Erratum to
"Roughness-Induced Effect at Main order on the Reynolds Approximation"

SIAM Multiscale Model. Simul 8, 3, 997–1017, 2010

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A factor 2 has been omitted in one term when transforming the equations in oscillatory domain to equations in a strip.... This factor 2 exactly acts as a counterbalance to the new term we found in the limit. Hence, the result of the main Theorem is not true and the paper has not to be considered. The effect of rugosities will be at order one and not at main order.

More precisely, the main theorem has to be replaced by the following one.

Theorem 1 *Let $(u^\varepsilon, w^\varepsilon, p^\varepsilon)$ be a sequence of weak solutions of the Stokes system in the considered rough domain.*

The rescaled quantity $(u^\varepsilon, \frac{1}{\varepsilon}w^\varepsilon, \varepsilon^2 p^\varepsilon) \circ \big(x, h^\varepsilon(x)Z\big)$ two-scale converges to the weak solution (u_0, w_1, p_0) of

$$\begin{cases} -\eta \partial_Z^2 u_0 + h_1^2 \nabla_x p_0 = 0, \\ \partial_Z p_0 = 0, \\ div_x(h_1 u_0) + \partial_Z(w_1 - Z\nabla_x h_1 \cdot u_0) = 0, \end{cases}$$

in the rescaled domain. It means in particular that the classical Reynolds approximation holds true:

$$div_x \left(\frac{h_1^3}{12\eta} \nabla_x p_0 \right) = div_x \left(\frac{h_1}{2} u_b \right).$$

It means that the paper "Roughness-Induced Effect at Main order on the Reynolds Approximation", SIAM Multiscale Model. Simul 8, 3, 997–1017, 2010, "only" gives a new justification of the Reynolds equation, in the context of a rough domain where the main order roughness is perturbated by quite fast second order oscillations:

$$h^\epsilon(x) = \epsilon\big(h_1(x) + \epsilon h_2(x/\epsilon^2)\big).$$

Yet the derivatives of the former profile induce higher order terms in the perturbation h_2 than in the main profile h_1.

on avait dit dans l'intro que les perturbations traitees s'arretaient a l'ordre $\alpha < 2$. c'est vrai? du coup, ca au moins on l'aurait ameliore...

For the convenience of the reader, we provide below the corrected proof of the result. It follows of course the lines of the derivation in "Roughness-Induced Effect at Main order on the Reynolds Approximation", SIAM Multiscale Model. Simul 8, 3, 997–1017, 2010. Difference is the starting point, that is a correct system of rescaled equations at the microscopic scale.

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Mathematical justification

We introduce the variable Z defined by

$$Z = \frac{z}{h^\varepsilon(x)} = \frac{z}{\varepsilon(h_1(x) + \varepsilon h_2(\frac{x}{\varepsilon^2}))}$$

to rescale the Stokes problem in the fixed domain $\Omega = \omega \times (0,1)$. According to this change of variable, we look at the unknown $\mathbf{u}^\varepsilon = (u^\varepsilon, w^\varepsilon)$ and p^ε as

$$u^\varepsilon(x,z) = \widetilde{u^\varepsilon}(x,Z), \quad w^\varepsilon(x,z) = \widetilde{w^\varepsilon}(x,Z) \quad \text{and} \quad p^\varepsilon(x,z) = \widetilde{p^\varepsilon}(x,Z).$$

For the sake of simplicity, we omit the tilde in the new unknowns and we focus on standard Dirichlet boundary conditions $\mathbf{u}^\varepsilon = \mathbf{u}_b$ on $\partial\Omega$. Stokes system reduces to the following equations in Ω

$$\begin{aligned}
&-\eta\Delta_x u^\varepsilon + \eta\Big(\frac{\Delta_x h^\varepsilon}{h^\varepsilon} - 2\frac{|\nabla_x h^\varepsilon|^2}{{h^\varepsilon}^2}\Big) Z\partial_Z u^\varepsilon + 2\eta\frac{\nabla_x h^\varepsilon}{h^\varepsilon}\cdot Z\nabla_x\partial_Z u^\varepsilon - \eta\frac{|\nabla_x h^\varepsilon|^2}{{h^\varepsilon}^2} Z^2\partial_Z^2 u^\varepsilon - \eta\frac{1}{{h^\varepsilon}^2}\partial_Z^2 u^\varepsilon \\
&\qquad\qquad\qquad\qquad + \nabla_x p^\varepsilon - \frac{\nabla_x h^\varepsilon}{h^\varepsilon} Z\partial_Z p^\varepsilon = 0, \qquad (1)\\
&-\eta\Delta_x w^\varepsilon + \eta\Big(\frac{\Delta_x h^\varepsilon}{h^\varepsilon} - 2\frac{|\nabla_x h^\varepsilon|^2}{{h^\varepsilon}^2}\Big) Z\partial_Z w^\varepsilon + 2\eta\frac{\nabla_x h^\varepsilon}{h^\varepsilon}\cdot Z\nabla_x\partial_Z w^\varepsilon - \eta\frac{|\nabla_x h^\varepsilon|^2}{{h^\varepsilon}^2} Z^2\partial_Z^2 w^\varepsilon - \eta\frac{1}{{h^\varepsilon}^2}\partial_Z^2 w^\varepsilon \\
&\qquad\qquad\qquad\qquad + \frac{1}{h^\varepsilon}\partial_Z p^\varepsilon = 0, \qquad (2)\\
&\mathrm{div}(u^\varepsilon) - \frac{\nabla_x h^\varepsilon}{h^\varepsilon}\cdot Z\partial_Z u^\varepsilon + \frac{1}{h^\varepsilon}\partial_Z w^\varepsilon = 0. \qquad (3)
\end{aligned}$$

The estimates derived in in "Roughness-Induced Effect at Main order on the Reynolds Approximation", SIAM Multiscale Model. Simul 8, 3, 997–1017, 2010, remain true. We thus assert that there exist limit functions $p_0 \in L^2(\Omega; L^2(\mathbb{T}^{d-1}))$, $u_0 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$, $w_0 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$ such that

$$\varepsilon^2 p^\varepsilon \overset{2}{\rightharpoonup} p_0, \qquad u^\varepsilon \overset{2}{\rightharpoonup} u_0 \qquad \text{and} \qquad w^\varepsilon \overset{2}{\rightharpoonup} w_0,$$

with moreover $\nabla_X u_0 = 0$, $\nabla_X w_0 = 0$. Since there was no mistake in the divergence equation in our former work, proofs of $\nabla_X p_0 = 0$, $\partial_Z p_0 = 0$, $w_0 = 0$ and of the limit divergence equation

$$h_1\,\mathrm{div}_x(u_0) - \nabla_x h_1 \cdot Z\partial_Z u_0 + \partial_Z w^1 = 0,$$

remain unchanged.

All the modifications have to be performed when passing to the limit in the momentum equation. We thus detail this part of the proof.

We begin by the auxiliary results. In view of the estimates derived for u^ε, we define the "anisotropic" two-scale limit $u^1 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$ such that $\nabla_x(\varepsilon u^\varepsilon) \overset{2}{\rightharpoonup} \nabla_X u^1$. It satisfies

Lemma 0.1 *The function u^1 is such that $\Delta_X u^1 = \dfrac{\Delta_X h_2}{h_1} Z\partial_Z u_0$.*

Proof: On the one hand, we multiply the divergence equation by $\varepsilon\phi(x, Z, x/\varepsilon^2)$ with $\phi \in \mathcal{D}(\Omega; \mathcal{C}^1(\mathbb{T}^{d-1}))$. We obtain

$$\int_\Omega \varepsilon\,\mathrm{div}(u^\varepsilon)\phi^\varepsilon - \int_\Omega \nabla_X h_2^\varepsilon \cdot Z\partial_Z u^\varepsilon \frac{\phi^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} - \int_\Omega \varepsilon\nabla_x h_1 \cdot Z\partial_Z u^\varepsilon \frac{\phi^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} + \int_\Omega \partial_Z w^\varepsilon \frac{\phi^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} = 0.$$

We recall that $\varepsilon\partial_Z u^\varepsilon \overset{2}{\rightharpoonup} 0$ and $w^\varepsilon \overset{2}{\rightharpoonup} 0$. Passing to the limit in the latter relation, we get

$$\int_\Omega \int_{\mathbb{T}^{d-1}} \big(\mathrm{div}_X(u^1) - \frac{1}{h_1}\nabla_X h_2 \cdot Z\partial_Z u_0\big)\phi \, dX\, dx\, dZ = 0,$$

that is

$$\mathrm{div}_X(u^1) = \frac{1}{h_1}\nabla_X h_2 \cdot Z\partial_Z u_0. \qquad (4)$$

On the other hand, we multiply Equation (1) by $\varepsilon^3\phi(x,Z,x/\varepsilon^2)$ and we integrate by parts. We obtain

$$\begin{aligned}
&\int_\Omega \eta\varepsilon\nabla u^\varepsilon\cdot(\varepsilon^2\nabla_x\phi^\varepsilon+\nabla_X\phi^\varepsilon)+\int_\Omega \eta\varepsilon^3\frac{\Delta_x h_1+\frac{1}{\varepsilon^3}\Delta_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon}Z\partial_Z u^\varepsilon\cdot\phi^\varepsilon\\
&-\int_\Omega 2\eta\varepsilon^2\frac{|\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}\varepsilon Z\partial_Z u^\varepsilon\cdot\phi^\varepsilon-2\int_\Omega\frac{\eta\varepsilon^2}{h_1+\varepsilon h_2^\varepsilon}(\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon)\cdot\varepsilon\nabla u^\varepsilon\partial_Z(Z\phi^\varepsilon)\\
&+\int_\Omega\eta\varepsilon^2\frac{|\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}\varepsilon\partial_Z u^\varepsilon\cdot\partial_Z(Z^2\phi^\varepsilon)+\int_\Omega\frac{\eta}{(h_1+\varepsilon h_2^\varepsilon)^2}\varepsilon\partial_Z u^\varepsilon\cdot\partial_Z\phi^\varepsilon\\
&-\int_\Omega\varepsilon^3 p^\varepsilon\big(\operatorname{div}_x(\phi^\varepsilon)+\frac{1}{\varepsilon^2}\operatorname{div}_X(\phi_\varepsilon)\big)+\int_\Omega\frac{1}{h_1+\varepsilon h_2^\varepsilon}\varepsilon^2 p^\varepsilon\varepsilon\big(\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon\big)\cdot\partial_Z(Z\phi^\varepsilon)=0. \qquad (5)
\end{aligned}$$

We choose ϕ in the form $\phi=\operatorname{curl}_X\psi$ to cancel the term containing $\operatorname{div}_X(\phi^\varepsilon)$. Passing to the limit in the other terms, using $\partial_Z p_0=0$, we get

$$\int_\Omega\int_{\mathbb{T}^{d-1}}\nabla_X u^1\cdot\nabla_X\phi+\int_\Omega\int_{\mathbb{T}^{d-1}}\frac{\Delta_X h_2}{h_1}Z\partial_Z u_0\phi=0,$$

for any $\phi=\operatorname{curl}_X\psi$. It follows that

$$\operatorname{curl}_X(\Delta_X u^1)=\frac{1}{h_1}\nabla_X^\perp(\Delta_X h_2)\cdot Z\partial_Z u_0$$

and then

$$\operatorname{curl}_X u^1=\frac{1}{h_1}\nabla_X^\perp(h_2)\cdot Z\partial_Z u_0. \qquad (6)$$

We infer the result of the lemma from (4)-(6), using the formula $\Delta_X u^1=\nabla_X\mathrm{div_X}u^1-\nabla_X^\perp\operatorname{curl}_X u^1$ and $\nabla_X\nabla_X h_2-\nabla_X^\perp\nabla_X^\perp h_2=(\Delta_X h_2)\mathrm{Id}$. Note that our mistake in "Roughness-Induced Effect at Main order on the Reynolds Approximation", SIAM Multiscale Model. Simul 8, 3, 997–1017, 2010, has no influence on this result thanks to the choice of the test function $\epsilon^3\phi^\epsilon$. □

The pressure being such that $\lim_{\varepsilon\to 0}\int_\Omega\varepsilon p^\varepsilon\operatorname{div}_X(\phi)=0$ for any $\phi\in L^2(\Omega;H^1(\mathbb{T}^{d-1}))$, we now have sufficient tools to pass to the limit in the momentum equation. We multiply Equation (1) by $\varepsilon^2\phi(x,Z)$ with $\phi\in\mathcal{D}(\Omega)$. Since $\Delta h/h-|\nabla h|^2/h^2=\operatorname{div}(\nabla h/h)$, we obtain:

$$\begin{aligned}
&\int_\Omega\eta\varepsilon^2\nabla u^\varepsilon\cdot\nabla_x\phi^\varepsilon-\int_\Omega\frac{\eta\varepsilon}{h_1+\varepsilon\, h_2^\varepsilon}(\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2)\cdot\varepsilon\nabla u^\varepsilon\partial_Z(Z\phi)\\
&-\int_\Omega\eta\varepsilon^2\frac{|\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}Z\partial_Z u^\varepsilon\cdot\phi\\
&+\int_\Omega\eta\varepsilon^2\frac{|\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}\partial_Z u^\varepsilon\cdot\partial_Z(Z^2\phi)+\int_\Omega\frac{\eta}{(h_1+\varepsilon h_2^\varepsilon)^2}\partial_Z u^\varepsilon\cdot\partial_Z\phi\\
&-\int_\Omega\varepsilon^2 p^\varepsilon\operatorname{div}_x(\phi)+\int_\Omega\frac{1}{h_1+\varepsilon h_2^\varepsilon}\varepsilon^2 p^\varepsilon\big(\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon\big)\cdot\partial_Z(Z\phi)\\
&-\int_\Omega\frac{\eta\varepsilon^2}{h_1+\varepsilon\, h_2^\varepsilon}(\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2)\,Z\,\partial_Z u^\varepsilon\nabla_x\phi=0. \qquad (7)
\end{aligned}$$

The correction corresponds to the new term in the second line of the former relation. We have:

$$\lim_{\varepsilon\to 0}\int_\Omega\frac{1}{h_1+\varepsilon h_2^\varepsilon}\varepsilon p^\varepsilon\nabla_X h_2^\varepsilon\cdot\partial_Z(Z\phi)=\lim_{\varepsilon\to 0}\int_\Omega\varepsilon p^\varepsilon\operatorname{div}_X\big(\frac{h_2^\varepsilon}{h_1}\partial_Z(Z\phi)\big)=0.$$

Passing to the limit in (7), we get, using that $u_0=u_0(x,Z)$ and $\partial_Z p_0=0$,

$$\begin{aligned}
-\eta\int_\Omega\int_{\mathbb{T}^{d-1}}\frac{1}{h_1}\nabla_X h_2\cdot\nabla_X u_1\,\partial_Z(Z\phi)-\eta\int_\Omega\int_{\mathbb{T}^{d-1}}\frac{|\nabla_X h_2|^2}{(h_1)^2}Z\partial_Z u_0\,\phi+\eta\int_\Omega\int_{\mathbb{T}^{d-1}}\frac{|\nabla_X h_2|^2}{(h_1)^2}\partial_Z u_0\,\partial_Z(Z^2\phi)\\
+\eta\int_\Omega\int_{\mathbb{T}^{d-1}}\frac{1}{(h_1)^2}\partial_Z u_0\cdot\partial_Z\phi-\int_\Omega\int_{\mathbb{T}^{d-1}}p_0\mathrm{div}_x\phi=0.
\end{aligned}$$

Now, one easily checks that the former relation is the energy formulation corresponding to

$$-\eta\Big(\int_{\mathbb{T}^{d-1}} h_2\Delta_X h_2\Big)\frac{1}{h_1}Z\partial_Z(Z\partial_Z u_0) - \eta\Big(\int_{\mathbb{T}^{d-1}} |\nabla_X h_2|^2\Big)\frac{1}{h_1^2}Z^2\partial_Z(\partial_Z u_0)$$
$$-\eta\Big(\int_{\mathbb{T}^{d-1}} |\nabla_X h_2|^2\Big)\frac{1}{h_1^2}Z\partial_Z u_0 - \frac{\eta}{h_1^2}\partial^2_{ZZ}u_0 + \nabla_x p_0 = 0.$$

Integrating by parts the integral terms, one finally gets

$$-\frac{\eta}{h_1^2}\partial^2_{ZZ}u_0 + \nabla_x p_0 = 0,$$

which corresponds to the classical Reynolds approximation. Theorem 1 is proved.

Remark 1 *Of course the next order approximation may be calculated to see the roughness-induced effect at order one of the perturbation* h_2*.*