Optimal pollution control with distributed delays

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HIGHLIGHTS

- Pollution control model with distributed delays reflecting space/time heterogeneity.
- Analysis of two-dimensional systems of mixed type functional differential equations.
- Full localization of roots of dynamic system with general advance and delay kernels.
- Hopf bifurcation theorem with general advance and delay kernels.
- Sensitivity of dynamics to distributions' parameters.

ABSTRACT

We present a model of optimal stock pollution control with general distributed delays in the stock accumulation dynamics. Using generic functional forms and a distribution structure covering a wide range of distributions, we solve analytically the complex dynamic system that arises from the introduction of these distributed delays. From a theoretical standpoint, our contribution extends the dynamic optimization literature that focused on single discrete delays and develops an original method to address control problems written as mixed type functional differential equations with general kernels. Our results show the qualitative impact of acknowledging these distributed delays on the optimal pollution paths dynamics. We study analytically the properties of the dynamics and we identify the conditions for the occurrence of limit cycles. This theoretical work contributes to the design of efficient environmental policies in the presence of complex delays.

1. Introduction

Since the seminal contributions of Keeler et al. (1972) and Plourde (1972), partial equilibrium stock pollution control models have been discussed and enriched in various ways with the introduction of uncertainty, multiple pollutants, irreversibility, technological change, etc. However, apart from a few exceptions presented below, this vast literature systematically assumes that the time of emission is tantamount to the time of contamination. This assumption leaves out a crucial aspect of many pollution problems that feature significant delays in the accumulation process. For instance, the contamination of aquifers by leaching nitrates from agricultural sources can occur several decades later (Kim et al., 1993), which in some cases explains why reductions in nitrogen-loaded inputs are not immediately followed by a decrease in downstream water pollution (Grimval et al., 2000).

From a theoretical point of view, the addition of these delays to the standard optimal stock pollution control framework modifies the properties of the optimal pollution path. Winkler (2011) studies the properties of a model with a discrete delay depending on whether the objective function is separable or not. Using a separable objective function in a model with heterogeneous polluters, Bourgeois and Jayet (2011) show that longer time lags lead to a higher optimal pollution stock at the steady state and that this effect is amplified by asymmetric information. The common feature of these contributions is that they use a single discrete delay, assuming that an emission at time \( t \) will reach entirely and systematically the pollution stock at time \( t + \tau \). This kind of delay merely translates the dynamic path and leaves its mathematical properties relatively unaffected.

These models with a discrete delay imply nonetheless that the accumulation process is perfectly homogeneous and they ignore

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the possibility of differentiated accumulation velocity of polluting emissions. Site-specific conditions, such as soil heterogeneity in the case of water contamination by nutrients or temperature and pressure variations in the case of greenhouse gases, can cause significant variability in the time frame of pollution. To better capture the intricate lags phenomena, the time of accumulation of these pollutants should in fact be distributed along a time interval following the emission. Such distributed delays cast light on the challenging task of assessing the link between the time and amount of emissions and the time and intensity of the damage they trigger. They raise significant technical difficulties which, contrary to the single discrete delay model, cannot be overcome easily even for a separable objective function. The application of Pontryagin’s principle to this model with distributed delays gives rise to a system of optimality conditions that includes at the same time leads and lags, turning the system into Mixed Type Functional Differential Equations (MFDE). In a different field, the model developed by Buonorno and d’Onofrio (2013) to analyze the optimal conduct of politicians when public awareness of their honesty is delayed in time also presents some mathematical similarities to the one we address. Nevertheless we study a general distribution of delays while they focus on two specific delay kernels, which leads to significant differences in the resolution of the problem that are discussed afterwards.

Our aim is to characterize analytically these complex dynamics and the stability conditions of a model using general functional forms. To do so we extend the approach used by Bocekkine et al. (2005, 2010) who apply Pontryagin’s approach to vintage capital\(^1\) and we resort to an original method to address the MFDE at stake. Our main contribution to the optimal control literature consists in establishing analytically several properties that enable us to locate the roots of the characteristic equations of this range of models, and thus to characterize the qualitative properties of the dynamics. This is made possible by our use of a general delays kernel which covers a wide range of distribution structures. Consequently, our results enrich economic theory by showing that when a truly general form for delays is considered in dynamic problems (beyond simple discrete delays or exponential kernels as it is usually done), a very large range of qualitative properties for the dynamics can be encountered, including limit cycles. This theoretical contribution finds significant applications in the design and calibration of environmental policies but could also be applied beyond this field to various other economic dynamic systems exhibiting similar delays (advertising, capital building, ...).

Section 2 presents the optimal stock pollution control model with general delays. Section 3 is devoted to the study of the dynamics: we write the dynamics as an MFDE and we analyze the properties of the characteristic equations. We then derive properties in terms of optimal trajectories, studying the impact of the various parameters involved in the economic model. Section 4 concludes.

2. Introducing distributed delays in the standard pollution control model

We consider the introduction of delays in a standard dynamic partial equilibrium model including a representative producer/polluter and the environmental damage sustained by society. The standard social planner problem is

\[
\max_{p(t)} \int_0^{\infty} \left[ f(p(t)) - D(c(t)) \right] e^{-\rho t} dt
\]

where \(f(p(t))\) is the private benefit derived from the emissions \(p(t), D(c(t))\) is the environmental damage caused by the pollution stock \(c(t)\) and \(\rho\) is the social discount rate with \(\rho \in [0, 1]\). \(f\) and \(D\) have the standard properties of the literature: \(f\) positive, non-decreasing, concave, defined over \(\mathbb{R}^+\) and respecting the Inada conditions and \(D\) increasing, convex and such that \(D(0) = 0\).

We consider a general expression of the pollution accumulation process that allows for various forms of pollution diffusion. The accumulation equation can be written, with \(\alpha > 0\) being the natural decay rate of pollution, as

\[
\dot{c}(t) = -\alpha c(t) + \theta \int_{t-\tau_1}^{t-\tau_2} p(u) \mu(t-u) du,
\]

where \(0 \leq \tau_1 < \tau_2 < \infty\) and \(\mu(.)\) is a probability density function on \([\tau_1, \tau_2]\) such that \(\int_{\tau_1}^{\tau_2} \mu(u) du = 1\).

\(\theta (\theta \in [0, 1])\) is the "technological" factor reflecting the portion of the pollution generated by the economic activity that will leak into the environment.\(^2\) This factor thus depends on local productive and environmental conditions. The cleaner the production process or the better calibrated the fertilizer application, the lower \(\theta\).

Expression (2) is convenient to embrace a wide range of distribution structures. The choice of function \(\mu(.)\) will depend on the specific accumulation process of the problem considered. Pollution emitted at time \(t\) will be released in several weighted loads, distributed \(a\) priori across a time interval \([t + \tau_1, t + \tau_2]\). More precisely, pollution emitted at time \(t\) will reach the stock at time \(t + \tau + u\), with weight \(\mu(t + u)\), where \(u \in [\tau_1, \tau_2]\). Expressing the delayed accumulation process \(\mu(s)ds\) as a probability density function allows us to encompass a large range of complex pollution problems characterized by significant site-specific heterogeneity in the pollutants velocity (Gaines and Gaines, 1994) that convert a spatial heterogeneity into a temporal one.

Before solving the general problem, let us give a few examples of possible distribution specifications. It is clear that if \(\mu(v) = \delta(v),\) where the delay \(\tau\) is strictly positive and \(\tau \in [\tau_1, \tau_2],\) the dynamic reflects the standard homogeneously delayed accumulation process such as is found in Winkler (2008). Another relevant example is

\[
\mu(v) = e^{-\vartheta v} \int_{\tau_1}^{\tau_2} e^{-\vartheta s} ds \text{ for } v \in [\tau_1, \tau_2].
\]

Here the parameter \(\vartheta\) sets the repartition of the pollution load in time: the higher \(\vartheta\) the earlier the emissions reach the stock within the time interval \([\tau_1, \tau_2]\).

3. Optimal pollution control with distributed delays

In order to highlight the specificities of our results with distributed delays, let us briefly recall the salient properties of the optimal pollution control problem with discrete delays such as they have been characterized by Winkler (2008, 2011). Under the assumption of a separable utility function such as the one in (1), the core properties of the benchmark optimal control model are preserved in the presence of discrete delays. The latter operates a mere “translation” of the steady state and of the corresponding optimal path towards a higher pollution stock. The saddle path property, the monotonicity of the optimal path and even the finite dimension of the stable manifold are maintained. Furthermore, the longer the delay the higher the pollution stock and the emission level at the steady state.

\(^1\) Another strand of literature uses the Hamilton-Jacobi-Bellman equation to solve delayed models for which a closed form can be obtained (Federico et al., 2010).

\(^2\) In the case of nitrate contamination \(\vartheta\) is tantamount to the portion of the fertilizers applied that are not assimilated by the crops and other local organisms. For polluting gases it corresponds to the portion of gas produced that has not been captured by end of pipe abatement devices.
Let us now address the general problem

\[
\max_{p(t)} \int_0^{\infty} \left[ f'(p(t)) - D(c(t)) \right] e^{-\rho t} dt
\]

s.t. \[ \dot{c}(t) = -\alpha c(t) + \theta \int_{t-\tau}^{t-\tau_1} p(u) \mu(t-u) du, \]
\[ c(t) \geq 0, \ \beta \geq p(t) \geq 0, \]
\[ c(\tau^0) = c_0 (\tau^0) \in C([-\tau, 0]) \quad \text{given for } \tau \in (t_2, 0], \]
\[ p(c) = p_0 (c) \in \mathcal{C}^\theta ([-\tau^1, 0)) \quad \text{given for } \tau \in (t_2, 0), \]

where \( \mathcal{C}([-\tau, 0]) \) denotes the set of continuous functions on \( [-\tau, 0) \) and \( \mathcal{C}^\theta ([-\tau^1, 0)) \) denotes the set of continuous functions \( y(\cdot) \) on \([-\tau, 0)\) such that \( \lim_{t \to 0^-} y(t) \) exists and is finite.

In contrast to the single discrete delay model, no change of variables, as done by Winkler (2011), can turn the system into a standard problem without delays.

3.1. First order conditions

We define optimal solutions as in Boucekkine et al. (2005).

**Definition 1.** A trajectory \((p(t), c(t)), t \geq 0\) with \(p(t)\) piecewise continuous and \(c(t)\) piecewise differentiable, is admissible if it satisfies (2), constraints \(c(t) \geq 0\) and \(p(t) \geq 0\), and if the objective function converges.

A trajectory \((p(t), c(t)), t \geq 0\) is an optimal solution if it is admissible and optimal in the set of admissible trajectories.

Existence of optimal solutions follows from Boucekkine et al. (2005). The next lemma provides necessary conditions along the optimal path. Like Boucekkine et al. (2005), we resort to traditional tools, using Lagrangian perturbation, to derive these conditions.

**Lemma 2.** Let \((p(t), c(t)), t \geq 0\) be an optimal solution for \(t \geq 0\). There exists a piecewise differentiable function \(x(t)\), which corresponds to the current value shadow price of pollution, and a piecewise continuous function \(w_1(t)\) such that

\[
f'(p(t)) + w_1(t) = -\theta \int_{t+\tau_1}^{t+\tau_2} x(u) e^{-\rho(t-u)} \mu(u-t) du,
\]

\[
\dot{x}(t) = D'(c(t)) + x(t) (\alpha + \rho),
\]

with the slackness condition

\[
p(t) \geq 0, \ w_1(t) \geq 0, \ w_1(t) p(t) = 0,
\]

and the transversality condition

\[
\lim_{t \to \infty} x(t) c(t) e^{-\rho t} = 0.
\]

**Proof.** The proof is given in Appendix A.1. ■

In the remaining part of this paper we will study the optimal interior trajectories, for which \(p(t) > 0\).

The first order condition (3) expresses in a framework with distributed delays the traditional trade-off between the marginal benefit triggered by an additional unit of pollution and the marginal damage caused by this pollution, valued by its shadow price. In the benchmark pollution control model without delays, this trade-off takes place between simultaneous emissions, since the emissions at time \(t\) reach the stock at the same instantaneous time \(t\). However, in the presence of delays this trade-off compares the marginal benefit obtained from emissions at time \(t\) with the damages they cause in the time interval \([t + \tau_1, t + \tau_2]\). In order to value the damages caused by the portion \(\mu(t - t)\) at each time \(u\) during this period, we must use the current value shadow price of pollution at that time, \(x(u)\), and we must discount it accordingly over the period \([t, u]\). Hence the right hand term of (3).

### 3.2. Steady state and sensitivity analysis

From (3) we obtain

\[
p(u) = \int_{u+\tau_2}^{u+\tau_1} \int_{u+\tau_2}^{u+\tau_1} x(s) e^{-\rho(t-u)} \mu(s-u) ds du,
\]

which we can substitute in the dynamics to get the following system with both leads and lags:

\[
\begin{align*}
\dot{c}(t) &= -\alpha c(t) + \theta \int_{t-\tau}^{t-\tau_1} p(u) \mu(t-u) du, \\
\dot{x}(t) &= D'(c(t)) + x(t) (\alpha + \rho).
\end{align*}
\]

**Lemma 3.** There exists a unique steady state \((c^*, x^*)\) and

\[
\frac{dc^*}{dt} > 0, \quad \frac{d\dot{c}^*}{dt} < 0.
\]

**Proof.** According to (5), a steady state satisfies

\[
\begin{align*}
\alpha c^* &= \theta \int_{t_1}^{t_2} f^{-1}(s) \int_{t_1}^{t_2} x(s) e^{-\rho(s-t)} \mu(s-t) ds du, \\
x^* &= -D'(c^*).
\end{align*}
\]

Given the properties of \(f\) and \(D\), there exists a unique \(c^*\) solving system (6), that is to say, satisfies

\[
H(c; \tau_1, \tau_2) = 0,
\]

with

\[
H(c; \tau_1, \tau_2) = \frac{1}{\theta} \int_{t_1}^{t_2} f^{-1}(s) \int_{t_1}^{t_2} \left( \frac{\alpha c}{\theta} - D'(c) \right) ds du.
\]

Therefore the steady state \((x^*, c^*)\) exists and is unique. ■

**Lemma 3** shows that the existence of a unique steady state is preserved in the presence of distributed delays. In addition, by applying the implicit function theorem to Eq. (7) we obtain the following comparative statics results:

\[
\begin{align*}
\frac{dc^*}{d\tau_1} &= \frac{1}{\theta} \int_{t_1}^{t_2} \frac{\mu(t_1)}{\theta} f^{-1}(s) \left( \frac{\alpha c}{\theta} - D'(c) \right) ds du > 0, \\
\frac{dc^*}{d\tau_2} &= \frac{1}{\theta} \int_{t_1}^{t_2} \frac{\mu(t_2)}{\theta} f^{-1}(s) \left( \frac{\alpha c}{\theta} - D'(c) \right) ds du < 0.
\end{align*}
\]

The economic interpretation of these comparative statics is quite intuitive. If we look at the delayed accumulation mechanism from the perspective of the interval \([t - \tau_2, t - \tau_1]\), the amount of pollution reaching the stock at time \(t\) will be entirely determined by the flows emitted over this period. By reducing this interval, an increase in \(\tau_1\), all things being equal, will diminish the total pollution received at time \(t\), and thus eventually lead, in our discounted framework, to a higher optimal pollution level \(c^*\) at the steady state. An increase in \(\tau_2\) has the opposite effect: it increases earlier pollution which implies settling at a lower \(c^*\). The impact of a delay variation is similar to the case of discrete delays as it strengthens the discounting effect on environmental damages while private benefits remain unaffected, which allows a higher steady state pollution level.
3.3. Analytical characterization of the dynamics

We are now going to study the local dynamics in the neighborhood of the steady state and determine the conditions under which the latter is a saddle point or may display cyclical dynamics. The stability properties of the local dynamics in the absence of a center manifold relies on a theorem for MFDE similar to the Hartman–Groban theorem for ODEs that has been established in d’Albis et al. (2012). The existence of the cyclical dynamics is based on a Hopf bifurcation’s theorem proved in Hupkes et al. (2008).

For the rest of our analysis we define

\[ B = \theta^2 \frac{D''_{x}}{f''_{x}}, \]

where \( f''_{x} = f''(p^{*}) \) and \( D''_{x} = D''(e^{*}) \).

Our economic interpretation of the dynamic properties of the system will revolve around the value of the (negative) parameter \( B \) compared to the cutoffs \( B^H \) and \( B^L \) defined below.

Linearizing (5) around the steady state yields the following system:

\[
\begin{cases}
\dot{\lambda} (t) = -\alpha \lambda - \theta^2 \int_{-\tau_{1}}^{\tau_{1}} \sum_{t=1}^{2} x(s) e^{-\rho(s-t) u} du, \\
\dot{\lambda} (t) = D'' c (t) + x (t) (\alpha + \rho).
\end{cases}
\]  

(8)

We compute the characteristic equation \( \Delta(\lambda) = 0 \) of system (8) where

\[
\Delta(\lambda) = \det \begin{pmatrix}
\lambda + \alpha & \theta^2 \int_{-\tau_{1}}^{\tau_{1}} \mu(s) \int_{-\tau_{1}}^{\tau_{1}} e^{-\rho(s-t) u} \mu(u) du \\
-D'' & \lambda + (\alpha + \rho)
\end{pmatrix}
\]

\[
= (\lambda + \alpha) (\lambda - (\alpha + \rho)) + B \int_{-\tau_{1}}^{\tau_{1}} \mu(s) e^{-\rho s} ds \times \int_{-\tau_{1}}^{\tau_{1}} e^{-(\rho - \lambda) u} \mu(u) du.
\]

The stability properties of the system will depend on the location of the complex roots of the characteristic equation \( \Delta(\lambda) = 0 \). Characteristic equations of MFDEs are known to have an infinite number of complex isolated roots with positive and negative real parts. Moreover, in our case it can be easily proved by replacing \( \lambda \) by \( (\rho - \lambda) \) in the characteristic equation that these roots are symmetric along the axis \( \xi = \frac{\rho}{2} \).

3.3.1. Preliminary results: location of roots

In this paragraph, we present the location of roots according to the values of the structural parameter \( B \). The results we obtained are presented in Lemmas 4 and 5 and are summarized in Fig. 1.

We define \( \delta_{(0, \rho)} = \{ \lambda \in \mathbb{C} : \Re(\lambda) \in [0, \rho] \} \).

**Lemma 4.** There exist \( B^0 \) and \( B_{\min} \), with \( B_{\min} \leq B^0 \) such that:

- if \( B < B^0 \), the characteristic equation \( \Delta(\lambda) = 0 \) has no real roots.
- if \( B > B^0 \), \( \Delta(\lambda) = 0 \), has four real roots and these roots are outside the set \( \delta_{(0, \rho)} \).
- if \( B > B_{\min} \), \( \Delta(\lambda) = 0 \) has no complex roots in \( \delta_{(0, \rho)} \).

**Proof.** The proof is given in Appendix A.2. \( \blacksquare \)

The case \( B = B^0 \) is a non-generic case with a root \( \lambda_D = \lambda(B^0) \) solving \( \Delta(\lambda_D) = 0 \) and \( \Delta'(\lambda_D) = 0 \).

**Lemma 5.** There exist triplets \( (\mu; \tau_{1}; \tau_{2}) \) such that for sufficiently small \( B \) the characteristic equation admits complex roots in \( \delta_{(0, \rho)} \).

**Proof.** The proof is given in Appendix A.3. \( \blacksquare \)

For such triplets \( (\mu; \tau_{1}; \tau_{2}) \), there thus exist roots such that \( \Re(\lambda) \in [0, \rho] \) when \( B \) is very small. Since there are no roots in \( \delta_{(0, \rho)} \) for \( 0 > B > B_{\min} \), there exists at least one \( B < B_{\min} \) such that the characteristic equation admits pure imaginary roots. Let us call \( B^H \) the highest value of \( B \) for which pure imaginary roots exist.

If \( (\mu; \tau_{1}; \tau_{2}) \) is such that the characteristic equation admits no pure imaginary roots for any \( B \), we write \( B^H = -\infty \).

3.3.2. Saddle point configuration and stability

We first look at the conditions for a saddle configuration.

**Lemma 6.** If \( B > B^H \), the steady state is a saddle point.

**Proof.** A saddle configuration arises whenever the characteristic equation has no pure imaginary roots. The result thus follows directly from the definition of \( B^H \). \( \blacksquare \)

In order to characterize more precisely the saddle property, we can reformulate the problem as it is done in (10). We thus notice that the MFDE in system (5) depends only on the delay \( \tau_{1} - \tau_{2} \) and on the advance \( \tau_{2} - \tau_{1} \). This property relies on the fact that the problem can be broken down into two phases. Indeed, knowing \( p(\sigma) \) for \( \sigma \in [-\tau_{2}, 0] \) enables us to compute \( c(t) \) for \( t \in [0, \tau_{1}] \) and to isolate the corresponding damages over which no control can be exerted. Using the change of variable \( p(t - \tau_{1}) = q(t) \), the problem can then be rewritten as

\[
\int_{0}^{\tau_{1}} -D(c(t)) e^{-\rho t} dt + \max_{q(\cdot)} \int_{\tau_{1}}^{\infty} f(q(t)) e^{\rho t} - D(c(t)) e^{-\rho t} dt \]

s.t. \( \dot{\xi}(t) = -\alpha \xi(t) + \theta_{12} \int_{-\tau_{1}}^{\tau_{1}} q(u + t) \mu(u - t) du \),

\[
c(\sigma) = c_{0}(\sigma) \in E([-\tau_{2} + 2\tau_{1}, \tau_{1}]),
\]

\[
q(\sigma) = q_{0}(\sigma) \in E^{b}([-\tau_{2} + 2\tau_{1}, \tau_{1}]) \text{ given for } \sigma \in [-\tau_{2} + 2\tau_{1}, \tau_{1}].
\]

We can thus focus on the second term of the program (10). Although the problem depends highly on initial conditions being given on \([-\tau_{2}, 0]\) for the state variable and \([-\tau_{2}, 0]\) for the control variable, the long run dynamic can be reformulated in terms of initial conditions on an interval of length \( \tau_{2} - \tau_{1} \). Taking this into consideration, the MFDE (5) that arises from the first order conditions of the reformulated problem corresponds to an operator mathematically operating on state \( C([-\tau_{2} + 2\tau_{1}, \tau_{1}]) \).

According to Lemma 6, if \( B \) is such that there are no imaginary roots, there exist two sets \( S \) and \( U \), such that

\[
S \cap U = C([-\tau_{2} + 2\tau_{1}, \tau_{1}]),
\]

where \( S \subset C([-\tau_{2} + 2\tau_{1}, \tau_{1}]) \) is the set of initial functions leading to convergent solutions as time tends to infinity, and \( U \subset C([-\tau_{2} + 2\tau_{1}, \tau_{1}]) \) is the set of initial functions leading to convergent solutions as time tends to minus infinity (Mallet-Paret and Verduyn-Lunel, to appear). This property means that the dynamics can be projected on a stable manifold which is of infinite dimension. Unlike the case studied by d’Albis et al. (2012), which was one dimensional, we are not yet able to characterize more accurately this saddle path decomposition. However, the location of roots gives information about this dynamics: the dynamics on the stable manifold can either be monotonic or may display damped oscillations, as is stated in the following lemma.

**Lemma 7.** If \( B < B^0 \), the optimal path displays damped oscillations in the neighborhood of the steady state. Otherwise, the optimal path may be monotonic.
Proof. The proof relies on Lemma 4 according to which there exists a unique scalar $B^D$ such that $\Delta(\lambda) = 0$ has respectively zero or four real roots if $B < B^D$ or $B > B^D$. ■

Lemma 7 implies that when we study the monotonicity of optimal paths in this complex framework of distributed delays we are faced with two possibilities. If $B < B^D$ then the optimal path will display an oscillatory behavior, as there are no real roots. However, if $B > B^D$, the optimal path will be characterized by oscillations in the short term but it will eventually converge monotonically towards our unique steady state $(c^*, x^*)$. If the real root is the root with the greatest real part among the set of roots spanning the stable manifold, convergence is monotonic. If there exist complex roots with real part greater than the real root’s, damped oscillations take place. In the second case, the appropriate shadow price, implemented through a Pigovian tax for example, will set the system on the optimal pollution path that will reach the desirable steady state in the long run, despite possible initial oscillations.

3.3.3. Hopf bifurcation

Let us now study the case where pure imaginary roots exist and the potential consequences in terms of limit cycles.

Lemma 8. If $B^H > -\infty$ but small enough, the optimal path gives rise to a Hopf bifurcation when $B$ is in the neighborhood of $B^H$.

Proof. We have already seen that for $B = B^H$ there exist pure imaginary roots. We show in Appendix A.4 that in addition these roots are simple and that they cross the imaginary axis transversally. The conditions for the application of the Hopf bifurcation theorem are thus satisfied (Hupkes et al., 2008). ■

Lemma 8 provides an interesting addition to the literature on limit cycles in an infinite dimensional control setting as it sheds some light on the key structural parameters thought to cause these cycles. In terms of environmental policy, our model shows that if the above conditions hold, a cyclical policy around the steady state will be optimal, alternating pollution accumulation and pollution reduction phases through significant variations of the optimal emission level.

3.4. Discussion

We can illustrate the previous results in Fig. 2 and establish the spectrum of the dynamic behavior of the optimal pollution path depending on the value of $B$. Cyclical policies around the steady state will be optimal in the limit cycles regime ($B < B^H$ and $B^H$
small enough). Such limit cycles rarely arise in standard optimal pollution control problems, except in the presence of a catastrophic risk (Cropper, 1976) or adjustment costs (Wirl, 1999).

In order to discuss the operational power of our model let us address the key economic parameters that determine the regime of the optimal path. The latter depends heavily on the value of $B$. To keep our economic interpretation as clear as possible, we shall distinguish within $B$ two components: the technological parameter $\theta^2$ and the preference (negative) ratio $\frac{\rho}{\tau}$. Technological parameter. It is straightforward that a higher $\theta$ leads to a lower $B$. As a result, optimal policies derived from our model will be (locally) monotonic ceteris paribus in a setting where the polluting by-product of the economic activity is low, and for less efficient technologies or less favorable site conditions, the likelihood of oscillations, and then of limit cycles, increases.

Preference ratio. In order to relate our results to the literature on limit cycles more easily, we will consider our preference ratio $\frac{\rho}{\tau}$ as an indicator of the concavity of our separable objective function. High absolute values for this ratio, that is to say a highly convex damage function and/or a highly concave benefit function, reflect weakly green social preferences that attribute a significant value to marginal environmental damages but are not so strong as to altogether prevent pollution. These weakly green preferences have been identified by Wirl (1999) in a model without delays as potential determinants of limit cycles in two dimensional control problems. Our analytical characterization thus confirms and completes these previous results by identifying the conditions of occurrence of limit cycles in infinite dimension control problems that are made more complex by the introduction of distributed delays that disturb the time frame of the model. If the preferences are less green, then the optimal policy will be stable, although it might involve damped oscillations.

If we separate the sources of the concavity of our separable objective function, we observe that the case corresponding to a monotonic optimal policy is the case of a damage function that is not too convex and a benefit function that is concave enough. This situation fits quite well various cases of pollution when the profit of the polluters depends only partially on the amount of pollution emitted and when the damages do not increase too steeply with the stock. This kind of objective function can represent the case of nitrate contamination from agricultural sources. Indeed the marginal benefit of farmers decreases in the amount of fertilizers used while the marginal environmental damage does not increase much once a concentration threshold has been reached.

**Fig. 2.** Dynamic properties of the optimal path depending on $B$ (with $B^0$ small enough).
Applying Fubini’s theorem, we obtain
\[ \int_0^t \xi (t) \int_{t_1}^{t_2} \delta \mu (u) \mu (t - u) \, dt \, du \]
\[ = \int_0^h \delta \mu (u) \int_{u + t_1}^{u + t_2} \xi (t) \mu (t - u) \, dt \, du, \]
which we substitute in the expression of \( \delta V \), to obtain the necessary conditions with respect to the control and the co-state variable. We then substitute \( x (t) = \xi (t) e^{\varphi t} \), where \( x (t) \) is the current value shadow price of pollution, and \( w_1 (t) = \varphi_1 (t \, e^{\varphi t}) \).

Since \( c (h) \geq 0, \varphi_2 (h) \) is sign-constrained if \( c (h) = 0 \). In this case, \( c (h) \geq 0 \). We thus need \( \xi (h) \geq 0 \) to satisfy \( \delta V_0 \leq 0 \). Therefore the optimal solution must satisfy \( \xi (h) \geq 0 \) and \( \xi (h) = c (h) \).

### A.2. Proof of Lemma 4

#### A.2.1. Real roots

Let us first focus on the real roots of \( \Delta (\lambda) = 0 \). As the roots are symmetric about the axis \( \text{Re}(\lambda) = \frac{p}{2} \), we can study exclusively the roots smaller than \( \frac{p}{2} \).

We start by noticing that \( \Delta \left( \frac{p}{2} \right) < 0 \), and \( \lim_{\rho \to \infty} \Delta (\lambda) = - \infty \). Moreover,

\[
\Delta^\prime (\lambda) = 2 \lambda - \rho + B \int_t^{t_2} \mu (s) e^{-\frac{\lambda}{s}} \times \int_t^{t_2} (u - s) e^{-\frac{\lambda}{s}} u \mu (u) \, du \, ds.
\]

Therefore the roots of \( \Delta^\prime (\lambda) = 0 \) are \( \frac{p}{2} \) and the roots of \( \psi (\lambda) = 0 \), where \( \psi \) is given by

\[
\psi (\lambda) = 2 + B \int_t^{t_2} \mu (s) e^{-\frac{\lambda}{s}} \times \int_t^{t_2} (u - s) e^{-\frac{\lambda}{s}} u \mu (u) \, du \, ds.
\]

and we have

\[
\psi'' (\lambda) = B \int_t^{t_2} \mu (s) e^{-\frac{\lambda}{s}} \times \int_t^{t_2} (u - s) e^{-\frac{\lambda}{s}} u \mu (u) \, du \, ds < 0.
\]

As \( \psi (\lambda) = \psi (\rho - \lambda) \), the roots of \( \psi \) are also symmetric about the line \( \text{Re} (\lambda) = \frac{p}{2} \). The sign of \( \psi (\lambda) = 2 + B \int_t^{t_2} \mu (s) e^{-\frac{\lambda}{s}} \frac{t_2 - t}{t_2 - s} e^{-\frac{\lambda}{s}} u \mu (u) \, du \, ds \) depends on \( B \). For small enough it is negative, otherwise it can be positive.

Let us first show that for small enough, the characteristic equation admits no real roots. Indeed, if \( B < B_0 \) then \( \Delta (\lambda, B) \) has two maxima \( \lambda_1 > \lambda_2 \), and a minimum at \( \frac{p}{2} \), with \( \lambda_1 < \frac{p}{2} < \lambda_2 \). We can then find that for \( B \) close to zero, \( \Delta (\lambda) = 0 \) admits at least one real root close to \( - \alpha \) and another one close to \( \alpha + \rho \).

Given that \( \int_t^{t_2} \mu (s) e^{-\lambda} > 0 \) and \( \int_0^{t_2} \mu (s) e^{-(s - \alpha)^2} > 0 \), we have \( \Delta (\lambda, B) > \Delta (\lambda, B^0) \) for \( B > B^0 \).

Since \( \Delta (\lambda, B) > 0 \) has no real root for \( B \) close to \( 0 \), this implies that there exists \( B^0 \) such that \( \Delta (\lambda, B^0) = 0 \) and \( B^0 \) is unique (because \( \Delta (\lambda, B) > \Delta (\lambda, B^0) \) for \( B > B^0 \)).

For \( B > B^0 \) we thus have no real roots while for \( B > B^0 \), there are two real roots smaller than \( \frac{p}{2} \), which amounts to four real roots due to the symmetry.

We also note that \( \Delta (0) < 0 \) and, for \( B > B^0 \), \( \Delta (0) < 0 \), thus, when these real roots exist, the two of them which satisfy \( \lambda < \frac{p}{2} \) also satisfy \( \lambda < 0 \), which implies that all the real roots are outside the set \( \delta (0, \rho) \).

#### A.2.2. Complex roots

Let us start by considering the complex roots when \( B > B^0 \). We have shown in A.2.1 that in that case the characteristic equation \( \Delta (\lambda) \) has two negative real roots, which we denote \( x_3 \) and \( x_2 \), with \( x_3 = \rho - x_2 \).

Let us consider \( \psi (\lambda) = \frac{\Delta (\lambda)}{(\lambda - x_2)(\lambda - \rho - x_2)} \). An easy but fastidious computation (based on the same idea as the one used in the computation of \( \psi \) in A.2.1) shows that

\[
\psi (\lambda) = 1 + B \left( \int_t^{t_2} \nu (s) (e^{(\lambda - x_2)^2 z} - e^{(\lambda - x_2)^2}) \frac{1}{\mu (u)} du \right).
\]

Since we proved in A.2.1 that \( \Delta (\lambda) < 0 \) for \( \lambda \in [x_3, \rho - x_2] \) and \( \Delta (\lambda) > 0 \) for \( \lambda \in [x_3, x_2] \), we can deduce that \( \psi (\lambda) > 0 \) for \( \lambda \in [x_3, \rho - x_2] \). The roots of \( \Delta (\lambda) = 0 \) are the roots of \( \psi (\lambda) = 0 \) as well as \( \lambda = x_2 \) and \( \lambda = \rho - x_2 \).

Let us consider \( \lambda = p + iq \), with \( p \in [x_3, \rho - x_3] \). If it were a root of \( \psi (\lambda) = 0 \), it would solve

\[
(-B) \left( \int_t^{t_2} \nu (s) (e^{(x_2 - x_2^2)} - e^{(x_2 - x_2^2)}) \mu (u) du \right) dz - \int_t^{t_2} \mu (u) du = 1.
\]

and thus

\[
(-B) \left( \int_t^{t_2} \nu (s) (e^{(x_2 - x_2^*)} - e^{(x_2 - x_2^*)}) \mu (u) du \right) dz - \int_t^{t_2} \mu (u) du = 1.
\]

which contradicts the fact that \( \psi (\lambda) > 0 \) for \( \lambda \in [x_3, \rho - x_3] \). There are thus no complex roots in \( \delta (0, \rho) \) when \( B > B^0 \).

Let us now look at the complex roots when \( B < B^0 \). We will prove the existence of \( B_{\text{min}} \) such that there are no complex roots in \( \delta (0, \rho) \) when \( B > B_{\text{min}} \). We have already proved that for \( B > B^0 \), there are no complex roots in \( \delta (0, \rho) \). Since the characteristic equation is continuous in \( B \), if complex roots were to exist in \( \delta (0, \rho) \) for some value of \( B \), it would imply that there is a value of \( B \) for which pure imaginary roots exist and cross the imaginary axis from left to right. If pure imaginary roots did exist, they would necessarily satisfy the following conditions obtained by solving equation

If \( B > B_0 \), then \( \psi \left( \frac{p}{2} \right) > 0 \). This means that \( \Delta (\lambda, B) \) has two maxima \( \lambda_1 \) and \( \lambda_2 \), and a minimum at \( \frac{p}{2} \), with \( \lambda_1 < \frac{p}{2} < \lambda_2 \). We can note that for \( B \) close to zero, \( \Delta (\lambda) \equiv 0 \) admits at least one real root close to \( - \alpha \) and another one close to \( \alpha + \rho \).

Given that \( \int_t^{t_2} \mu (s) e^{-\lambda} > 0 \) and \( \int_0^{t_2} \mu (s) e^{-(s - \alpha)^2} > 0 \), we have \( \Delta (\lambda, B) > \Delta (\lambda, B^0) \) for \( B > B^0 \).

Since \( \Delta (\lambda, B) > 0 \) has no real root for \( B \) close to \( 0 \), this implies that there exists \( B^0 \) such that \( \Delta (\lambda, B^0) = 0 \) and \( B^0 \) is unique (because \( \Delta (\lambda, B) > \Delta (\lambda, B^0) \) for \( B > B^0 \)).
\(\Delta (iq) = 0\) and splitting the real and imaginary parts:

\[
q^2 + (\alpha + \rho) \alpha = B \int_{t_1}^{t_2} \int_{s_1}^{s_2} \mu (s) e^{-\rho \mu (u)} \times \cos (q (u - s)) duds,
\]

(11)

\[
q \rho = B \int_{t_1}^{t_2} \int_{s_1}^{s_2} \mu (s) e^{-\rho \mu (u)} \sin (q (u - s)) duds.
\]

(12)

The necessary condition (12) can be rewritten as

\[
\frac{\rho}{B} = \chi (q),
\]

(13)

with

\[
\chi (q) = \frac{1}{q} \left( \int_{t_1}^{t_2} \int_{s_1}^{s_2} \mu (s) e^{-\rho \mu (u)} \sin (q (u - s)) duds \right).
\]

(14)

Next, we prove that \(\chi (q)\) is greater than some (negative) constant \(\tilde{\chi}\). Let us rewrite (14) as

\[
\chi (q) = \frac{1}{q} \left( \int_{t_1}^{t_2} \int_{s_1}^{s_2} \mu (s) e^{-\rho \mu (u)} \sin (q (u - s)) duds \right).
\]

If \(u - s < 0\), we have \(\frac{\sin (q (u - s))}{q} > (u - s)\). The previous expression can thus be rewritten as

\[
\chi (q) > \int_{t_1}^{t_2} \int_{s_1}^{s_2} \mu (s) e^{-\rho \mu (u)} (u - s) duds.
\]

Next, we are going to prove that these pure imaginary roots are simple roots. If \(q_0\) were a double root, it would necessarily solve \(\Delta' (iq) = 0\), that is to say

\[
\text{Re} \Delta' (iq) = 0, \\
\text{Im} \Delta' (iq) = 0,
\]

with

\[
\frac{\text{Re} \Delta' (iq)}{q} = 2 \int_{t_1}^{t_2} \int_{s_1}^{s_2} (u - s)^2 e^{-\rho \mu (s)} \times \cos (q (u - s)) \mu (u) duds,
\]

\[
\frac{\text{Im} \Delta' (iq)}{q} = \int_{t_1}^{t_2} \int_{s_1}^{s_2} (u - s) e^{-\rho \mu (s)} \sin (q (u - s)) \mu (u) duds.
\]

However

\[
\frac{\text{Im} \Delta' (iq)}{q} = 2 \int_{t_1}^{t_2} \int_{s_1}^{s_2} (u - s)^2 e^{-\rho \mu (s)} \mu (u) duds.
\]

According to the assumption \(-2 \left( \int_{t_1}^{t_2} (u - s) e^{-\rho \mu (s)} \mu (u) duds \right)^{-1} > 0\), we thus have

\[
\frac{\text{Im} \Delta' (iq)}{q} < 0,
\]

which contradicts the necessary condition above.

Second, let us prove that these roots cross the imaginary axis transversally. Considering \(\Delta (\lambda) = 0\) as a function of parameter \(B\), we can totally differentiate the characteristic equation to get

\[
\Delta' (\lambda) d\lambda = - \left( \int_{t_1}^{t_2} \mu (s) e^{-\lambda \mu (u)} duds \right) d\lambda ,
\]

thus

\[
\frac{d\lambda}{d\rho} = \frac{2 \lambda - \rho + \int_{t_1}^{t_2} (u - s)^2 e^{-\rho \mu (u)} \mu (u) duds}{\int_{t_1}^{t_2} \mu (s) e^{-\rho \mu (u)} duds}.
\]
\[ B_{\min} = \min \left\{ \frac{B_0}{\tau_2} \int_{\tau_1}^{\tau_2} \mu(s) e^{-\rho u} \mu(u)(u - s) \, du - 1.218 \int_{\tau_1}^{\tau_2} \mu(s) e^{-\rho u} \mu(u)(u - s) \, du \right\} \]

Box I.

Since \( \lambda \) is a root of \( \Delta(\lambda) = 0 \), the previous equation can be rewritten as

\[
\left( \frac{d\lambda}{dB} \right)^{-1} = B \frac{2\lambda - \rho + B \int_{\tau_1}^{\tau_2} (u - s) \mu(s) e^{-\rho u} \mu(u)(u - s) \, du}{(\lambda + \alpha)(\lambda - \rho - \alpha)}.
\]

Let us now study the sign of \( \frac{d\lambda}{dB} \). We have

\[
\text{sign} \left( \frac{d\lambda}{dB} \right) = \text{sign} \left( \frac{d\lambda}{dB} \right){^{\left| B = b^H \right.}}.
\]

Hence, using Eq. (15) above,

\[
\text{sign} \left( \frac{d\lambda}{dB} \right){^{\left| B = b^H \right.}} = \text{sign}(M),
\]

where \( M \) is defined as

\[
M = (q_\alpha + \alpha(\alpha + \rho)) \left( -\rho + B^H \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} \mu(s) e^{-\rho u} \mu(u)(u - s) \cos(q_0(u - s)) \, du \right)
+ 2q_\alpha^2 \rho \left( 2 + B^H \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} \mu(s) e^{-\rho u} \mu(u)(u - s)^2 \sin(q_0(u - s)) \, du \right).
\]

Therefore, if

\[
B^H < \min \left\{ -2 \left( \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} (u - s)^2 e^{-\rho u} \mu(s) \mu(u)(u - s) \, du \right)^{-1}, \right. \\
\rho \left( \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} e^{-\rho u} \mu(s) \mu(u)(u - s) \, du \right)^{-1} \right\},
\]

then

\[
\text{sign} \left( \frac{d\lambda}{dB} \right){^{\left| B = b^H \right.}} < 0.
\]

References


