xi_2|$ for any $t \in (0, c)$, we assert that

$$\int_0^c |\xi_1 + t(\xi_2 - \xi_1)|^{p-2}\,\mathrm{d}t \geqslant \frac{c}{(|\xi_1| + |\xi_2|)^{2-p}}.$$

Assertion (iii) of the lemma follows. Lemma 5.1 is proved.

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