Business cycle fluctuations and learning-by-doing externalities in a one-sector model

Hippolyte d'Albis¹,*, Emmanuelle Augeraud-Veronb, Alain Vendittic,1

¹ Paris School of Economics, University Paris 1, CES, 106 bd de l’Hôpital, 75013 Paris, France
² LMA, University of La Rochelle, Avenue Michel Crépeau, 17042 La Rochelle, France
³ Aix-Marseille University (Aix-Marseille School of Economics), CNRS, EHESS & EDHEC, Centre de la Vieille Charité, 2 rue de la Charité, 13002 Marseille, France

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A B S T R A C T

We consider a one-sector Ramsey-type growth model with inelastic labor and learning-by-doing externalities based on cumulative gross investment (cumulative production of capital goods), which is assumed, in accordance with Arrow (1962), to be a better index of experience than the average capital stock. We prove that a slight memory effect characterizing the learning-by-doing process is enough to generate business cycle fluctuations through a Hopf bifurcation leading to stable periodic orbits. This is obtained for reasonable parameter values, notably for both the amount of externalities and the elasticity of intertemporal substitution. Hence, contrary to all the results available in the literature on aggregate models, we show that endogenous fluctuations are compatible with a low (in actual fact, zero) wage elasticity of the labor supply.

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1. Introduction

Learning-by-doing externalities in the production process are well-known as a source of economic growth. In this paper, we show that they could also explain economic fluctuations. To do so, we depart significantly from most existing contributions in the literature and notably from Romer (1986) that use the existing capital stock as a proxy for experience. We assume, in accordance with Arrow (1962), that cumulative gross investment (cumulative production of capital goods) is a better index of experience. More precisely, we measure the learning-by-doing effects with the whole gross investment process over a given period of time. This assumption, which will importantly shape the equilibrium dynamics, represents a memory effect suggesting that investments made a long time ago do not have the same impact on the index of experience as recent ones. This can be justified by the finite life expectancy of the workers, the job turn-over of employees, and also by the fact that the diffusion’s speed of innovations in the economy is increasing. However, considering that the structural parameters are evaluated on a yearly basis, and that the time interval characterizing the memory process will be small, our paper clearly focuses on business cycle fluctuations.

We study a continuous-time4 aggregate model with inelastic labor and learning-by-doing externalities in the production process. For computational convenience, we represent the memory process as a one-hoss shay depreciation: the weight of a given vintage in the index of experience is one during a given time interval, then zero. The equilibrium path is described by a system of functional differential equations that is similar to those that can be found in the vintage capital literature. The main difficulty that emerges in the course of the analytical resolution of those models is that the optimality conditions are formulated as a system of mixed functional differential equations in which delayed (the capital stock)

* Corresponding author. Tel.: +33 1 4407 8199; fax: +33 1 4407 8232.
E-mail address: dalbis@univ-paris1.fr (H. d’Albis).

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² d’Autume and Michel (1993) consider the original formulation by Arrow in which society’s stock of knowledge, measured as the cumulative gross investment over an infinite past, acts as an externality in the production of all firms. They prove that endogenous growth can occur as a result of this.
³ Nevertheless, in the present paper, we study a simple infinitely-lived representative individual. For the dynamics of an economy with learning-by-doing externalities and a continuum of finitely-lived individuals, see d’Albis and Augeraud-Véron (2009).
⁴ It is possible to consider a discrete time framework, but this would yield a large scale dynamical system for which the analytical characterization is difficult.

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and advanced (the shadow price) terms are considered simultaneously. In our paper, introducing a lagged capital stock through an Arrow-type externality leads to optimality conditions formulated as a system of delay functional differential equations. This simpler framework notably allows us to provide results on local stability and determinacy.

The main purpose of our paper is to show the existence of a Hopf bifurcation and to establish the conditions under which the equilibrium path converges towards the periodic solution in an aggregate growth model with learning-by-doing externalities. In particular, we prove that persistent endogenous fluctuations can occur, first without considering endogenous labor and external effects coming from the labor supply, and second with standard CES preferences and technology characterized by small values for the elasticity of intertemporal substitution in consumption and a capital–labor elasticity of substitution in line with recent empirical estimates. We hence demonstrate that a simple aggregate model may generate business cycle fluctuations under plausible parameterization of the fundamentals and a low amount of externalities based on an Arrow-type learning-by-doing process. Moreover, we show that the equilibrium, be it convergent toward the steady state or the periodic cycle, is locally determinate, i.e. unique.

The economic intuition for such fluctuations is the following. Assume, for instance, that the initial level of experience is low. Then, private returns to capital are high as well as the level of investment. This increases the experience and reduces the returns to capital. Investments are then slowed down. However, due to the memory function, experience is reduced, which subsequently increases the return to capital. Permanent fluctuations are then possible, whereas they are ruled out with Romer (1986)’s assumption.

Our paper also proposes an analysis of the local determinacy around the steady-state. Even though this exercise is intuitively similar to the ones that are performed with systems of ordinary differential equations, one has here to deal with a continuum of initial values and with a stable manifold whose dimension is infinite. We show that the steady-state cannot be indeterminate and give a condition for local determinacy. However, nonexistence of equilibrium paths cannot be excluded in our framework.

Our paper contributes to the theory of endogenously-driven fluctuations that studies extensions of the Romer (1986) optimal growth model. In this framework, the aggregate dynamics are characterized by a unique monotonically convergent equilibrium path and business cycle fluctuations can only be obtained if exogenous shocks on the fundamentals are introduced. Contrary to this, multisector optimal growth models easily exhibit endogenous fluctuations without any stochastic perturbation. However, depending on whether time is discrete or continuous, the number of goods matters. In a discrete-time model, the consideration of two sectors with both consumption goods and investment goods is sufficient to generate period-two cycles through a flip bifurcation as shown by Benhabib and Nishimura (1985). 6 In a continuous-time model, Benhabib and Nishimura (1979) show that at least three sectors with one consumption good and two capital goods need to be considered to generate endogenous fluctuations through a Hopf bifurcation. 7 It is worth noting here that Hopf bifurcations explain endogenous fluctuations better than flip bifurcations. They are more likely to imply persistent, positively auto-correlated endogenous fluctuations (see Dufourt et al., 2009), while the latter generate several counterfactual characteristics in the simulated time series.

More recently, endogenous fluctuations through the existence of local indeterminacy and sunspot equilibria have been shown to occur even in one-sector models. Building on the work by Romer (1988), Benhabib and Farmer (1994) consider a Ramsey-type continuous-time aggregate model augmented to include economy-wide externalities in the production function measured by the aggregate stock of capital and total labor, which are assumed to be a proxy for some learning-by-doing process. It is indeed assumed that by using capital over time, agents increase their experience and are thus able to increase their productivity. Within such a framework, Benhabib and Farmer (1994) show that local indeterminacy and fluctuations derived from agents’ self-fulfilling expectations can occur, but not through a Hopf bifurcation. However, besides external effects in production with large enough increasing returns at the social level, the basic model also has to be increased by the consideration of endogenous labor supply, whose wage elasticity is sufficiently high, i.e., close enough to infinity. Since the elasticity of the aggregate labor supply is usually shown to be low, 8 it follows that the occurrence of local indeterminacy relies on parameter values that do not match empirical evidence.

To overcome this problem, some papers have studied Ramsey-type aggregate models notably with vintage capital, whose dynamics are represented by functional differential equations that are similar to the one we study. As initially shown by Kalecki (1935), some production lag is a possible source of aggregate fluctuations. Benhabib and Rustichini (1991) and Boucekkine et al. (1997) show that vintage capital leads to oscillatory dynamics governed by replacement echoes. 9 More recently, Bambi (2008) considers an endogenous growth model based on some AK technology with time-to-build, and shows that dampened fluctuations occur, but that persistent endogenous fluctuations through a Hopf bifurcation are ruled out. A similar result has been obtained by Bambi and Licandro (2005) in the Benhabib and Farmer (1994) model augmented to include time-to-build. However, once utility and production functions are nonlinear, Hopf bifurcations can occur (see Benhabib and Rustichini (1991), 10 Rustichini (1989) and 11 Asea and Zak (1999) 12).

The paper is organized as follows: Section 2 presents the model and defines the intertemporal equilibrium. Section 3 contains the main results on the existence of a Hopf bifurcation, the local stability properties of the periodic orbits, the local determinacy of the steady state, and presents a numerical example. Section 4 contains concluding comments and proofs are provided in a final Appendix.

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6 The consumption goods sector needs to be more capital intensive than the investment goods sector.

7 The optimal path necessarily converges monotonically to the steady state in two-dimensional continuous-time models. See Hart (1987) for a proof of this result in general autonomous control problems with one state variable.

8 The optimal path necessarily converges monotonically to the steady state in two-dimensional continuous-time models. See Hart (1987) for a proof of this result in general autonomous control problems with one state variable.

9 See also Boucekkine et al. (2005), Boucekkine et al. (1999) and Fabbri and Gozzi (2008).

10 Benhabib and Rustichini (1991) mention the possibility of a Hopf bifurcation in an aggregate model with non-linear utility and vintage capital but do not provide any formal proof of its existence and do not discuss the stability of the bifurcating solutions.

11 Rustichini (1989) considers a two-sector optimal growth model in which delays are introduced on both the control (investment and output) and state (capital stock) variables and derives a system of mixed functional differential equations. He shows that endogenous fluctuations can occur through a Hopf bifurcation.

12 Asea and Zak (1999) consider an exogenous growth model with time-to-build and claim that the steady state can exhibit Hopf cycles. However, their result is puzzling because a time-to-build assumption should lead to a system of mixed functional differential equations with both delay and advance, whereas they only consider delay in their model.
2. The model

2.1. The production structure

Let us consider a perfectly competitive economy in which the final output is produced using capital K and labor L. Although production at the firm level takes place under constant return-to-scale, we assume that each of the many firms benefits from positive externalities due to learning-by-doing effects. We consider indeed that by using capital over time, agents increase their experience and are thus able to increase their productivity. Contrary to most contributions in the literature following Romer (1986), in which the average level of capital is used as a proxy of experience, we assume, in accordance with Arrow (1962), that cumulative gross investment (cumulative production of capital goods) is a better index of experience. "Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962, p. 157). However, like Romer (1986), we consider that these learning-by-doing effects enter the production process as externalities. In what follow, we consider a model with no endogenous growth but our framework could be easily extended to constant return to scale in the accumulation factor.

The production function of a representative firm is thus

\[ f(k, e) = A[k^{\alpha} + (1 - \alpha)e^{-\beta e}]^{-\frac{1}{\gamma}} \]  
(1)

with \( \alpha \in (0, 1) \), \( \beta \in (0, 1) \), \( \gamma > -1 \) and \( A > 0 \). At the optimum of the firm, the interest rate \( r(t) \) and the wage rate \( w(t) \) then satisfy:

\[ r(t) = f_1(k(t), e(t)) - \delta, \]
\[ w(t) = f(k(t), e(t)) - k(t)f_1(k(t), e(t)) \]  
(2)

with \( \delta \geq 0 \) the depreciation rate of capital.

We also compute the share of capital in total income:

\[ s(k, e) = \frac{f_1(k, e)}{f(k, e)} = \frac{\alpha}{\alpha + (1 - \alpha)e^{-\beta e}k^\alpha} \in (0, 1) \]  
(3)

the elasticity of capital–labor substitution \( \sigma = 1/(1 + \nu) \), and the following share and elasticity related to the externalities \( e \):

\[ \varepsilon_e(k, e) = \frac{f_2(k, e)}{f(k, e)} = \beta(1 - s(k, e)), \]
\[ \varepsilon_{ke}(k, e) = \frac{f_{2e}(k, e)}{f_1(k, e)} = \frac{\beta(1 - s(k, e))}{\sigma}. \]  
(4)

The share \( \varepsilon_e(k, e) \) provides a measure of the size of the externalities and \( \varepsilon_{ke}(k, e) \) is the elasticity of the rental rate of capital with respect to \( e \). Note that the restriction \( \beta \in (0, 1) \) implies that the externalities are small enough to be compatible with a demand for capital which is decreasing with respect to the rental rate and that the marginal productivity of capital is an increasing function of the externalities. We will also assume in the following that \( \nu > 0 \), i.e. \( \sigma \in (0, 1) \). This restriction ensures that over the business cycles, the labor share is countercyclical while the capital share is procyclical.\(^\text{13}\)

Assuming a stationary population, the dynamics of capital per capita can be written as follows:

\[ k(t) = f(k(t), e(t)) - \delta k(t) - c(t) \]  
(5)

where \( c(t) \) denotes the consumption per capita. As explained previously, we assume that the externalities are generated by a learning-by-doing process in the sense described by Arrow (1962), and correspond to the per capita cumulative gross investment.

**Assumption 1.** \( e(t) = \int_{t-\tau}^{t} (k(s) + \delta k(s)) \, ds \geq 0, \text{ with } \tau \geq 0. \)

The parameter \( \tau \) is exogenous and represents a memory effect. We assume indeed that workers improve their experience by using capital over time but their memory is bounded in the sense that after some time \( \tau \), experiences that are too old are forgotten. It is worth noting that a different formulation for the depreciation of memory could be considered. For instance Boucekkine et al. (2006) assume an exponential depreciation rate.

**Remark 1.** The formula in Assumption 1 encompasses the Ramsey (1928) model when \( \delta = \tau = 0 \) and the Romer (1986) model when \( \delta = 0, \tau \rightarrow +\infty \) and \( \lim_{t \rightarrow -\infty} k(t) = 0 \) as particular cases.

2.2. Preferences and intertemporal equilibrium

The representative infinitely-lived individual supplies a fixed amount of labor \( l = 1 \) and derives utility from consumption \( c \) according to a CARA function such that

\[ u(c) = \frac{c^{1-\gamma}}{1-\gamma} \]  
(6)

with \( \varepsilon_e \in (0, +\infty) \) that denotes the elasticity of intertemporal substitution in consumption.

The intertemporal maximization program of the representative agent is given by:

\[ \max_{c(t), k(t)} \int_{t=0}^{+\infty} e^{-\rho t} u(c(t)) \, dt \]  
(7)

s.t. \( \dot{k}(t) = f(k(t), e(t)) - \delta k(t) - c(t) \),
\( k(t) = k_0(t) \text{ for } t \in [-\tau, 0] \) and \( [e(t)]_{t<0} \) given

\[ \max_{k(t)} \int_{t=0}^{+\infty} e^{-\rho t} u(f(k(t), e(t)) - \delta k(t) - \dot{k}(t)) \, dt \]  
(8)

s.t. \( (k(t), \dot{k}(t)) \in \mathcal{D}([e(t)]_{t\geq0}) \),
\( k(t) = k_0(t) \text{ for } t \in [-\tau, 0] \) and \( [e(t)]_{t\geq0} \) given

with

\[ \mathcal{D}([e(t)]_{t\geq0}) = \left\{ (k(t), \dot{k}(t)) \in \mathbb{R}_+ \times \mathbb{R} | f(k(t), e(t)) - \delta k(t) - \dot{k}(t) \geq 0, \forall e(t) \geq 0 \right\} \]

being the convex set of admissible paths. An interior solution to problem (8) satisfies the Euler equation

\[ \left[ f(k(t), e(t)) - \delta \dot{k}(t) + f_k(k(t), e(t)) \dot{k}(t) - \dot{k}(t) \right] u''(c(t)) + f_l(k(t), e(t)) \dot{k}(t) - \delta - \rho u'(c(t)) = 0 \]  
(9)

and the transversality condition

\[ \lim_{t \rightarrow +\infty} u'(c(t)) k(t) e^{-\rho t} = 0 \]  
(10)

for all given \( e(t) \). At the individual level, a solution of the Euler equation (9) is thus a path of capital stock parameterized by a given path of externalities, namely \( k(t, [e(s)]_{s \geq 0}) \). At the aggregate level, as the externalities are defined according to Assumption 1, an
equilibrium path is the solution of a fixed-point problem defined such that

\[ e(t) = \int_{t-\tau}^{t} \left( \dot{k} \left( s, [e(z)]_{z \geq 0} \right) + \delta k \left( s, [e(z)]_{z \geq 0} \right) \right) ds \quad (11) \]

for all \( t \geq 0 \). Assuming that such a fixed-point problem has a solution, the capital dynamics are characterized by the following non-linear functional differential equation with distributed delays. By deriving Eqs. (5) and (9) as well as the non-linear functional differential equation with distributed delays.

\[ k \text{ parameter defined above, we obtain:} \]

\[ \ddot{k}(t) = \left[ f_1\left( k(t), k(t) - k(t - \tau) + \delta \int_{t-\tau}^{t} k(s) ds \right) - \delta \dot{k}(t) \right]  \]

\[ + \delta \int_{t-\tau}^{t} k(s) ds - \delta - \rho \]

(12) together with the transversality condition (10).

We now study the existence of an interior steady state in the neighborhood of which an equilibrium path exists by continuity.

2.3. The steady state and the characteristic equation

We consider the dynamics in the neighborhood of the steady state. Along a stationary path \( k(t) = \bar{k} \) for any \( t \geq 0 \), Assumption 1 implies \( e(t) = \bar{e} = \delta \tau k \) provided that \( \tau \ll +\infty \). An interior steady state is thus a \( k \) that solves:

\[ f_1(\bar{k}, \delta \tau k) = \delta + \rho \]

(13) and the corresponding stationary consumption level is

\[ \bar{c} = f(\bar{k}, \delta \tau \bar{k}) - \delta \bar{k} > 0. \]

(14) We immediately obtain:

**Proposition 1.** Under Assumption 1, there exists a unique steady state

\[ \bar{k} = (\delta \tau)^{\frac{\rho}{\beta}} \frac{\alpha A}{\alpha + \beta} \left( \frac{c_{\alpha}}{\tau \rho} \right)^{\frac{1}{1-\alpha}}. \]

(15)

We easily derive from (3) and (4) that at the steady state:

\[ s = \alpha \left( \frac{\delta + \rho}{\beta} \right)^{\frac{1}{\tau \rho}}, \quad \epsilon_s = \beta(1 - s), \]

\[ \epsilon_{ke} = \frac{\beta(1 - s)}{\sigma}. \]

(16)

The elasticity of intertemporal substitution in consumption \( \epsilon_s \) will be used as the bifurcation parameter. Let us now establish the characteristic equation.

**Lemma 1.** The characteristic equation is \( D(\lambda) = 0 \) with

\[ D(\lambda) = \lambda^2 - \rho \lambda - \frac{\epsilon_s(1-s)(\delta + \rho)[\delta(1-s) + \rho]}{s \sigma} \]

\[ - \frac{\beta(1-s)(\delta + \rho)}{\sigma} \left[ \frac{\lambda^2 + \frac{1}{\sigma} [1 - \epsilon_s(1-s) + \rho]}{s \sigma} \right] \]

\[ - \epsilon_s[\delta(1-s) + \rho] \int_{-\tau}^{0} e^{\lambda s} ds = 0. \]

When \( \tau \in (0, +\infty) \), the characteristic equation is transcendental and there exist an infinite number of roots, some of them being complex with negative real part. When \( \tau \not\in (0, +\infty) \), one may in some cases derive characteristic polynomials that are easy to study. For instance, when \( \tau = 0 \), there is no externality and the characteristic equation becomes:

\[ D(\lambda) = \lambda^2 - \rho \lambda - \frac{\epsilon_s(1-s)(\delta + \rho)[\delta(1-s) + \rho]}{s \sigma}. \]

There are two real roots of opposite sign and the steady-state is saddle-point stable. Similarly, if \( \delta = 0 \), \( \tau \to +\infty \) and \( \lim_{\tau \to \infty} k(t) = 0 \), the characteristic equation becomes

\[ D(\lambda) = \lambda^2 - \rho \lambda - \frac{\epsilon_s(1-s)^2}{s \sigma}. \]

and again there are two real roots of opposite sign.

When there is no externality (\( \tau = 0 \)), the formulation corresponds to a standard optimal growth model with one state variable, the capital stock, and one forward variable, the associated implicit price, or equivalently the time derivative of the capital stock \( k(t) \). One negative, i.e. stable, characteristic root is then associated to the state variable, while one positive, i.e. unstable, characteristic root is associated with the forward variable. Starting from a given initial value for the capital stock \( k(0) \), the unique optimal path is obtained from the existence of a unique value for the implicit price which allows to select the unique converging path.

When externalities are present (\( \tau > 0 \)), while there is still one state variable, we have now to define a continuum of initial values for the capital stock, i.e. \( k(t) = k_0(t) \) for \( t \in [-\tau, 0] \). Moreover, as we will show in the next section, the initial condition for the forward variable is no longer a real number but is now a continuously differentiable function of time \( \tau \) that has also to be defined on \( t \in [-\tau, 0] \).

Because of the delays, the spectral decomposition features an infinite number of characteristic roots with negative real parts. Let us now establish an important result.

**Lemma 2.** \( D(\lambda) = 0 \) has at least a positive real root.

The existence of characteristic roots with positive real part implies that any path is unstable in the initial state of continuously differentiable functions on \( [-\tau, 0] \). As the transversality condition rules out divergent paths, the initial conditions for the forward variable should be chosen such that the equilibrium path belongs to the space associated with the stable characteristic roots. As it will be shown below, the restrictions imposed on this choice will then depend on the (finite) number of unstable characteristic roots.

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14 In a continuous-time framework, the existence of a solution of this kind of fixed-point problem is a difficult issue. Simple cases in endogenous growth settings have nevertheless been studied by Romer (1986) and d’Albis and Le Van (2006).

15 This property can be shown using standard methods.

16 From a mathematical point of view, the initial conditions have to be chosen to belong to the direct sum of the stable space and the center space.

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3. Endogenous business cycle fluctuations

The occurrence of business cycle fluctuations is obtained through the existence of a Hopf bifurcation, which generates periodic cycles. The analysis is conducted in two steps: first, we study the existence of a Hopf bifurcation and provide conditions for the occurrence of locally stable periodic cycles. Second, we analyze the local determinacy property of the steady state and the bifurcating cycles.

3.1. Hopf bifurcation: existence and stability

In this first part of the analysis, we propose conditions that ensure the existence of a critical value \( \epsilon_c > 0 \) for the elasticity of intertemporal substitution in consumption such that when \( \epsilon = \epsilon_c^H \), a pair of purely imaginary roots is the solution of the characteristic equation. Let us then consider the following lower bound:

\[
\xi_c = \frac{\delta \sigma \beta}{(1-s)(\delta + \rho) (\beta - \frac{c H}{\delta})} \tag{18}
\]

is positive if \( \tau < 2\beta/\delta \). We introduce the following restrictions.

**Assumption 2.** \( \epsilon_c > \xi_c > 0 \).

We provide the following result.

**Theorem 1.** Under Assumptions 1 and 2, there exists a critical value \( \epsilon_c^H \) such that when \( \epsilon = \epsilon_c^H \), a Hopf bifurcation occurs generically.

We prove that endogenous business cycle fluctuations are compatible with small externalities as \( \epsilon_c^H \) is bounded above by \((1-s)/\sigma\), but cannot be obtained for arbitrarily low elasticities of intertemporal substitution in consumption. In Section 3.3, we nevertheless show that the critical value \( \epsilon_c^H \) can remain compatible with plausible values. It is also worth noting that this result still applies to endogenous labor as long as the wage elasticity of the labor supply remains low enough.

We are now interested in the stability and the direction of the periodic orbit. Indeed, different cases may occur: the periodic orbit may arise in the right neighborhood of \( \epsilon_c^H \) and be stable, or may arise in the left neighborhood of \( \epsilon_c^H \) and be unstable.

We use the methodology of Hassard et al. (1981) to compute coefficients that determine the Hopf bifurcation direction and the stability properties of the bifurcating periodic solution. Our strategy can be described as follows: we write our system of delay functional differential equations as a system of ordinary differential equations but on a particular space (of functions \( C^1([-\tau, 0], \mathbb{R}^2) \)), on which we define a bilinear form. We look for the tangent space of the central manifold. We project the solution of the delay functional differential equations system on this tangent space and look at the dynamics that are described by an ordinary differential equation. Some coefficients of the Taylor approximation of this ordinary differential equation give the conditions for stability.

Let \( y(t) = k(\tau) - K \) and let us write Eq. (12) by considering the variable \( y \) instead of \( k \). The resulting dynamic system admits 0 as a steady state. Let \( \varphi = (\varphi_1, \varphi_2) \) with \( y(t) = \varphi_2(t) \) and \( dy/y \) becomes \( \varphi_1(t) \). System (12) becomes:

\[
\frac{d\varphi_1(t)}{dt} = f_1(\varphi_2(t) + K, X(t)) - \delta \varphi_1(t)
+ f_2(\varphi_2(t) + K, X(t)) \times \varphi_1(t) - \varphi_1(t) + \delta [\varphi_2(t) - 2\varphi_2(t) - \varphi_2(t)]
- \epsilon_c (f(\varphi_2(t) + K, X(t), - \delta_\varphi_2(t) - \delta_\varphi_1(t))
	imes f_1(\varphi_2(t) + K, X(t)) - \delta_\varphi_1(t)) \tag{19}
\]

where

\[
X(t) = \varphi_2(t) - \varphi_2(t) + \delta t K + \delta \int_{t-\delta}^{t} \varphi_2(s) ds.
\]

Let \( \epsilon_\varphi = \epsilon_c^H + \epsilon \). System (19) can be written as:

\[
\dot{\varphi}(t) = G(\varphi, \varphi_t). \tag{20}
\]

A Taylor expansion of the neighborhood of the steady state allows us to split this system into linear and nonlinear parts. Let \( z(t) = \varphi\), be a solution of Eq. (20) when \( \epsilon = 0 \). Following Hassard et al. (1981), the projection of this dynamical system on the center manifold gives the following equation

\[
\dot{z}(t) = \mu_0 + g(z, \varphi) \tag{21}
\]

which allows to discuss the stability of the limit cycle. Let us indeed consider a Taylor expansion of \( g(z, \varphi) \) such that: \( g(z, \varphi) = g_0(z, \varphi) + g_1(z, \varphi) + g_2(z, \varphi_2) + h.o.t. \) From the terms \( g_{02}, g_{11}, g_{20}, g_{21} \), applying the main results of Hassard et al. (1981), we can compute three coefficients \( \mu_1, \mu_2, \) and \( \mu_3 \) which will allow to characterize the bifurcation.

**Theorem 2.** There exist three coefficients \( \mu_1, \mu_2, \) and \( \mu_3 \) such that the stability of the periodic solution is determined as follows.

1. \( \mu_1 \) determines the direction of the Hopf bifurcation. If \( \mu_1 > 0 \) then the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for \( \epsilon_c > \epsilon_c^H \).
2. \( \mu_2 \) determines the stability of the bifurcating periodic solutions. If \( \mu_2 < 0 \) then the bifurcating periodic solutions are stable.
3. Period \( T \) of the periodic solutions is given by

\[
T = \frac{2\pi}{\omega_0} \left[ 1 + \mu_3 (\epsilon_c - \epsilon_c^H)^2 + 0 ((\epsilon_c^H - \epsilon)^4) \right].
\]

In Appendix A4, we show how to compute these coefficients explicitly and provide, in Section 3.3, some numerical illustrations.

3.2. Local determinacy

Externalities often generate local indeterminacy, i.e. the existence of a continuum of equilibrium paths from a given initial stock of capital. On a more formal basis, this is obtained when the number of stable characteristic roots is larger than the number of pre-determined variables, or equivalently, when the number of unstable characteristic roots is lower than the number of forward variables.

The main difference of our framework with the standard optimal growth model hinges on the initial condition associated with the forward variable, i.e. the time derivative of the capital stock \( k(t) \). Contrary to the framework with ordinary differential equations, the initial value in the presence of delays is a continuously differentiable function \( k_0(t) \) that is defined over the interval \([-\tau, 0] \). Therefore, \( k_0(t) \) has to be compatible with the fact that the initial value for the capital stock is given by \( k(t) = k_0(t) \) for \( t \in [-\tau, 0] \). As a result the only degree of freedom to compute the initial condition is to define a value for \( k_0(0) \). Now, using the result obtained in Lemma 2 such that there exists at least one positive real root, we define the equilibrium path on a particular sub-space. Denoting by \( n \geq 1 \) the (finite) number of unstable characteristic roots, it can be shown that the function \( k_0(t) \) also has to satisfy \( n \) constraints. We then get the following result.

**Proposition 2.** If \( n = 1 \), the steady state is locally determinate, i.e. there exists a unique equilibrium path, while there does not exist any equilibrium path when \( n > 1 \).
Under Assumptions 1 and 2, the existence of the critical value $\epsilon^H_1$ allows us now to discuss more precisely the local determinacy property of the steady state. Indeed, we can provide simple conditions on the share of capital $s$ and the delay $\tau$ to get $n = 1$ when $\epsilon_1 \in (\epsilon^H_1, \epsilon^H_2)$. 

**Assumption 3.** $s \in (0, 1/2)$

**Theorem 3.** Under Assumptions 1–3, there exists $\tau \in (0, 2\beta / \delta)$ such that when $\tau \in (0, \tau)$ and $\epsilon_1 \in (\epsilon^H_1, \epsilon^H_2)$, the steady state is locally determinate.

Theorem 3 is based on the fact that, when $\tau \in (0, \tau)$ and $\epsilon_1 \in (\epsilon^H_1, \epsilon^H_2)$, the characteristic equation admits exactly one root with positive real part, i.e. $n = 1$. As a result, there does not exist any fluctuations based on self-fulfilling expectations although external effects occur in the production process.

The local determinacy of periodic cycles can be also characterized. When $\epsilon_1$ is in a right neighborhood of $\epsilon^H_2$, we get $n = 3$ and the steady state is now unstable. But using similar arguments, we conclude that if $\mu_1 > 0$ and $\mu_2 < 0$, the periodic cycle occurs when $\epsilon_1 > \epsilon^H_2$ and is locally determinate, in the sense that there exists a unique initial value for the forward variable such that the equilibrium converges toward the limit cycle.

### 3.3. A numerical illustration

Considering a yearly calibration, we assume that the fundamental parameters are set to the following values: $\nu = 1$, $\alpha = 0.5$, $\delta = 0.1$, $\rho = 0.0808$, and $\tau = 0.1$. It follows that the share of capital is, as usual, $s = 0.3$ and the elasticity of capital–labor substitution is $\sigma = 0.5$. Such a value for $\sigma$ is in line with recent empirical estimates which show that $\sigma$ is in the range of $0.4$–$0.6$. The size of externalities in all the following simulations is contained between $0.15$ and $0.20$, an interval which is in line with the estimations of Basu and Fernald (1997), although the latter considers a different production function. We also note that the learning-by-doing process is based on a rather small memory lag $\tau$. But, in the following, we show that this small departure from the standard Ramsey model is enough to generate endogenous business cycle fluctuations.

Recent empirical estimates of the elasticity of intertemporal substitution in consumption provide divergent views. Some authors like Campbell (1999) and Vissing-Jorgensen (2002) argue that it is definitely below one, while others, like Mulligan (2002) and Gruber (2006) repeatedly obtained estimates above the unity. However, all acknowledge that the elasticity should belong to the range $0.2$–$2.2$. In the table that follows, we present the Hopf bifurcation value $\epsilon^H_0$ and the period of the cycle $T$ computed for various combinations of parameters $\beta$ and $\epsilon_1$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\epsilon_1$ (%)</th>
<th>$\epsilon^H_0$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.286</td>
<td>20</td>
<td>0.6</td>
<td>12.35</td>
</tr>
<tr>
<td>0.245</td>
<td>17.2</td>
<td>1</td>
<td>4.03</td>
</tr>
<tr>
<td>0.243</td>
<td>17</td>
<td>1.2</td>
<td>3.3</td>
</tr>
<tr>
<td>0.24</td>
<td>16.8</td>
<td>2</td>
<td>2.17</td>
</tr>
</tbody>
</table>

In each configuration, the bifurcating periodic orbit solutions exist when $\epsilon_1 > \epsilon^H_1$ (i.e. $\mu_1 > 0$) and are orbitally stable (i.e. $\mu_2 < 0$).

Moreover, for any $\epsilon_1$ in the right neighborhood of $\epsilon^H_1$, the period of the bifurcating solutions is proportional to $T$. Note that the lower the amount of externalities $\epsilon_1$, the higher the Hopf bifurcation value $\epsilon^H_2$ and the lower the period of the cycle $T$.

These numerical illustrations prove that with standard values of the fundamental parameters, persistent endogenous fluctuations easily arise with an elasticity of intertemporal substitution in consumption that is sufficiently high but that still remains compatible with the recent empirical estimates of Campbell (1999). It is also worth noting that similar results still apply to bifurcation values such that $\epsilon^H_2 \in (0.5, 2.1)$ when different sizes of externalities are considered, or with endogenous labor as long as the wage elasticity of the labor supply remains low enough, a property that is compatible with the empirical studies of the labor market.\(^{20}\)

Note finally that using a low value for the parameter $\tau$, namely $\tau = 0.1$, has allowed us to prove that only a small departure from the standard Ramsey model is enough to generate endogenous fluctuations. All our results remain valid for longer memory processes. In particular, using the same values for the structural parameters, the previous numerical results are basically not affected as long as $\tau < 0.7$. With much longer memory processes, we lose the existence of Hopf bifurcation but, due to the complex roots we obtain medium term swings that eventually vanish.

### 4. Concluding comments

We have considered a one-sector Ramsey-type growth model with inelastic labor and learning-by-doing externalities based on cumulative gross investment, which is assumed, in accordance with Arrow (1962), to be a better index of experience than the Romer-type formulation based on the instantaneous aggregate capital stock. We have proven that a slight memory effect characterizing the learning-by-doing process and a small amount of externality are enough to generate business cycle fluctuations through a Hopf bifurcation if the elasticity of intertemporal substitution is high enough but remains within limits compatible with recent empirical estimates. Moreover, contrary to all the results available in the literature on aggregate models, we have shown that endogenous fluctuations are compatible with a zero (or at least low enough) wage elasticity of the labor supply. Finally, we have shown that the equilibrium, be it convergent toward the steady state or the limit cycle, is always locally unique.

### Acknowledgments

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\(^{17}\) We have also checked that in a neighborhood of those parameters, the fixed point as defined by (11) exists and satisfies $k(t) > 0$ and $s(t) > 0$.

\(^{18}\) See Chirinko (2008) for a review of many studies that have attempted to estimate the elasticity of capital–labor substitution using various econometric methods.

\(^{19}\) Similar results can be obtained with larger values of $\tau$ as long as $\tau < 0.7$.

\(^{20}\) See, for instance, Blundell and MaCurdy (1999) for microelasticities and Rogerson and Wallenius (2009) for macroelasticities.
Appendix

A.1. Proof of Lemma 1

Let us use the following notations:
\[ f = f (\bar{k}, \delta \tau \bar{k}) , \quad f_1 = f_1 (\bar{k}, \delta \tau \bar{k}) , \quad f_2 = f_2 (\bar{k}, \delta \tau \bar{k}) , \quad f_{11} = f_{11} (\bar{k}, \delta \tau \bar{k}) , \quad f_{12} = f_{12} (\bar{k}, \delta \tau \bar{k}) . \]

Linearizing system (12) around the steady state \( \bar{k} \) and defining \( k(t) = \bar{k} + \epsilon x(t) \) lead to
\[
\dot{x}(t) = [f_1 - \delta + f_2] x(t) - f_2 (t - \tau) + [f_2] - \epsilon \tau (f_1 + f_{12}) x(t) - [f_2] - \epsilon \tau f_{12} \int_{t-\tau}^{t} x(s) ds . \tag{22}
\]
The characteristic equation of \( D(\lambda) = 0 \) is obtained by replacing \( x(t) = x(0)e^{\lambda t} \) and rearranging using \( f_1 = \delta + \rho \) together with the shares and elasticities (3)-(4).

A.2. Proof of Lemma 2

From the definition of \( D(\lambda) \), we get \( \lim_{\lambda \to \infty} D(\lambda) = +\infty \), and \( D(0) < 0 \). The result follows.

A.3. Proof of Theorem 1

Adding the extra root \( \lambda = 0 \) and letting \( \Delta(\lambda) = \lambda D(\lambda) \), the characteristic polynomial can then be written as a third-order quasi-polynomial
\[
\Delta(\lambda) = P(\lambda) + Q(\lambda) e^{-\lambda \tau} \tag{23}
\]
with
\[
P(\lambda) = \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 , \quad Q(\lambda) = q_2 \lambda^2 + q_1 \lambda + q_0
\]
and
\[
q_0 = -\frac{e_{\tau} (1 - s)(\delta + \rho)[\delta (1 - s) + \rho]}{s \tau} , \quad q_1 = \frac{\beta (1 - s)(\delta + \rho)}{s \tau} \left[ 1 - \frac{e_{\tau} [\delta (1 - s) + \rho]}{s \tau} \right] , \quad q_2 = \frac{\beta (1 - s)(\delta + \rho)}{s \tau} = - \frac{(\rho + p_2)}{s \tau} \tag{24}
\]
\[
p_1 = -\frac{e_{\tau} (1 - s)(\delta + \rho)[\delta (1 - s) + \rho]}{s \tau} + q_1 .
\]

The proof of Theorem 1 is given through the next three lemmas.

Lemma A.1. Under Assumption 2, there exists \( q > 0 \) such that \( |Q(iq)| = 0 \).

Proof. We study the occurrence of imaginary roots of the characteristic equation. Let \( \lambda = p + iq \) and then rewrite equation \( \Delta(\lambda) = 0 \) such that:
\[
-\epsilon i^3 + p_2 \epsilon o^2 + iop_1 + p_0 + (-q_2 \epsilon o^2 + iq_1 o + q_0) \times (\cos (\omega \tau) - i \sin (\omega \tau)) = 0.
\]

We are looking for \( \omega_0 > 0 \) such that \( Q(i\omega_0) = 0 \). Separating real and imaginary parts, we have
\[
p_2 \omega_0^2 - p_0 = (q_0 - q_2 \omega_0^2) \cos (\omega_0 \tau) + q_1 \omega_0 \sin (\omega_0 \tau) - \omega_0^2 + p_0 \omega_0 = (q_0 - q_2 \omega_0^2) \sin (\omega_0 \tau) - \omega_0 \omega_0 \cos (\omega_0 \tau) .
\]
Squaring both sides of the previous equations and summing them yield
\[
\omega_0^4 + (p_2^2 - 2p_1 - q_2^2) \omega_0^2 + (p_1^2 + 2p_0 \rho - q_1^2) = 0
\]
which is rewritten as follows:
\[
x^2 + 2\eta x + \psi = 0 \tag{25}
\]
where \( x = \omega_0^2 \) and, using the shares and elasticities (3)-(4), we obtain \( \eta \) and \( \psi \) as given in Box I. The discriminant of (25) is \( \Delta = \eta^2 - \psi \) and the roots are \( x_{1,2} = -\eta \pm \sqrt{\eta^2 - \psi} \). The first condition for the existence of a real root is \( \Delta \geq 0 \). Then there are two cases depending on the sign of \( \psi \):
- If \( \psi < 0 \) then \( \Delta \geq 0 \) and there exists a unique positive real root for any sign of \( \eta \);
- If \( \psi > 0 \), then the existence of a positive real root requires \( \eta < 0 \) and \( \eta^2 \geq \psi \).

Assumption 2 implies \( \psi < 0 \). It follows that the positive root of \( x^2 + 2\eta x + \psi = 0 \) is \( x_1 = -\eta + \sqrt{\eta^2 - \psi} \). As \( \eta \) and \( \psi \) are functions of \( \epsilon_{\tau} \), let us denote \( \omega_0 = \omega (\epsilon_{\tau}) = \sqrt{\eta^2} \).

We also have
\[
\cos (i\omega_0 \tau) = \omega_0^4 + a_0^2 (q_0 - q_2 \omega_0^2)^2 \left( q_0 - q_2 \omega_0^2 \right)^2 \tag{26}
\]
\[
\sin (i\omega_0 \tau) = \omega_0^2 q_2 + a_0^2 (p_0 q_0 - p_1 q_1 + q_0 p_2) - p_0 q_0
\]
It follows that the bifurcation value \( \epsilon_{\tau}^H \) is obtained as the value of \( \epsilon_{\tau} \) that solves the following equation:
\[
\cos (\epsilon_{\tau}^H \sqrt{\eta^2}) \equiv G_1 (\epsilon_{\tau}) = \frac{x_1^2 (q_1 - p_2 q_2) + x_2 (p_0 q_2 - p_1 q_1 + q_0 p_2) - p_0 q_0}{(q_0 - q_2 \omega_0^2)^2}
\]
Recall from Assumption 2 that \( \epsilon_{\tau}^H < +\infty \). We can easily show that
\[
\lim_{\epsilon_{\tau} \to -\infty} G_1 (\epsilon_{\tau}) = \lim_{\epsilon_{\tau} \to +\infty} G_2 (\epsilon_{\tau}) = 1 .
\]
We can also compute a series expansion of \( G_2 (\epsilon_{\tau}) \) in order to compute the limit when \( \epsilon_{\tau} \to +\infty \). We obtain \( G_1 (\epsilon_{\tau}) \) as given in Box II. It follows that under Assumption 2, \( \lim_{\epsilon_{\tau} \to +\infty} G_2 (\epsilon_{\tau}) = -1^+ \).

Moreover, as \( \epsilon_{\tau} \) increases from \( \epsilon_{\tau} \) to \( +\infty \), the function \( G_2 (\epsilon_{\tau}) \) oscillates continuously between 1 and 1. It follows that there necessarily exist an infinite number of solutions to Eq. (27) for \( \epsilon_{\tau} \) large enough.

Let us then consider the smallest solution of Eq. (27), denoted by \( \epsilon_{\tau}^H \), which corresponds to the Hopf bifurcation value such that \( \pm i\omega_0 \) is an imaginary root of (23).

Lemma A.2. \( \pm i\omega_0 \) is generically a simple root.

Proof. If we suppose by contradiction that it is not a simple root, we have
\[
P' (i\omega_0) + (Q' (i\omega_0) - \tau Q (i\omega_0)) e^{-i\omega_0 \tau} = 0.
\]
Separating imaginary and real parts, and squaring each member, we have:

\[
\omega_0^2 (\tau^2 q^2 - 9) + \omega_0^2 (6p_1 - 4p_2^2 + 2\tau q_2(q_1 - \tau q_0) \\
+ (2q_2 - \tau q_1)^2) + (q_1 - \tau q_0)^2 - p_1^2 = 0.
\]

As it is also a root of the characteristic equation, we also have

\[
\omega_0^2 + (p_2^2 - 2p_1 - q_1^2) \omega_0^2 + (p_1^2 + 2p_0\rho - q_1^2) = 0.
\]

That implies the equation given in Box III. Such equality is nongeneric. ◻

Let \( \epsilon_i^H \) be the value for \( \epsilon_i \) for which we have an imaginary root.

**Lemma A.3.** Re \( \left( \frac{d\lambda(\epsilon_i)}{d\epsilon_i} \right)_{|\epsilon_i=\epsilon_i^H} \neq 0. \)

**Proof.** This proof is computational and consists in three steps. In step 1, we compute \( \frac{d\lambda}{d\epsilon_i} = \frac{D_2(\lambda,\epsilon_i,\epsilon_i^H)}{D_1(\lambda,\epsilon_i,\epsilon_i^H)} \); in step 2, we evaluate

\[
\text{Re}\left( \frac{d\lambda}{d\epsilon_i} \right) = \text{Re}\left( \frac{D_2(\lambda,\epsilon_i,\epsilon_i^H)}{D_1(\lambda,\epsilon_i,\epsilon_i^H)} \right);
\]

in step 3, we use the specific form of the pure characteristic root \( \lambda_0 \) to conclude.

Step 1. Let us differentiate the following equation according to \( \epsilon_i \), noting that \( \epsilon_i \) only appears in \( \frac{d\lambda}{d\epsilon_i} = -\epsilon_i \epsilon_i^c \).

\[
\Delta(\lambda, \epsilon_i) = \left( \lambda^3 + p_2 \lambda^2 + p_1 (\epsilon_i) \lambda + p_0 (\epsilon_i) \right) \\
+ (q_2 \lambda^2 + q_1 (\epsilon_i) \lambda + q_0 (\epsilon_i)) e^{-\lambda \tau} = 0.
\]

As the root we consider is a simple root, we can use the implicit function theorem

\[
\frac{d\lambda}{d\epsilon_i} \left|_{\epsilon_i=\epsilon_i^H} \right. = \frac{\left( p_1 (\epsilon_i^H) \lambda + p_0 (\epsilon_i^H) \right) + \left( q_1 (\epsilon_i^H) \lambda + q_0 (\epsilon_i^H) \right) e^{-\lambda \tau}}{\left( \left( p'(\lambda) + (Q'(\lambda) - \tau Q(\lambda)) \right) e^{-\lambda \tau} \right)},
\]

\[
q'(\epsilon_i^H) = -\frac{p_1\epsilon_i^c e^{-\lambda \tau} \left( 1 - 1 + s + \rho \right)}{\left( \left( 1 - 1 + s + \rho \right) \right)},
\]

\[
p'(\epsilon_i^H) = \left( \frac{\left( 1 - 1 + s + \rho \right) e^{-\lambda \tau}}{s - \rho} \right) \left( \frac{p_1\epsilon_i^c \left( 1 - 1 + s + \rho \right)}{s - \rho} \right)
\]

To simplify the computation, we can rewrite the numerator of \( \left( \frac{d\lambda}{d\epsilon_i} \right)_{|\epsilon_i=\epsilon_i^H} \) as \( N(\lambda, \epsilon_i^H) \):

\[
N(\lambda, \epsilon_i^H) = \frac{-\lambda^2 \left( \lambda + p_2 + q_2 e^{-\lambda \tau} - \frac{1 - e^{-\lambda \tau}}{\lambda} \right) \delta q_2}{\epsilon_i^H}.
\]

Indeed:

\[
N(\lambda, \epsilon_i^H) = \left( \frac{p_1\epsilon_i^c + \epsilon_i^c (\delta + \rho)}{\epsilon_i^c} \right) \lambda + \frac{p_0\epsilon_i^c}{\epsilon_i^c} \lambda + \left( \frac{q_1\epsilon_i^c \lambda + q_0\epsilon_i^c}{\epsilon_i^c} \right) e^{-\lambda \tau}
\]

\[
+ \left( 1 - e^{-\lambda \tau} \right) \left( \frac{p_1\epsilon_i^c \lambda + p_0\epsilon_i^c}{\epsilon_i^c} \right)
\]

\[
+ \left( \left( \left( 1 - 1 + s + \rho \right) \right) \delta q_0 \right).
\]

Step 2. We substitute \( \lambda = i\omega_0 \) respectively in the numerator and in the denominator of \( \left( \frac{d\lambda}{d\epsilon_i} \right)_{|\epsilon_i=\epsilon_i^H} \). Substitution in the numerator yields

\[
N(i\omega_0, \epsilon_i^H) = a + ib
\]

where

\[
a = \frac{a^2}{\epsilon_i^H} \left( p_2 + q_2 \cos(\omega_0\tau) - \frac{\sin(\omega_0\tau)\delta q_2}{\omega} \right)
\]

\[
b = \frac{a^2}{\epsilon_i^H} \left( \omega_0 - q_2 \sin(\omega_0\tau) + \frac{1 - \cos(\omega_0\tau)}{\omega_0} \delta q_2 \right).
\]

Substitution of \( \lambda = i\omega_0 \) in the denominator \( D(\lambda, \epsilon_i^H) \) of \( \left( \frac{d\lambda}{d\epsilon_i} \right)_{|\epsilon_i=\epsilon_i^H} \) yields

\[
D(i\omega_0, \epsilon_i^H) = c + id
\]

where

\[
c = - \left( 3 + \tau p_2 \right) \omega_0^2 + \left( p_1 + \tau p_0 \right) \omega_0 + 2q_2 \omega_0 \sin(\omega_0\tau)
\]

\[
d = - \tau \omega_0^2 + \left( 2p_2 + \tau p_1 \right) \omega_0
\]

\[
- \sin(\omega_0\tau) q_1 + 2q_2 \omega_0 \cos(\omega_0\tau).
\]

We then compute

\[
\frac{D_2 D_1}{D_1 D_1} \left( i\omega_0, \epsilon_i^H \right) = \frac{ac + bd}{c^2 + d^2}.
\]

As \( i \omega_0 \) is a simple root, \( c^2 + d^2 \neq 0 \). Let us then compute \( ac + bd \) and express it as a function of \( \omega_0 \) that we will denote by \( H(\omega_0) \):

\[
H(\omega_0) = p_1 p_2 + q_1 q_2 + \tau \rho_0 p_2 + 2 \delta \rho_0 q_2 + \tau \delta \rho_1 q_2 - \tau \rho_0 \rho_1 p_2 + \tau \rho_0 q_2 - \tau \rho_0 \rho_1 q_2
\]

where \( \rho_0, \rho_1, q_2 \) are independent of \( \epsilon_2 \). This equation has only two real roots, but we have seen in proof of Lemma A.1 that there were a large number of \( \epsilon^2 \) for which the Hopf bifurcation occurs. Thus, the transversality crossing condition is satisfied. \( \square \)

A.4. Proof of Theorem 2

We follow the procedure described by Hassard et al. (1981) and we provide various lemmas to derive the dynamics of the system on the center manifold.

**Lemma A.4.** System (20) can be written as the following functional differential equation in \( C \):

\[
\dot{\psi}(t) = \Lambda_x \psi(t) + F(\epsilon, \psi(t), \phi(t))
\]

where \( \phi(t) = \phi(t + \theta) \) and \( \Lambda_x : C \to \mathbb{R}^2 \) is given by \( \Lambda_x \psi = \mathbf{L}(\epsilon^H + \epsilon) \phi(0) + R(\epsilon^H + \epsilon) \phi(-\tau) + M(\epsilon^H + \epsilon) \int_{-\tau}^0 \phi(u) du \) with

\[
L(\epsilon_c) = \begin{bmatrix} -p_2 & -p_1 \\ 1 & 1 \end{bmatrix}, \quad R(\epsilon_c) = \begin{bmatrix} -q_2 & -q_1 \\ 0 & 0 \end{bmatrix}, \quad M(\epsilon_c) = \begin{bmatrix} 0 & 0 \\ -p_0 & 0 \end{bmatrix}.
\]

the coefficients \( p_i, q_i \) being defined in (24), and \( F(\epsilon, \psi) = G(\epsilon, \psi) - \Lambda_x \psi \).

**Proof.** The equation in the \( y \) variable is written as follows:

\[
\dot{y}(t) = \begin{bmatrix} f_1(y(t) + \bar{K}, y(t) - y(t - \tau)) \\ + \delta \bar{K} + \delta \int_{t-\tau}^{t} y(s) ds - \delta \end{bmatrix}
\]

\[
+ f_2(y(t) + \bar{K}, y(t) - y(t - \tau) + \delta \bar{K} + \delta \int_{t-\tau}^{t} y(s) ds) \times [y(t) - y(t - \tau) + \delta [y(t) - y(t - \tau)]]
\]

\[
- \epsilon_c \begin{bmatrix} f(y(t) + \bar{K}, y(t) - y(t - \tau)) \\ + \delta \int_{t-\tau}^{t} y(s) ds - \delta \bar{K} - \delta \dot{y}(t) - \dot{y}(t) \end{bmatrix}
\]

\[
+ \delta \int_{t-\tau}^{t} y(s) ds - \delta \bar{K} - \delta y(t) - \dot{y}(t)
\]

The linearization of the system at \((0, 0, 0)\) is

\[
\dot{\psi}_1(t) = [\rho + f_2] \psi_1(t) + f_2 \psi_1(t - \tau)
\]

\[
+ [f_2 \delta - \epsilon_c \bar{K}_1] \psi_2(t)
\]

\[
- [\bar{K} \phi_3 - \bar{K} \bar{K} \bar{K}_2] \psi_2(t - \tau)
\]

\[
\dot{\psi}_2(t) = \psi_1(t)
\]

\[
\dot{\psi}_3(t) = \psi_2(t).
\]

Let \( F : \mathbb{R} \times C \to \mathbb{R} \) and denote the partial derivatives of \( F \) as \( f_i = f_i(\bar{K}, \delta \bar{K}), f_{ij} = f_{ij}(\bar{K}, \delta \bar{K}), f_{ijkl} = f_{ijkl}(\bar{K}, \delta \bar{K}), f_{ijklm} = f_{ijklm}(\bar{K}, \delta \bar{K}), i, j, k, l = 1, 2 \).

The following lemma gives a Taylor expansion up to order three of \( F(\epsilon, \psi_{1t}(0), \psi_{1t}(-\tau), \phi_{2t}(0), \phi_{2t}(-\tau), \int_{-\tau}^{t} \phi_{2t}(u) du) \).

**Lemma A.5.** Let \( (\psi_{1t}(0), \psi_{1t}(-\tau), \phi_{2t}(0), \phi_{2t}(-\tau), \int_{-\tau}^{t} \phi_{2t}(u) du) = (x_1, x_2, x_3, x_4, x_5) \). Then

\[
F(\epsilon, \psi_{1t}(0), \psi_{1t}(-\tau), \phi_{2t}(0), \phi_{2t}(-\tau), \int_{-\tau}^{t} \phi_{2t}(u) du)
\]

\[
= \sum_{i=1}^{5} \sum_{j=1}^{5} a_{ij} x_i x_j + \sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{m=1}^{5} a_{ijm} x_i x_j x_m
\]
with \( a_0 \) and \( a_{j-1} \) some coefficients that depend on the second, third and fourth order derivatives of the production function evaluated at the steady state.\(^{21}\)

We can also formulate \( \Lambda \varphi \) as follows:

\[
\Lambda \varphi = \int_{-\tau}^{0} d\eta(u) \varphi(u)
\]

with

\[
d\eta (\varepsilon, u) = L (\varepsilon) \delta (u) + R (\varepsilon) \delta (u + \tau) + M (\varepsilon) \, du.
\]

To determine the normal form, the projection method is used as in Hassard et al. (1981). We first need to compute the eigenvector relative to the eigenvalue \( i\omega_0 \). Instead of writing the delay dynamic system, we use the infinitesimal generator expression, as is usually done for delay functional differential equations. For \( \varphi \in C^1 ([-\tau, 0], \mathbb{R}^2) \), let us define

\[
A (\varepsilon) \varphi = \begin{cases} 
\frac{d\varphi (\theta)}{d\theta}, & \text{if } \theta \in [-\tau, 0] \\
L (\varepsilon) \varphi (0) + R (\varepsilon) \varphi (-\tau) + M (\varepsilon) \int_{-\tau}^{0} \varphi (u) \, du, & \text{if } \theta = 0
\end{cases}
\]

and

\[
G (\varepsilon) \varphi = \begin{cases} 
\{0, & \text{if } \theta \in [-\tau, 0] \\
F (\varepsilon, \varphi), & \text{if } \theta = 0
\end{cases}
\]

It follows that (28) is equivalent to

\[
\dot{\varphi}_1 = A (\varepsilon) \varphi_1 + G (\varepsilon) \varphi_2.
\]

To compute the normal form on the center manifold, we use the projection method, which is based on the computation of the eigenvector relative to \( i\omega_0 \) and the corresponding adjoint eigenvector. The computation of the adjoint eigenvector requires the definition of the adjoint space and the adjoint operator of \( A (\varepsilon) \).

We define the adjoint space \( C^* \) of \( C^1 \) by \( C^* : = \{ \chi : [0, \tau] \to \mathbb{R}^2 \} \) with the adjoint operator \( \Lambda^* (\varepsilon) \).

\[
\Lambda^* (\varepsilon) \chi = \begin{cases} 
- \frac{d\chi (\sigma)}{d\sigma}, & \text{for } \sigma \in [0, \tau] \\
\int_{-\tau}^{0} d\eta^* (\varepsilon, t) \chi (-t), & \text{for } \sigma = 0.
\end{cases}
\]

**Remark 2.** As \( d\eta (\varepsilon, t) \) is real, we have \( d\eta^* (\varepsilon, t) = d\eta (\varepsilon, t) \).

We consider the bilinear form

\[
\langle v, u \rangle = v^T (0) u (0) - \int_{0}^{\theta} \int_{0}^{\theta} v^T (\xi - \theta) d\eta (\varepsilon) \, u (\xi) \, d\xi
\]

\[
= \left( v^T (0) u (0) - \int_{0}^{\theta} v^T (\xi + \tau) R (\varepsilon) u (\xi) \, d\xi \right.
\]

\[
- \left. \int_{0}^{\theta} \int_{0}^{\theta} v^T (\xi - \theta) M (\varepsilon) u (\xi) \, d\xi \, d\theta \right)
\]

The following lemma now provides a basis for the eigenspace and the adjoint eigenspace.

---

\(^{21}\) The expressions of these coefficients are available upon request.
Let \( \eta(0) = u = (u_1, u_2)^t \), the previous expression is rewritten as follows:

\[
\begin{align*}
-u_1 \left( p_2 + q_2 e^{-i\omega \tau} \right) + u_2 &= i \omega_0 u_1 \\
-u_1 \left( p_1 + q_1 e^{-i\omega \tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u} du \right) &= i \omega_0 u_2.
\end{align*}
\]

Substituting \( u_2 \) in the first expression we have

\[
\begin{align*}
-u_1 \left( p_2 + q_2 e^{-i\omega \tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u} du \right)
+ \left( p_1 + q_1 e^{-i\omega \tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u} du \right)
= 0
\end{align*}
\]

\[
-u_1 \left( p_1 + q_1 e^{-i\omega \tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u} du \right) = i \omega_0 u_2.
\]

\( u_1D(i\omega_0) = 0 \)

\[
-u_1 \left( p_1 + q_1 e^{-i\omega \tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u} du \right) = i \omega_0 u_2.
\]

As \( i \omega_0 \) is a root of the characteristic equation, we can choose \( u_1 \) as we want, so \( u \) can be rewritten as follows:

\[
u = u_1 \left( \frac{1}{i} \left( p_1 + q_1 e^{-i\omega \tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u} du \right) \right).
\]

We now compute \( u_1 \) thanks to equation \( (q^*, q) = 1 \), which leads to:

\[
\Pi_1 = \begin{bmatrix}
-\omega_0 & -1 & \omega_0 \tau \left( i\omega_2 \omega_0 + q_1 \right) e^{-i\omega \tau} \\
\omega_0 & 0 & -\omega_0 \tau \left( i\omega_2 \omega_0 + q_1 \right) e^{-i\omega \tau} \\
\omega_0 & 0 & 0
\end{bmatrix}^{-1}.
\]

\textbf{Remark 3.} Computations lead to \((q^*, \bar{q}) = 0\).

Let \( \varphi(t) \) be a solution of Eq. \((30)\) when \( \varepsilon = 0 \). We associate a pair \((z, w)\) where \( z(t) = (q^*, \varphi_1) \).

\textbf{Solutions} \( \varphi_1(\theta) \) on the central manifold are given by

\[
\varphi_1 = w(z, \bar{z}, \theta) + z(t) q(\theta) + \bar{z}(t) \bar{q}(\theta).
\]

\textbf{Let us denote} \( w(t, \theta) = w(z, \bar{z}, \theta) \), where \( z \) and \( \bar{z} \) are local coordinates for the central manifold in directions \( q^* \) and \( \bar{q^*} \), and \( F_0(z, \bar{z}) = F(0, w(z, \bar{z}, 0) + 2Re(z(\bar{z}q)(0))) \). Hassard et al. \(1981\) then show that the dynamic on the center manifold is given by

\[
z(t) = i \omega_0 + g(z, \bar{z})
\]

and in the neighborhood of \( \varepsilon^\varepsilon \) the complete dynamic of the system is given by

\[
\begin{align*}
\dot{w}(t, \theta) &= A_0 w(t, \theta) - 2Re(\hat{g}(z, \bar{z}) q(\theta)) & \text{if } \theta \in [-\tau, 0) \\
\dot{w}(t, \theta) &= A_0 w(t, \theta) - 2Re(\hat{g}(z, \bar{z}) q(\theta)) + \dot{F}_0(z, \bar{z}) & \text{if } \theta = 0.
\end{align*}
\]

Let

\[
w(z, \bar{z}, \theta) = w_20(\theta) z^2 + w_11(\theta) z \bar{z} + w_02(\theta) \frac{\bar{z}^2}{2} + \text{h.o.t.}
\]

\textbf{Lemma A.7.}

\[
F_0(z, \bar{z}) = \left( \frac{\phi_{20}}{2} z^2 + \frac{\phi_{11}}{2} z \bar{z} + \frac{\phi_{02}}{2} \frac{\bar{z}^2}{2} + \frac{\psi_{21}}{2} \frac{\bar{z}^2}{2} + \text{h.o.t.} \right)
\]

with \( \phi_{20}, \phi_{11}, \phi_{02}, \phi_{21} \) some complex functions of the coefficients \( a_i \) and \( d_{ij} \) derived in Lemma A.5.\(\footnote{The expressions of these functions are available upon request.}

\textbf{Proof.} We know from the proof of Lemma A.4 that

\[
F \left( \delta; \varphi_1(0), \varphi_1(-\tau) , \varphi_{20}(0), \varphi_{21}(0) \right) \frac{\partial (u_1(u))}{\partial (u_1(u))} = \sum_{i=1}^{5} \sum_{j=1}^{5} a_{ij} x_i x_j + \sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{m=1}^{5} a_{ijm} x_i x_j x_m.
\]

As on the central manifold we have:

\[
\varphi_1(\theta) = \varphi_1(\theta) + 2Re(\hat{q}(\theta) z(t))
\]

and \( q(\theta) = (i \omega_0, 1)^T e^{i\omega \theta} \), coefficients of the solution can be expressed as:

\[
\varphi_{20}(\theta) = \frac{w_20(\theta) z^2}{2} + w_{11}(\theta) z \bar{z} + \frac{w_{02}(\theta) \bar{z}^2}{2}
+ (t) i \omega_0 e^{i\omega \theta} = \tau(t) i \omega_0 e^{i\omega \theta} + O(z, \bar{z})
\]

\[
\varphi_{21}(\theta) = \frac{w_20(\theta) z^2}{2} + \frac{w_{11}(\theta) z \bar{z}}{2} + \frac{w_{02}(\theta) \bar{z}^2}{2}

\]

\( \tau(t) e^{i\omega \theta} + \tau(t) \bar{z} e^{i\omega \theta} + O(z, \bar{z}) \).

We use the preceding formula to compute \( (\varphi_{20}(\cdot))_{p=1,3} \). Then replacing \((\varphi_{20}(\cdot))_{p=1,3} \) in \((33)\) we obtain coefficients \( \phi_{20}, \phi_{02}, \phi_{11} \) and \( \phi_{21} \) as in the lemma. \(\square\)

It is worth noting that \( \phi_{20}, \phi_{02}, \phi_{11} \) are obtained as constants, while \( \phi_{21} \) depends on \( w_0(\cdot), \bar{w}_1(\cdot) \), which will be computed later on.

\textbf{Lemma A.8.} \( g_{20} = \Pi_1 \varphi_{20}, g_{11} = \Pi_1 \varphi_{11}, g_{02} = \Pi_1 \varphi_0, \) and \( g_{21} = \Pi_1 \varphi_{21} \).

\textbf{Proof.} As

\[
\bar{q}(\theta) = \Pi_1 \left( \frac{1}{i} \left( p_1 + q_1 e^{-i\omega \tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u} du \right) \right)
\]

\( \Pi_1 \) and using \( g(z, \bar{z}) = \bar{q}^*(\theta) F_0(z, \bar{z}) \) we obtained easily the result of this lemma. \(\square\)

To end the computation of coefficients \( g_{02}, g_{11}, g_{21} \), we need to compute \( w_{11}(\theta) \) and \( w_{20}(\theta) \).

\textbf{Lemma A.9.}

\[
w_{20}(\theta) = E_1 e^{i\omega \theta} + \frac{\bar{q}_{20}}{q_0} q(0) e^{i\omega \theta} + \frac{\bar{q}_{20}}{q_0} q(0) e^{i\omega \theta}
\]

\[
w_{11}(\theta) = -\frac{\bar{q}_{11}}{q_0} q(0) e^{i\omega \theta} + \frac{\bar{q}_{11}}{q_0} q(0) e^{-i\omega \theta} + E_2
\]

with

\[
E_1 = \left( \frac{2i \omega_0}{\Delta (2i \omega_0)} \right) \phi_{20}
\]

\[
E_2 = \left( \frac{2i \omega_0}{\Delta (2i \omega_0)} \right) \phi_{21}
\]

\textbf{Proof.} We rewrite
\[
\dot{w}(t, \theta) = Aw - 2\Re (g(z, \zeta) q(\theta)) \quad \text{if } \theta \in [-\tau, 0)
\]
\[
\dot{w}(t, 0) = Aw - 2\Re (g(z, \zeta) q(0)) + F_0(z, \zeta) \quad \text{if } \theta = 0
\]
as
\[
\dot{w} = Aw + H(z, \zeta, \theta). \tag{34}
\]
And we consider a Taylor expansion of \(H(z, \zeta, \theta) = H_{20} z^2 + H_{11} z \zeta + H_{00} \zeta^2 + \text{h.o.t. As}
\]
\[
H(z, \zeta, \theta) = -g(z, \zeta) q(\theta) - \overline{g(z, \zeta)} \overline{q(\theta)} \quad \text{if } \theta \in [-\tau, 0)
\]
we have
\[
H_{20}(\theta) = -g_{20} q(\theta) - \overline{g_{20}} \overline{q(\theta)} \quad \text{if } \theta \in [-\tau, 0)
\]
\[
H_{11}(\theta) = -g_{11} q(\theta) - \overline{g_{11}} \overline{q(\theta)} \quad \text{if } \theta \in [-\tau, 0).
\]

Moreover, we have on the central manifold
\[
w(z, \zeta, \theta) = w_{20} z^2 + w_{11} z \zeta + w_{00} \zeta^2 + \text{h.o.t.}
\]
which implies that
\[
dw(z, \zeta, \theta) = w_{20} z^2 + w_{11} (z \zeta + \zeta z) + w_{00} \zeta^2 + \text{h.o.t.}
\]
This is rewritten as follows, as \(\dot{z} = i \omega z + g(z, \zeta)
\]
\[
dw(z, \zeta, \theta) = 2 w_{20} z^2 + \overline{\omega_0} \overline{z} + \overline{w_{11}} \zeta \overline{z} + \overline{w_{00}} \overline{z}^2 + \overline{\text{h.o.t.}}
\]
which is rewritten as follows, using the definition of operator \(A,
\]
\[
\dot{w}_{20}(\theta) = 2i \omega_0 w_{20}(\theta) + g_{20}(\theta) - \overline{g_{20}} \overline{q}(\theta)
\]
\[
A w_{11}(\theta) = -g_{11} q(\theta) + \overline{g_{11}} \overline{q}(\theta)
\]

\text{Solving this equation, we obtain}
\[
w_{20}(\theta) = E_1 e^{i \omega_0 \theta} + \frac{i \omega_0}{q_0} q(0) e^{i \omega_0 \theta} + \frac{\overline{\omega_0}}{3q_0} \overline{q}(0) e^{-i \omega_0 \theta}.
\]
In a similar way, we have
\[
w_{11}(\theta) = g_{11} q(\theta) + \frac{\overline{g_{11}}}{q(\theta)} e^{i \omega_0 \theta} + E_2
\]
where \(E_1 \) and \(E_2 \) can be determined with initial conditions
\[
H(z, \zeta, 0) = -2 \Re (g(z, \zeta) q(0)) + f_0(z, \zeta)
\]
that is
\[
H_{20}(0) = -g_{20} q(0) - \overline{g_{20}} \overline{q}(0) + \left( \frac{\phi_{20}}{0} \right)
\]
\[
H_{11}(0) = -g_{11} q(0) - \overline{g_{11}} \overline{q}(0) + \left( \frac{\phi_{11}}{0} \right)
\]

Remembering the definition of \(A \) and
\[
(2i \omega_0 - A) w_{20}(\theta) = -g_{20} q(\theta) - \overline{g_{20}} \overline{q}(\theta)
\]
\[
Aw_{11}(\theta) = -g_{11} q(\theta) + \overline{g_{11}} \overline{q}(\theta)
\]
we have
\[
2i \omega_0 w_{20}(0) + g_{20} q(0) + \overline{g_{20}} \overline{q}(0) = L(e_i \omega_0)(0) + R(e_i \omega_0)(-\tau)
\]
\[
+ M(e_i \omega_0) \int_{-\tau}^0 w_{20}(u) du + \left( \frac{\phi_{20}}{0} \right)
\]
\[
2i \omega_0 w_{20}(0) = E_1 + \frac{i \omega_0}{q_0} q(0) + \frac{\overline{\omega_0}}{3q_0} \overline{q}(0)
\]
\[
w_{20}(0, -\tau) = E_1 e^{-i \omega_0 \tau} + \frac{i \omega_0}{q_0} q(0) e^{i \omega_0 \tau} + \frac{\overline{\omega_0}}{3q_0} \overline{q}(0) e^{-i \omega_0 \tau}.
\]

Using the fact that \(L(e_i \omega_0)(0) - R(e_i \omega_0)(-\tau) - M(e_i \omega_0) \int_{-\tau}^0 q(u) du = i \omega_0 q(\theta) \) and that \(q(\theta) \) is the eigenvector of \(A \) according to \(i \omega_0 \), we derive
\[
\left( 2i \omega_0 - L(e_i \omega_0) - R(e_i \omega_0) e^{-i \omega_0 \tau} - M(e_i \omega_0) \int_{-\tau}^0 e^{i \omega_0 \tau} du \right) E_1 = \left( \frac{\phi_{20}}{0} \right)
\]
which implies
\[
\left[ \left( 2i \omega_0 + p_2 + q_2 e^{-2 i \tau \omega} \right) E_1 + \left[ p_1 + q_1 e^{-2i \tau \omega} + p_0 \int_{-\tau}^0 e^{2 i \omega_0 \tau} du \right] E_2 \right] = \phi_{20}
\]
\[
2i \omega_0 E_1^2 - E_1 = 0
\]
and thus
\[
E_1 = \frac{\phi_{20}}{\Delta(2i \omega_0)}
\]
\[
E_1 = 2i \omega_0 \frac{\phi_{20}}{\Delta(2i \omega_0)}.
\]

Similarly, we have
\[
\left[ \left[ (p_2 + q_2 e^{-2i \tau \omega}) E_2 + \left[ p_1 + q_1 e^{-2i \tau \omega} + p_0 \int_{-\tau}^0 e^{2 i \omega_0 \tau} du \right] E_2 \right] E_2 \right] = \phi_{11}
\]
which implies
\[
E_2 = \frac{-\phi_{11}}{(p_2 + q_2 e^{-2i \tau \omega})}
\]
\[
E_2^2 = 0.
\]
Consider Lemma A.3 and let us denote $\lambda^* (c^H) = (dx/dc_t)|_{c_t=\alpha t}$. From all these computations we derive the formula

$$C_1 = \frac{i}{2 \omega_0} \left( \frac{2 \omega_0 g_{11} - 2 |g_{11}|^2 - \frac{1}{3} \frac{\omega_2}{\omega_3} |g_{12}|^2}{\omega_0 \lambda^* (c^H)} \right) + \frac{\omega_1}{2},$$

$$\mu_1 = \frac{-\text{Re} (C_1)}{\text{Re} (\lambda^* (c^H))},$$

$$\mu_2 = \frac{-\text{Im} (C_1) + \lambda^* \text{Im} (\lambda^* (c^H))}{\omega_0}.$$

(37)

and the result follows from Hassard et al. (1981). □

A.5. Proof of Proposition 2

Let us linearize Eq. (12) around the steady state and rewrite the dynamics as

$$\begin{align*}
\dot{k}(t) &= y(t) \\
\dot{y}(t) &= \alpha \int_0^t k(t + x) \phi(x) dx + bk(t) + a\omega(t)
\end{align*}$$

with

$$\begin{align*}
a &= \frac{\rho + \beta (1 - 1/\delta + \rho)}{1 - \beta (1 - 1/\delta + \rho)}
\end{align*}$$

and

$$\begin{align*}
b &= \frac{\alpha \int_0^\infty e^{-\eta s} \phi(x) dx - \beta \lambda - a}{1 - \beta (1 - 1/\delta + \rho)}.
\end{align*}$$

Initial conditions for these dynamics are such that $k(t) = k_0(t)$ and $y(t) = y_0(t)$, both functions in $C([-\tau, 0])$. Moreover for $t \in [-\tau, 0]$, we need to have $y(t) = 0$. It follows that $k(t)$ has to be in $C^1([-\tau, 0]) \cap C([-\tau, 0])$, and that it constrains $y_0(t)$ for $t \in [-\tau, 0]$. The characteristic matrix is

$$J(\lambda) = \begin{pmatrix} \lambda & -1 \\ -\alpha \int_0^\infty e^{-\eta s} \phi(x) dx - \beta & \lambda - a \end{pmatrix}.$$

Let $n \geq 1$ be the number of eigenvalues with $\text{Re} (\lambda) \geq 0$. Assume first that we have a simple eigenvalue $\lambda_0 > 0$ and that this is the only eigenvalue with $\text{Re} (\lambda) \geq 0$, i.e. $n = 1$. The criteria for an initial condition $(k_0(\cdot), y_0(\cdot))$ with $t \in [-\tau, 0]$ to be continued to a bounded solution for $t \geq 0$ are then given by

$$Q_0(k_0(\cdot), y_0(\cdot)) = 0$$

where $Q_0$ is the spectral projection (the reasoning is the same if we have a set $\Sigma$ with a finite number $n > 1$ of roots with positive real part, except that we have to consider the spectral projection on the set vect $(e^s, \theta \in \Sigma)$).

According to the spectral projection formula given by equation (IV.3.3) in Diekmann et al. (1995), then

$$(Q_0(k_0(\cdot), y_0(\cdot))) (\theta^t) = e^{i\theta t} \left( \left( k_0(0), y_0(0) \right) \int_0^{-\theta t} \phi(x) e^{i\theta x} e^{-\eta s} k_0(u) dudx \right)(\theta^t)$$

where $H$ has to be now determined. We are looking for some $H$ of the form $H = pq^t$, with some vectors $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$, such that

$$\begin{align*}
\begin{pmatrix} f(\lambda_0) & \alpha t \\
f(\lambda_0)^t & 0
\end{pmatrix} & = 0
\end{align*}$$

where $f(\lambda_0)$ is the transpose of $f(\lambda_0)$. We thus find that $q = (\lambda_0 - a, 1)$ and $p = (1, \lambda_0)$. Explicitly $H$, $Q_0$ $(k_0(\cdot), y_0(\cdot)) = 0$ implies that

$$\begin{align*}
y_0(0) + (\lambda_0 - a) k_0(0) \\
+ \alpha \int_0^{-\theta t} \phi(x) e^{i\theta x} \int_0^\infty e^{-\eta s} k_0(u) dudx = 0.
\end{align*}$$

(38)

Considering now that $k(t) = k_0(t)$ is given for $t \in [-\tau, 0]$ and that $y_0(t)$ must satisfy $y_0(t) = k_0(t)$ for $t \in [-\tau, 0]$, when $n = 1$ the only degree of freedom to solve (locally) our problem is to choose $y(t) \in C([-\tau, 0])$ with the condition that $y_0(0)$ is specified as $y_0(0) = -(\lambda_0 - a) k_0(0) - \alpha \int_0^{-\theta t} \phi(x) e^{i\theta x} \int_0^\infty e^{-\eta s} k_0(u) dudx$. On the contrary, if $n > 1$ we derive, applying the same argument as above, $n$ constraints similar to (38). But as $y_0(0)$ is the only unknown, there does not exist any solution. □

A.6. Proof of Theorem 3

For $\tau = 0$ the characteristic equation admits one positive real root and one negative real root. Assuming that $\tau > 0$ implies that an infinite number of roots with negative real part will now exist. Our aim here is to prove that for a nonempty set of values of $\tau$, the characteristic equation admits only one positive real root.

To prove this, we show that there exists $\tau > 0$ such that for $0 < \tau < \tau$, there are no pure imaginary roots. According to the proof of Lemma A.1, under Assumption 2, if there exists a pure imaginary root $\omega_0$, $\omega_0 > 0$, it has to solve

$$\begin{align*}
\begin{pmatrix} A(\varepsilon_c) + \tau \varepsilon_c \omega_c + \tau \omega_c & \chi_0 \\
\omega_0 & x
\end{pmatrix} &= 0
\end{align*}$$

$$\begin{align*}
\omega_0 &= x.
\end{align*}$$

Let us consider the critical value $\varepsilon_c$ as defined by (18). We get

$$\begin{align*}
\eta(\varepsilon_c) &= \frac{1 - s - \rho \delta}{\alpha \delta \tau^2} \left( \delta \tau - \beta \right)
\end{align*}$$

$$\begin{align*}
+ \frac{\beta (1 - s - \rho \delta)^2}{\delta \tau^2} + \frac{\rho^2}{2}.
\end{align*}$$

$$\begin{align*}
\delta \tau - \beta
\end{align*}$$

$$\begin{align*}
\varepsilon_c(\delta) = \varepsilon_c(\delta) \left( 1 - s - \rho \delta \right) - \delta \tau
\end{align*}$$

$$\begin{align*}
\frac{\beta}{\tau^2} \left( \delta \tau^2 + \frac{\beta}{\tau^2} \right) - \frac{1}{\tau} + O(1).
\end{align*}$$

It follows that there exists $\tau_1$, such that for $\tau \in (0, \tau_1)$, we have

$$(1 - s - \rho \delta)^2 - \delta \tau \left( 1 - s - \rho \delta \right) > 0$$

if

$$\delta \tau - \beta \left( 1 - s - \rho \delta \right) > 0$$

and

$$\tau \leq \frac{1}{2} \left( \frac{1}{\rho} - 1 \right).$$

If $s \in (0, 1/2)$, the previous condition is always satisfied. We then conclude that when $\tau \in (0, \tau_1)$, $\eta(\varepsilon_c) > 0$. Let us now consider $\tau \equiv \min(\tau_1, 2\beta/\delta)$. According to Theorem 1, for $\varepsilon_c \in [\varepsilon_c, \varepsilon_c]$, the characteristic polynomial admits no pure imaginary roots. Thus for $\tau \in (0, \tau)$ and $\varepsilon_c \in [\varepsilon_c, \varepsilon_c]$, there exists only one root with positive real part which corresponds to the unique positive real root. □

References

