SEAWATER INTRUSION PROBLEM IN A FREE AQUIFER: DERIVATION OF A SHARP–DIFFUSE INTERFACES MODEL AND EXISTENCE RESULT

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Abstract. We derive a new model for seawater intrusion phenomenons in free aquifers. It combines the efficiency of the sharp interface approach with the physical realism of the diffuse interface one. The three-dimensional problem is reduced to a two-dimensional model involving a strongly coupled system of pdes of parabolic type describing the evolution of the depths of the two free surfaces.

Key words. Seawater intrusion; free boundaries; nonlinear parabolic partial differential equation; system of strongly coupled partial differential equations; initial and boundary value problem.

AMS subject classifications. 35R35, 35K20, 76S05, 76T05, 76E19

1. Introduction.

Groundwater is a major source of water supply. In coastal zones there exist hydraulic exchanges between fresh groundwater and seawater. They are slow in "natural conditions" and thus are often forgotten and replaced by a quasi-equilibrium between two fluid layers (Ghyben–Herzberg approximation). The picture fails in case of more drastic conditions due to meteorological events or to human interventions. Intensive extraction of freshwater leads for instance to local water table depression causing problems of saltwater intrusion in the aquifer. We thus need efficient and accurate models to simulate the displacement of saltwater front in coastal aquifer for the optimal exploitation of fresh groundwater.

We refer to the textbooks [6]–[8] for general informations about seawater intrusion problems. Beyond the above mentioned Ghyben–Herzberg static model, the existing models for seawater intrusion may be classified in three categories:

Hidden diffuse interfaces: This is the physically correct approach. Fresh and salt water are two miscible fluids. Due to density contrast they tend to separate into two layers with a transition zone characterized by the variations of the salt concentration. Moreover the aquifer has to be considered as a partially saturated porous medium. There is another transition zone between the completely saturated part and the dry part of the reservoir, the definition of the area of desaturation being difficult. Two "diffusive interfaces" are thus hidden in this kind of model. The approach is very heavy from theoretical and numerical points of view ([10], see also [4] when further assuming a saturated medium; see [1] for numerical recipes).

Hidden sharp interfaces: A first simplification consists in assuming that fresh and salt water are two immiscible fluids (see [10] in unsaturated media). Points where the salty phase disappears may be viewed as a sharp interface. Nevertheless the explicit tracking of the interfaces remains unworkable to implement without further assumptions.

Sharp or abrupt interfaces: This approach is also based on the hypothesis that the two fluids are immiscible. Moreover the domains occupied by each fluid are assumed

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to be separated by a smooth interface called sharp interface, no mass transfert occurs between the fresh and the salt area and capillary pressure's type effects are neglected. This approximation is often reasonable (see *e.g.* [6] and below).
Of course, this type of model does not describe the behavior of the real transition zone but gives informations concerning the movement of the saltwater front. The other price to pay for this simplified approach is the mathematical handling of free interfaces.

In the present work we essentially have chosen to adopt the (numerical) simplicity of a sharp interface approach. We compensate the mathematical difficulty of the analysis of the free interfaces by an upscaling procedure which allows to model the three-dimensional problem by a pdes system set in a two-dimensional domain. The originality and novelty is to mix this abrupt interface approach with a phase field approach, thus reinjecting in a new way the realism of diffuse interfaces models. We exploit here the specificity of the dynamics of the fluids in an aquifer for using such a model which was originally developed for phase transition phenomenons in binary fluids. We thus combine the advantage of respecting the physics of the problem and that of the computational efficiency. The two key assumptions are summarized as follows:

- There is no explicit mass transfer between freshwater and saltwater, thus separated by an abrupt interface. The free interfaces are treated by an upscaling procedure, with an obvious dimensional gain since the 3D reality is processed by a 2D model. Both the simplicity and the efficiency of the model lie in the fact that the mass exchanges are in fact "hidden" in the diffuse interface.
- We suppose the existence of a diffuse interface between fresh and salt water. This diffuse interface is modeled using a phase field approach, here an Allen–Cahn type model in fluid-fluid context.

The same process is applied to model the transition between the saturated and unsaturated zones.

From a theoretical point of view, the addition of the two diffusive areas has the following advantages : If they are both present, the system has a parabolic structure, it is thus no longer necessary to introduce viscous terms in a preliminary fixed point step for avoiding degeneracy as is in the demonstration of [17]. But the main point is we can demonstrate a more efficient and logical maximum principle from the point of view of physics, which is not possible in the case of sharp interface approximation (see for instance [14], [22]).

In the first part of the article, we model the evolution of the depth h of the interface between freshwater and saltwater and of the depth h_1 of the interface between the saturated and unsaturated zone. The derivation of the model is based on the coupling of Darcy's law and mass conservation principle written for freshwater and saltwater. We detail the assumptions allowing the vertical upscaling. A phase-field model is superimposed to mix the sharp and diffuse approaches. The resulting model consists in a system of strongly and nonlinearly coupled pdes of parabolic type. We state an existence result of variational solutions for this model completed by initial and boundary conditions in the second part of the paper. We apply a Schauder fixed point strategy to a regularized problem penalized by the velocity of the salt front. Then we establish uniform estimates allowing us to turn back to the original problem.

2. Derivation of the model. The basis of the modeling is the mass conservation law for each 'species' (fresh and salt water) coupled with the classical Darcy law

for porous media. Fluids and soil are considered to be weakly compressible.
For the three-dimensional description, we denote by (x, z), $x = (x_1, x_2) \in \mathbb{R}^2$, $z \in \mathbb{R}$, the usual coordinates.

2.1. Conservation laws. We begin with the conservation of momentum. In view of the (large) dimensions of an aquifer (related to the characteristic size of the porous structure of the underground), we consider a continuous description of the porous medium. The effective velocity q of the flow is thus related to the pressure P through the so-called Darcy law

$$q = -\frac{k}{\mu}(\nabla P + \rho g \nabla z),$$

where ρ and μ are respectively the density and the viscosity of the fluid, k is the permeability of the soil and g the gravitational acceleration constant. Introducing the hydraulic head Φ defined by

$$\Phi = \frac{P}{\rho_0 g} + z - h_{ref}, \tag{2.1}$$

we write the previous equation as follows:

$$q = -K\nabla\Phi - \frac{k}{\mu}(\rho - \rho_0) g \nabla z, \quad K = \frac{k\rho_0 g}{\mu}. \tag{2.2}$$

In this relation, the matrix K is the hydraulic conductivity which expresses the ability of the underground to conduct the fluid. We have denoted by ρ_0 the reference density of the fluid. In (2.1), $Z = z - h_{ref}$ is the elevation above a fixed datum level under the aquifer, $h_{ref} < 0$.

Next, the conservation of mass during displacement is given by the following equation

$$\partial_t(\phi\rho) + \nabla \cdot (\rho q) = \rho Q, \tag{2.3}$$

where ϕ is the porosity of the medium and Q denotes a generic source term (for production and replenishment).

2.2. State equation for the fluid compressibility. We consider that the fluid are compressible by assuming that pressure P is related to the density ρ as follows:

$$\frac{d\rho}{\rho} = \alpha_P dP. \tag{2.4}$$

The real number $\alpha_P \geq 0$ is the fluid compressibility coefficient. Further assuming $\alpha_P = 0$ we would recover the incompressible case.

2.3. State equation for the soil compressibility. We now introduce in the model the effects of the rock compressibility. This means ruling the dependence of the porosity with respect to the depth. A simple model is the Athy's one which reads

$$\phi(z) = \phi_0 e^{-Mz}, \qquad (\phi_0, M) \in \mathbb{R}_+^2.$$

Notice that no dependence of the porosity with the variation of the pressure is included in such a formula. A much more physical approach thus consists in deriving a differential equation for the porosity. First we denote by σ the total stress in the porous media and by σ_s the stress related to the skeleton. We have

$$\sigma = \phi P + (1-\phi)\sigma_s$$

where term ϕP accounts for the pressure effects. From Terzaghi's theory, the effective stress σ_e is defined by

$$\sigma_e = (1-\phi)(\sigma_s - P).$$

Assuming that the total stress does not change, we infer from $\sigma_e + P = \sigma$ that

$$d\sigma_e = -dP.$$

Let us now consider the variations of a given volume V of porous medium due to soil compressibility. If the grains of the porous rock are incompressible, the deformation is mainly produced by the rearrangement of the assembly of grains and the volume of the solid part $V_s = (1-\phi)V$ remains unchanged (*cf* [10]). We thus have

$$\frac{dV_s}{d\sigma_e} = -\frac{d\phi}{d\sigma_e}V + (1-\phi)\frac{dV}{d\sigma_e} = 0 \Leftrightarrow -\frac{1}{V}\frac{dV}{d\sigma_e} = \frac{1}{(1-\phi)}\frac{d\phi}{dP}. \tag{2.5}$$

Assuming small volume variations and low elastic behavior for the soil justifies the definition of the soil compressibility $\beta_P \in \mathbb{R}$ by

$$\beta_P = -\frac{1}{V}\frac{dV}{d\sigma_e}.$$

Equation (2.5) then reads

$$\beta_P = \frac{1}{1-\phi}\frac{d\phi}{dP}. \tag{2.6}$$

2.4. Hypothesis. Let us now list the assumptions on the fluid and medium characteristics but also on the flow which are meaningful in the context of seawater intrusion in an aquifer.

2.4.1. Hypothesis on the fluid and on the medium. Compressibility. First, we assume that the fluids (namely here fresh and salt water) *and* the soil are weakly compressible. It means that the densities of the fluids and the porosity of the medium weakly depend on the pressure variations, that is (in (2.4) and (2.6))

$$\alpha_P \ll 1, \quad \beta_P \ll 1. \tag{2.7}$$

Let us exploit the first assumption. In natural conditions and especially in an aquifer, one observes small fluid mobility (defined by the ratio k/μ). First consequence of the low compressibility of the fluid combined with the low mobility of fluid appears in the momentum equation. We perform a Taylor expansion with regard to P of the density ρ in the gravity term of the Darcy equation. Neglecting the terms weighted by $\alpha_P k/\mu \ll 1$ in (2.2), we get:

$$q = -K\nabla\Phi, \quad K = \frac{k\rho_0 g}{\mu}. \tag{2.8}$$

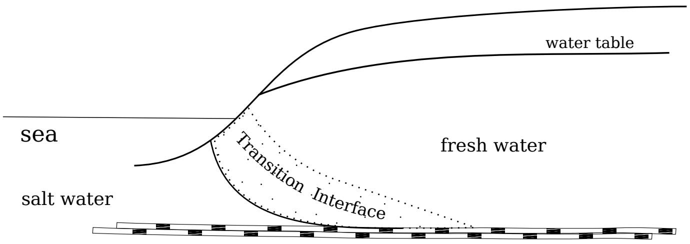


Fig. 1: Transition zone with variable salt concentration

Second consequence is $\nabla\rho \cdot q \ll 1$ which leads to the following simplification in the mass conservation equation (2.3):

$$\rho\partial_t\phi + \phi\partial_t\rho + \rho\nabla \cdot q = \rho Q.$$

Neglecting in this way the variation of density in the direction of flow is sometimes considered as an extra assumption called Bear's hypothesis (*cf* [1]). Here it follows from (2.7) and separating these assumptions seems questionable. Including (2.4) and (2.6), that is $\partial_t\rho = \rho\alpha_P\partial_t P$ and $\partial_t\phi = (1-\phi)\beta_P\partial_t P$ in the latter equation, we get

$$\rho\big((1-\phi)\beta_P + \phi\alpha_P\big)\partial_t P + \rho\nabla \cdot q = \rho Q.$$

Using the hydraulic head defined in (2.1) and the Darcy law (2.8) combined to $\rho > 0$, we finally write

$$S_0\partial_t\Phi - \nabla\cdot(K\nabla\Phi) = Q \quad \text{where} \quad S_0 = \rho_0 g\big((1-\phi)\beta_P + \phi\alpha_P\big). \tag{2.9}$$

The fluid storage coefficient S_0 characterizes the workable water volume. It accounts for the rock and fluid compressibility. In general, this coefficient is extremely small, once again due to (2.7).

Remark 1. *Model* (2.9) *is different from that of* Muskat *where density variations in the flow direction $\nabla\rho \cdot q$ are not neglected: $\rho S_0\partial_t\Phi - \nabla\cdot(\rho K\nabla\Phi) = \rho Q$.*

At this point, introducing specific index for the fresh (f) and salt (s) waters in (2.9) and using (2.8), we have derived the following model:

$$S_f\partial_t\Phi_f + \nabla\cdot q_f = Q_f, \quad q_f = -K_f\nabla\Phi_f, \tag{2.10}$$
$$S_f = \rho_f g\big((1-\phi)\beta_P + \phi\alpha_P\big),\ K_f = kg\rho_f/\mu_f,$$
$$S_s\partial_t\Phi_s + \nabla\cdot q_s = Q_s, \quad q_s = -K_s\nabla\Phi_s, \tag{2.11}$$
$$S_s = \rho_s g\big((1-\phi)\beta_P + \phi\alpha_P\big),\ K_s = kg\rho_s/\mu_s.$$

Remark 2. *Notice that due to the difference of reference quantities $\rho_f \neq \rho_s$, the model is density driven.*

2.4.2. Hypothesis on the flow. The following two assumptions are introduced for upscaling the 3D problem to a 2D model in the next subsection.

Sharp interfaces. The slow dynamics of the displacement in the aquifer let the fluids tend to the picture described in Figure 1. Of course, freshwater and saltwater are miscible. Therefore they are separated by a transition zone characterized by the variations of the salt concentration. Nevertheless the thickness of the transition zone is small compared to dimensions of the aquifer. We then assume that an abrupt interface separates two distinct domains, one for the saltwater and one for the freshwater. A sharp interface is also assumed separating the saturated and the dry parts of the aquifer, thus neglecting the thickness of the partially saturated zone. This latter free interface may be viewed as a moving water table. This approximation is justified because the thickness of the capillarity fringe is much smaller than the distance to the ground surface. We will alleviate these assumptions by re-including somehow mass transferts around interfaces in subsection 2.6 below.

Dupuit approximation (hydrostatic approach) Dupuit assumption consists in considering that the hydraulic head is constant along each vertical direction (vertical equipotentials). It is legitimate since one actually observes quasi-horizontal displacements when the thickness of the aquifer is small compared to its width and its length and when the flow is far from sinks and wells. This approximation is exact in the case of an homogeneous, isotropic and confined aquifer with constant thickness.

2.5. Upscaling procedure. We now use the approximations introduced in 2.4.2 to vertically integrate equations (2.10)–(2.11), thus reducing the 3D problem to a 2D problem.

2.5.1. Vertical integration. The aquifer is represented by a three-dimensional domain $\Omega \times (h_2, h_{max})$, $\Omega \subset \mathbb{R}^2$, function h_2 (respect. h_{max}) describing its lower (respect. upper) topography. For the sake of simplicity, we assume that the upper surface of the aquifer is at constant depth, $h_{max} \in \mathbb{R}$, and moreover that $h_{max} = 0$.

We denote by h_1 (respectively h) the depth of the free interface separating the freshwater layer and the dry part of the aquifer (respectively the saltwater layer). Since we do not consider very deep geologic formations, we assume that the pressure is constant and equal to the atmospheric pressure P_a in the upper dry part of the aquifer, that is between $z = h_1$ and $z = 0$. We impose pressure equilibrium at the boundary of each area, more precisely:

$$\left\{ \begin{array}{ll} \text{If } h_1 < h_{max} = 0 : & \Phi_{f|z=h_1} = P_a/\rho_f g + h_1 - h_{ref}. \\ \text{If } h_1 = h_{max} = 0 : & \Phi_{f|z=h_{max}} = P_a/\rho_f g - h_{ref}. \end{array} \right. \tag{2.12}$$

It follows that the right quantity for the hydraulic head Φ_f to be meaningful in the whole aquifer is $h_1^- = \inf(0, h_1)$. The upper head equilibrium condition (2.12) reads $\Phi_{f|z=h_1^-} = P_a/\rho_f g + h_1^- - h_{ref}$. Similar elements on the depth of the salt interface h lead to introduce $h^- = \inf(0, h)$.

Now we perform the vertical integration. We begin with the freshwater zone between depths h^- and h_1^-. We obtain

$$\int_{h^-}^{h_1^-} \left(S_f \partial_t \Phi_f + \nabla \cdot q_f \right) dz = \int_{h^-}^{h_1^-} Q_f \, dz, \quad B_f = h_1^- - h^-.$$

We denote by $\tilde{Q}_f$ the source term representing distributed surface supply of fresh water into the free aquifer:

$$\tilde{Q}_f = \frac{1}{B_f} \int_{h^-}^{h_1^-} Q_f \, dz.$$

Applying Leibnitz rule to the first term in the left-hand side yields:

$$\int_{h^-}^{h_1^-} S_f \partial_t \Phi_f dz = S_f \frac{\partial}{\partial t} \int_{h^-}^{h_1^-} \Phi_f dz + S_f \Phi_{f|z=h^-} \partial_t h^- - S_f \Phi_{f|z=h_1^-} \partial_t h_1^-.$$

We denote by $\tilde{\Phi}_f$ the vertically averaged hydraulic head

$$\tilde{\Phi}_f = \frac{1}{B_f} \int_{h^-}^{h_1^-} \Phi_f dz.$$

Because of Dupuit approximation, $\Phi_f(x_1, x_2, z) \simeq \tilde{\Phi}_f(x_1, x_2)$, $x = (x_1, x_2) \in \Omega$, $z \in (h^-, h_1^-)$, we have

$$\int_{h^-}^{h_1^-} S_f \partial_t \Phi_f dz = S_f B_f \partial_t \tilde{\Phi}_f.$$

We also have

$$\int_{h^-}^{h_1^-} \nabla \cdot q_f \, dz = \nabla' \cdot (B_f \tilde{q}'_f) + q_{f|z=h_1^-} \cdot \nabla(z - h_1^-) - q_{f|z=h^-} \cdot \nabla(z - h^-),$$

where $\nabla' = (\partial_{x_1}, \partial_{x_2})$, $q'_f = (q_{f,x_1}, q_{f,x_2})$ and the averaged Darcy velocity $\tilde{q}'_f = \frac{1}{B_f} \int_{h^-}^{h_1^-} q'_f \, dz$ is given by

$$\tilde{q}'_f = -\frac{1}{B_f} \int_{h^-}^{h_1^-} \left(K'_f \nabla' \Phi_f \right) dz = -\frac{1}{B_f} \int_{h^-}^{h_1^-} \left(K'_f \nabla' \tilde{\Phi}_f \right) dz = -\tilde{K}'_f \nabla' \tilde{\Phi}_f,$$

$$\tilde{K}'_f = \frac{1}{B_f} \int_{h^-}^{h_1^-} K'_f \, dz.$$

The averaged mass conservation law for the freshwater thus finally reads

$$\begin{aligned} S_f B_f \partial_t \tilde{\Phi}_f = {} & \nabla' \cdot (B_f \tilde{K}'_f \nabla' \tilde{\Phi}_f) - q_{f|z=h_1^-} \cdot \nabla(z - h_1^-) \\ & + q_{f|z=h^-} \cdot \nabla(z - h^-) + B_f \tilde{Q}_f. \end{aligned} \tag{2.13}$$

Similar computations in the saltwater layer give

$$\begin{aligned} S_s B_s \partial_t \tilde{\Phi}_s = {} & \nabla' \cdot (B_s \tilde{K}'_s \nabla' \tilde{\Phi}_s) + q_{s|z=h_2} \cdot \nabla(z - h_2) \\ & - q_{s|z=h^-} \cdot \nabla(z - h^-) + B_s \tilde{Q}_s, \end{aligned} \tag{2.14}$$

where $B_s = h^- - h_2$ is the thickness of the saltwater zone. In these equations, term $B_i \tilde{K}'_i$, $i = f, s$, may be viewed as the dynamic transmissivity of each layer. At this point, we have obtained an undetermined system of two pdes (2.13)–(2.14) with four unknowns, $\tilde{\Phi}_i$, $i = f, s$, h_1^- and h^-.

2.6. Fluxes and continuity equations across the interfaces. Our aim is now to include in the model the continuity and transfert properties across interfaces. As a consequence, we express the four flux terms appearing in (2.13)–(2.14) and we reduce the number of unknowns.

2.6.1. Fluxes across the interfaces. The present subsection is fundamental. Indeed our approach began like a sharp interface approach but it re-includes now existence of miscible zones, taking the form of two diffuse interfaces: one of characteristic thickness δ_1 between the dry and saturated zones and the other one of characteristic thickness δ_h between fresh and salt water. As mentioned in the introduction, this coupled sharp-diffuse interface approach is the new point relative to the existing literature that makes our work completely original. Furthermore the dynamics of these diffuse interfaces is ruled by phase field models.

Phase field models were first introduced for the description of phase transitions and solidification processes [9]. They are now largely used for modeling binary fluids transitions. Closer to our context, such models are employed to describe imbibition in porous media [12] (comparable to the upper interface in our setting). In phase separation problems the model typically contains a double-well potential in which the local minima correspond to the homogeneous stable states. Here we rather use a triple-well potential for respecting the primal sharp interface approach and for including the effect of this macroscopic front in the local phase field model. The energy functional also contains nonlocal terms involving the gradient (and possibly higher-order derivatives) of the phase field. In the present work we choose a simple model, namely a tristable Allen–Cahn type model.

• **Flux across the fresh-saltwater interface**

We introduce an order parameter F_h (the phase field) that "labels" the two "phases" (salt and fresh water) and the sharp interface:

$$F_h = \begin{cases} 0 & \text{in freshwater} \\ c_s/2 & \text{on sharp interface} \\ c_s & \text{in saltwater} \end{cases}$$

where c_s is *e.g.* the mean concentration of salt in the salty area. The sharp interface at time t corresponds to set $\{(x_1, x_2, z) \text{ s.t. } F_h(x_1, x_2, z, t) = c_s/2\}$. Function F_h satisfies an equation of Allen–Cahn type with three points of stability

$$\partial_t F_h + \vec{v} \cdot \nabla F_h - \delta_h \Delta' F_h + \frac{F_h(F_h - c_s/2)(F_h - c_s)}{\delta_h} = 0,$$

where we denote by $\vec{v}$ the velocity of the interface. Note that we have already neglected here the vertical diffusion with regard to the convective term. The characteristic size of the corresponding diffuse interface is $\delta_h > 0$ (see *e.g.* [3] for rigorous results). Since the stability set $\{F_h = c_s/2\}$ corresponds to the sharp interface of depth h^-, we have

$$F_h(x_1, x_2, z, t) = c_s/2 \ \Leftrightarrow\ z - h^-(x_1, x_2, t) = 0.$$

Differentiating (twice) $F_h(x_1, x_2, h^-(x_1, x_2, t), t) = c_s/2$ and including the result in the projection of the Allen–Cahn equation for $F_h = c_s/2$, we get

$$\partial_z F_h\big(-\partial_t h^- + \vec{v} \cdot \nabla(z - h^-) + \delta_h \Delta' h^-\big) + \delta_h \nabla' h^- \cdot \nabla' \partial_z F_h + \delta_h |\nabla' h^-|^2 \partial^2_{zz} F_h = 0.$$

The two last terms of the lefthand side of the latter relation may be neglected. Indeed they combine three low order quantities. First, of course, the diffusion parameter δ_h which is the characteristic size of the diffuse interface is small. Next point comes from the dynamics of the Allen–Cahn equation. A formal asymptotic analysis shows that the reaction term is dominant at small times, so that in the rescaled time scale

$t' = t/\delta_h^2$ the dynamics essentially lie in the ode $\partial_{t'} F_h = F_h(F_h - c_s/2)(F_h - c_s)$ and the values of F_h tend to the stable values thus creating steep transition layers. Then the propagation is associated with much slower time scale, convective and diffusive terms coming to balance with the reaction term near the stable surfaces, but the regular steep structure of the diffuse interface ensures small $\nabla'\partial_z F_h$ and order one $\partial^2_{zz} F_h$. Furthermore Dupuit's work [13] is based on the observation that in such a groundwater flow the slope of the interfacial surface is very small, that is $|\nabla' h^-| \ll 1$. For the same reason, function F_h heuristically behaves like a step function in the vertical direction and $\partial_z F_h \neq 0$. The latter equation thus gives:

$$-\partial_t h^- + \vec{v}\cdot\nabla(z-h^-) + \delta_h \Delta' h^- = 0. \tag{2.15}$$

We then turn back to the traditional sharp interface characterization. There is no mass transfer across the interface between fresh and salt water, *i.e.* the normal component of the effective velocity is continue at the interface $z = h^-$:

$$\Big(\frac{q_{f|z=h^-}}{\phi} - \vec{v}\Big)\cdot\vec{n} = \Big(\frac{q_{s|z=h^-}}{\phi} - \vec{v}\Big)\cdot\vec{n} = 0. \tag{2.16}$$

where $\vec{n}$ denotes the normal unit vector to the interface, $\vec{n} = |\nabla(z-h^-)|^{-1}\nabla(z-h^-)$.

Combining (2.15) and (2.16), we obtain :

$$\begin{aligned} q_{f|z=h^-}\cdot\nabla(z-h^-) = q_{s|z=h^-}\cdot\nabla(z-h^-) &= \phi(\partial_t h^- - \delta_h \Delta' h^-) \\ &= \phi\big(\mathcal{X}_0(-h)\partial_t h - \delta_h \nabla'\cdot(\mathcal{X}_0(-h)\nabla' h)\big) \end{aligned} \tag{2.17}$$

where we set

$$\mathcal{X}_0(h) = \left\{ \begin{array}{lll} 0 & \text{if} & h \le 0 \\ 1 & \text{if} & h > 0 \end{array} \right. .$$

Relation (2.17) is a regularized Stefan type boundary condition.

Remark 3. *For emphasizing once again the consistency with our primal sharp interface approach, we recall that rigorous asymptotic results let recover the sharp interface evolution equation. More precisely, if $\delta_h \to 0$, Allen–Cahn model tends to the classical Stefan problem, that is the classical modeling of the interface evolution given by a level-set equation $q_f(h^-)\cdot\nabla(z-h^-) = \phi\partial_t h^-$ (see [18]). If the Allen–Cahn equation is written as*

$$\partial_t F^\epsilon + \vec{v}\cdot\nabla F^\epsilon - \gamma\Delta F^\epsilon + \frac{F^\epsilon(F^\epsilon - c_s/2)(F^\epsilon - c_s)}{\epsilon\delta_h} = 0,$$

γ being a parameter related to the elasticity of the interface and to δ_h, when letting $\epsilon \to 0$ for any given γ, we get (see [2] and the references therein)

$$\partial_t F + \vec{v}\cdot\nabla F - \gamma\Delta F = 0.$$

• **Flux across the unsaturated-saturated interface**

We perform the same reasoning for the upper capillary fringe. Likewise, defining the phase function F_1 by

$$F_1 = \left\{ \begin{array}{lll} -1 & \text{in} & \text{unsaturated zone} \\ 0 & \text{at} & \text{sharp interface} \\ 1 & \text{in} & \text{saturated zone} \end{array} \right. ,$$

the sharp interface is characterized by $F_1(x_1,x_2,z,t)=0 \;\Leftrightarrow\; z-h_1^-(x_1,x_2,t)=0$. The leading order terms of the projection on $z=h_1^-$ of a tristable Allen–Cahn equation for a diffuse interface of characteristic size δ_1 give

$$-\partial_t h_1^- + \vec{v}\cdot\nabla(z-h_1^-) + \delta_1\Delta' h_1^- = 0. \tag{2.18}$$

We combine the latter equation with the relation ruling continuity of the normal component of the velocity

$$\Big(\frac{q_{f|z=h_1^-}}{\phi} - \vec{v}\Big)\cdot\vec{n} = 0,$$

and we obtain

$$\begin{aligned} q_{f|z=h_1^-}\cdot\nabla(z-h_1^-) &= \phi\big(\partial_t h_1^- - \delta_1\Delta' h_1^-\big) \\ &= \phi\big(\mathcal{X}_0(-h_1)\partial_t h_1 - \delta_1\nabla'\cdot\big(\mathcal{X}_0(-h_1)\nabla' h_1\big)\big). \end{aligned} \tag{2.19}$$

• **Impermeable layer at** $z=h_2$

If the lower layer is impermeable, there is no flux across the boundary $z=h_2$:

$$q_s(h_2)\cdot\nabla(z-h_2) = 0. \tag{2.20}$$

2.6.2. Continuity equations. Continuity relations now imposed on the interfaces will allow to properly reduce the number of unknowns in equations (2.13)-(2.14).

Dupuit approximation reads $\tilde\Phi_f \simeq \Phi_{f|z=h_1^-}$, that is

$$\tilde\Phi_f = \frac{P_a}{\rho_f g} + h_1^- - h_{ref}. \tag{2.21}$$

Bearing in mind the boundary condition (2.12) at the upper free interface and approximation $\Phi_{f|z=h_1^-} \simeq \Phi_{f|z=h^-}$, we have

$$\frac{P_a}{\rho_f g} + h_1^- - h_{ref} = \frac{P_{f|z=h^-}}{\rho_f g} + h^- - h_{ref} \;\Leftrightarrow\; P_{f|z=h^-} = P_a + \rho_f g(h_1^- - h^-).$$

Besides, the pressure is continuous at the interface between salt and fresh water. Since $P_{s|z=h^-} = \rho_s g(\Phi_{s|z=h^-} - h^- + h_{ref})$ and $\Phi_{s|z=h^-} \simeq \tilde\Phi_s$, it follows that

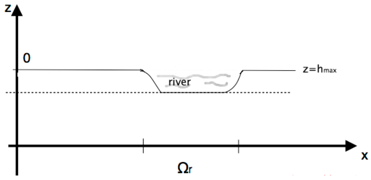
$$(1+\alpha)\,\tilde\Phi_s = \frac{P_a}{\rho_f g} + h_1^- + \alpha h^- - (1+\alpha)h_{ref}, \quad \alpha = \frac{\rho_s}{\rho_f} - 1. \tag{2.22}$$

Here parameter α characterizes the densities contrast.

Equations (2.21)-(2.22) allow us to avoid $\tilde\Phi_f$ and $\tilde\Phi_s$ in the final system.

2.6.3. Presence of other water ressources. Up to now, we have considered that the aquifer is surrounded by a dry zone. We suggest other settings in the present subsection.

• **Presence of a river in a part** $\Omega_r\times\{h_{max}\}$**,** $\Omega_r\subset\Omega$**, of the upper boundary:** Assume existence of a deflection in the upper bound of the aquifer containing a river. The river is in hydrodynamic equilibrium with the atmosphere, that is, if P_r is the pressure in the river:



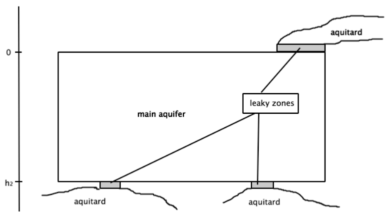
$$P_r(x_1,x_2,z) = P_a + \rho_f g(0-z),\ (x_1,x_2)\in\Omega_r,\ h_{max}\le z\le 0.$$

Hydraulic head of the river Φ_r is thus also constant with regard to z, just like Φ_f. The usual boundary condition at the interface between the aquifer and the river consists in prescribing the continuity of the hydraulic head. It reads

$$\Phi_{f|\{x\in\Omega_r,\, z=h_{max}\}} = \Phi_{r|\{x\in\Omega_r,\, z=h_{max}\}} = \frac{P_{r|z=h_{max}}}{\rho_f g} + h_{max} - h_{ref} = \frac{P_a}{\rho_f g} - h_{ref}.$$

Bearing in mind the general definition of $\tilde{\Phi}_f$ in the aquifer (see (2.21)), we can interpret the latter relation: when the free water interface touches the river, the model includes the river depth in the freshwater zone and h_1^- jumps from h_{max} to 0. In this case, the flux term is $q_{f|z=h_1^-=h_{max}} \cdot \nabla(z-h_1^-) = 0$. The same type of boundary condition holds true along the outflow face for freshwater along the bottom of the sea (with of course a term containing the density ratio ρ_s/ρ_f instead of $1/\rho_f$, see [8] section 9.7). But in this case, Dupuit assumption fails.

• **Presence of a weakly impermeable zone (aquitard):** Flux between the aquitard and the water contained in the aquifer consist in a leakage term q_L. The generic model for recharge and discharge is $q_L = (\Phi_{ext} - \Phi)/c_m$ where Φ (respect. Φ_{ext}) is the head on the aquifer's (respect. aquitard's) side of the semi-pervious 'membrane', resistance $c_m = \mathcal{O}(b_m/k_m)$ depending on the thickness b_m and on the permeability k_m ($k_m \ll k$) of the membrane (see *e.g.* [21]). This formulation allows to treat charge and discharge, depending on the ratio between Φ and Φ_{ext}. The boundary condition on the semi-permeable interface is $(-k\nabla\Phi)\cdot\vec{n} = q_L$, where $\vec{n}$ is the *outward* unit vector normal to the aquifer's boundary.



Here we can include a fresh leakage term q_{Lf} from the top of the aquifer to the fresh water when the aquifer is fully saturated (that is $h_1 = 0$) and a salty leakage term q_{Ls} from the bottom to the saltwater:

$$q_{f|z=h_1^-}\cdot\nabla(z-h_1^-) = q_{Lf} \text{ when } h_1^- = 0, \quad q_{s|z=h_2}\cdot\nabla(z-h_2) = -q_{Ls}. \tag{2.23}$$

This condition will be inserted below. More precisely, in view of (2.21), term q_{Lf} reads

$$\begin{aligned} q_{Lf}(x,h_1,h) &= \big(1-\chi_0(-h_1)\big)\chi_0(h_1^- - h^-)\frac{k_{mf}(x)}{b_{mf}(x)}\Big(\frac{P_{ext,f}(x)}{\rho_f g} + b_{mf}(x) - \frac{P_a}{\rho_f g} - h_1^-\Big) \\ &= \big(1-\chi_0(-h_1)\big)\chi_0(h_1^- - h^-)\frac{k_{mf}(x)}{b_{mf}(x)}\Big(\frac{P_{ext,f}(x)}{\rho_f g} + b_{mf}(x) - \frac{P_a}{\rho_f g}\Big). \end{aligned}$$

Indeed we specify that only fresh exchanges are allowed, thus the term $\chi_0(h_1^- - h^-)$, and that the semi-permeable zone is at depth $h_{max} = 0$, thus the term $(1-\chi_0(-h_1))$ (we consider here a phreatic aquifer: there is no leakage at the upper boundary unless the aquifer is fully saturated). We impose $k_{mf} = 0$ outside the aquitard's area. The same type of arguments and (2.22) leads to

$$\begin{aligned} &q_{Ls}(x,h_1,h) \\ &= \chi_0(h^- - h_2)\frac{k_{ms}(x)}{b_{ms}(x)}\Big(\frac{P_{ext,s}(x)}{\rho_s g} + h_2 - b_{ms}(x) - \frac{P_a}{\rho_f g(1+\alpha)} - \frac{h_1^-}{1+\alpha} - \frac{\alpha h^-}{1+\alpha}\Big). \end{aligned}$$

2.7. Conclusion: seawater intrusion model. We begin by some assumptions, essentially introduced for the sake of simplicity of the equations. The medium is supposed to be isotropic and the viscosity the same for the salt and fresh water. Using definition (2.2) for the permeabilities $\tilde{K}'_f$ and $\tilde{K}'_s$, it follows from $\mu_f = \mu_s$ that

$$\tilde{K}'_s = (1+\alpha)\tilde{K}'_f. \tag{2.24}$$

The two-dimensional model (2.13)–(2.14) now reads:

$$\begin{aligned} &S_f B_f \partial_t \tilde{\Phi}_f - \nabla' \cdot (B_f \tilde{K}'_f \nabla' \tilde{\Phi}_f) + q_{f|z=h_1^-} \cdot \nabla(z - h_1^-) \\ &\qquad -q_{f|z=h^-} \cdot \nabla(z-h^-) = B_f \tilde{Q}_f, \end{aligned} \tag{2.25}$$

$$\begin{aligned} &S_s B_s \partial_t \tilde{\Phi}_s - (1+\alpha)\nabla' \cdot (B_s \tilde{K}'_f \nabla' \tilde{\Phi}_s) - q_{s|z=h_2} \cdot \nabla(z - h_2) \\ &\qquad +q_{s|z=h^-} \cdot \nabla(z-h^-) = B_s \tilde{Q}_s. \end{aligned} \tag{2.26}$$

We can neglect the term $S_s B_s \partial_t \tilde{\Phi}_s$ because of the two following arguments. First the saltwater is confined since the bottom of the aquifer is assumed essentially impermeable:

$$\partial_t \Phi_s \ll 1.$$

Besides $S_s \ll 1$ because of the weak compressibility of the fluid and of the rock (see (2.7)), hence

$$S_s = \rho_s g\big((1-\phi)\beta_P + \phi\alpha_P\big) \ll 1.$$

We now choose to base the model on the salt mass conservation and on the total mass conservation. We thus write (2.26) using $S_s B_s \partial_t \tilde{\Phi}_s \simeq 0$, and the sum of (2.25) and (2.26):

$$\begin{aligned} &-(1+\alpha)\nabla' \cdot \big(B_s \tilde{K}'_f \nabla' \tilde{\Phi}_s\big) + q_{s|z=h^-} \cdot \nabla\big(z-h^-\big) - q_{s|z=h_2} \cdot \nabla\big(z-h_2\big) = B_s \tilde{Q}_s, \\ &S_f B_f \partial_t \tilde{\Phi}_f - \nabla' \cdot \big(B_f \tilde{K}'_f \nabla' \tilde{\Phi}_f\big) - (1+\alpha)\nabla' \cdot \big(B_s \tilde{K}'_f \nabla \tilde{\Phi}_s\big) + q_{f|z=h_1^-} \cdot \nabla\big(z-h_1^-\big) \\ &\qquad -q_{s|z=h_2} \cdot \nabla\big(z-h_2\big) = B_f \tilde{Q}_f + B_s \tilde{Q}_s. \end{aligned}$$

Once again for the sake of simplicity, we assume $\delta_h = \delta_1 := \delta$ (the diffuse interfaces widths are of the same order). We also reverse the vertical axis thus changing h_1 into $-h_1$, h into $-h$, h_2 into $-h_2$, z into $-z$. Bearing in mind that now $B_s = h_2 - h^+$, $B_f = h^+ - h_1^+$ and using (2.17), (2.19), (2.20) and (2.21)-(2.22), we write the latter system as:

$$(\mathcal{M}_1) \quad \left\{ \begin{aligned} &\phi \mathcal{X}_0(h)\partial_t h - \nabla' \cdot \big(\alpha \tilde{K}'_f (h_2 - h^+)\nabla' h\big) - \nabla' \cdot \big(\delta\phi \mathcal{X}_0(h)\nabla' h\big) \\ &\quad -\nabla' \cdot \big(\tilde{K}'_f \mathcal{X}_0(h_1)(h_2 - h^+)\nabla' h_1\big) - q_{Ls}(x,h_1,h) = -\tilde{Q}_s(h_2 - h^+), \\ &\mathcal{X}_0(h_1)\big(S_f(h^+ - h_1^+) + \phi\big)\partial_t h_1 - \nabla' \cdot \big(\tilde{K}'_f \mathcal{X}_0(h_1)\big((h^+ - h_1^+) + (h_2 - h^+)\big) \\ &\quad \nabla' h_1\big) - \nabla' \cdot \big(\delta\phi \tilde{K}'_f \mathcal{X}_0(h_1)\nabla' h_1\big) - \nabla' \cdot \big(\tilde{K}'_f \alpha (h_2 - h^+)\mathcal{X}_0(h)\nabla' h\big) \\ &\quad -q_{Lf}(x,h_1,h) - q_{Ls}(x,h_1,h) = -\tilde{Q}_f(h^+ - h_1^+) - \tilde{Q}_s(h_2 - h^+). \end{aligned} \right.$$

If we also use assumption (2.7) to neglect the storage coefficient S_f in the salty layer, the latter model reduces to:

$$(\mathcal{M}_2) \quad \left\{ \begin{aligned} &\phi \mathcal{X}_0(h)\partial_t h - \nabla' \cdot \big(\alpha \tilde{K}'_f (h_2 - h^+)\nabla' h\big) - \nabla' \cdot \big(\delta\phi \mathcal{X}_0(h)\nabla' h\big) \\ &\quad -\nabla' \cdot \big(\tilde{K}'_f \mathcal{X}_0(h_1)(h_2 - h^+)\nabla' h_1\big) - q_{Ls}(x,h_1,h) = -\tilde{Q}_s(h_2 - h^+), \\ &\phi \mathcal{X}_0(h_1)\partial_t h_1 - \nabla' \cdot \big(\tilde{K}'_f \mathcal{X}_0(h_1)\big((h^+ - h_1^+) + (h_2 - h^+)\big)\nabla' h_1\big) \\ &\quad -\nabla' \cdot \big(\delta\phi \tilde{K}'_f \mathcal{X}_0(h_1)\nabla' h_1\big) - \nabla' \cdot \big(\tilde{K}'_f \alpha (h_2 - h^+)\mathcal{X}_0(h)\nabla' h\big) \\ &\quad -q_{Lf}(x,h_1,h) - q_{Ls}(x,h_1,h) = -\tilde{Q}_f(h^+ - h_1^+) - \tilde{Q}_s(h_2 - h^+). \end{aligned} \right.$$

In $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$, leakage terms q_{Lf} and q_{Ls} are defined by
(2.27)

$$
\begin{aligned}
q_{Lf}(x,h_1,h) &= \big(1-\chi_0(h_1)\big)\chi_0(h-h_1)\frac{k_{mf}(x)}{b_{mf}(x)}\Big(\frac{P_{ext,f}(x)}{\rho_f g}+b_{mf(x)}-\frac{P_a}{\rho_f g}\Big),\\
q_{Ls}(x,h_1,h) &= \chi_0(h_2-h)\frac{k_{ms}(x)}{b_{ms}(x)}\Big(\frac{P_{ext,s}(x)}{\rho_s g}-h_2-b_{ms(x)}-\frac{P_a}{\rho_f g(1+\alpha)}\\
&\qquad +\frac{h_1^+}{1+\alpha}+\frac{\alpha h^+}{1+\alpha}\Big).
\end{aligned}
$$

Both systems give a $2D$ description for tracking a saltwater front in a free aquifer, the third dimension remaining in the model thanks to the unknowns h and h_1 which are the free interfaces depths.

Remark 4.
We remark that if $h_1=0$ in a subdomain Ω_0 with non null measure, the second equation of the above system (without the source terms $\tilde{Q}_f$, $\tilde{Q}_s$, q_{Lf} and q_{Ls}) gives $\nabla'\cdot\big(\tilde{K}_f'\alpha(h_2-h^+)\nabla' h\big)_{|\Omega_0}=0$, that is for any $\varphi\in V$:

$$
\begin{aligned}
&\int_{\partial\Omega_0}\alpha\big[\varphi(\tilde{K}_f'(h_2-h^+)\big]\nabla' h\cdot n-\int_{\Omega_0}\alpha\tilde{K}_f'(h_2-h^+)\nabla' h\cdot\nabla'\varphi=0,\\
\Leftrightarrow &\int_{\Omega_0}\nabla'\cdot\Big(\alpha\mathcal{X}_0(h_1)\tilde{K}_f'(h_2-h^+)\nabla h\Big)\varphi=0,
\end{aligned}
$$

where V is a well adapted space of test functions. Then substituting all the terms of the form $\nabla'\cdot\big(\tilde{K}_f'\alpha(h_2-h^+)\nabla' h\big)$ by $\nabla'\cdot\big(\alpha\mathcal{X}_0(h_1)\tilde{K}_f'(h_2-h^+)\nabla' h\big)$ does not change the physical meaning of the problem. We will use this point in the mathematical analysis below.

3. Mathematical setting and main results. We consider an open bounded domain Ω of $\mathbb{R}^2$ describing the projection of the aquifer on the horizontal plane. The boundary of Ω, assumed $\mathcal{C}^1$, is denoted by Γ. The time interval of interest is $(0,T)$, T being any nonnegative real number, and we set $\Omega_T=(0,T)\times\Omega$.

3.1. Some auxiliary results. For the sake of brevity we shall write $H^1(\Omega)=W^{1,2}(\Omega)$ and

$$V=H_0^1(\Omega),\ V'=H^{-1}(\Omega),\ H=L^2(\Omega).$$

The embeddings $V\subset H=H'\subset V'$ are dense and compact. For any $T>0$, let $W(0,T)$ denote the space

$$W(0,T):=\big\{\omega\in L^2(0,T;V),\ \partial_t\omega\in L^2(0,T;V')\big\}$$

endowed with the Hilbertian norm $\|\omega\|_{W(0,T)}=\big(\|\omega\|^2_{L^2(0,T;V)}+\|\partial_t\omega\|^2_{L^2(0,T;V')}\big)^{1/2}$. The following embeddings are continuous ([16] prop. 2.1 and thm 3.1, chapter 1)

$$W(0,T)\subset\mathcal{C}([0,T];[V,V']_{\frac{1}{2}})=\mathcal{C}([0,T];H)$$

while the embedding

$$W(0,T)\subset L^2(0,T;H) \tag{3.1}$$

is compact (Aubin's Lemma, see [20]). The following result by F. Mignot (see [15]) is used in the sequel.

Lemma 1. *Let $f : R \to R$ be a continuous and nondecreasing function such that $\limsup_{|\lambda|\to+\infty} |f(\lambda)/\lambda| < +\infty$. Let $\omega \in L^2(0,T;H)$ be such that $\partial_t\omega \in L^2(0,T;V')$ and $f(\omega) \in L^2(0,T;V)$. Then*

$$\langle \partial_t\omega, f(\omega)\rangle_{V',V} = \frac{d}{dt}\int_\Omega \Big(\int_0^{\omega(\cdot,y)} f(r)\,dr\Big)\,dy \ \text{ in } \mathcal{D}'(0,T).$$

Hence for all $0 \le t_1 < t_2 \le T$

$$\int_{t_1}^{t_2} \langle \partial_t\omega, f(\omega)\rangle_{V',V}\,dt = \int_\Omega \Big(\int_{\omega(t_1,y)}^{\omega(t_2,y)} f(r)\,dr\Big)\,dy.$$

3.2. Main results. We focus on the model with neglected storativity $(\mathcal{M}_2)$. For the complete model $(\mathcal{M}_1)$ we refer to [11]. We aim giving an existence result of physically admissible weak solutions for this model completed by initial and boundary conditions.

First we re-write the model $(\mathcal{M}_2)$ with some notational simplifications. The 'primes' are suppressed in the differentiation operators in $\mathbb{R}^2$. Source terms are denoted without 'tildes'. Permeability $\tilde{K}'_f$ is now denoted by K. We set $\alpha = 1$. We assume that depth h_2 is constant, $h_2 > 0$. We define some functions T_s and T_f by

$$T_s(u) = h_2 - u, \quad T_f(u) = u, \quad \text{for } u \in (0,h_2).$$

These functions are extended continuously and constantly outside $(0,h_2)$. We then consider the following set of equations in Ω_T:

$$\begin{aligned}\phi\partial_t h - \nabla\cdot\big(KT_s(h)\mathcal{X}_0(h_1)\nabla h\big) - \nabla\cdot\big(\delta\phi\nabla h\big) - \nabla\cdot\big(KT_s(h)\mathcal{X}_0(h_1)\nabla h_1\big)&\\ -\phi q_{Ls}(x,h_1,h) = -Q_sT_s(h),& \quad (3.2)\end{aligned}$$

$$\begin{aligned}\phi\partial_t h_1 - \nabla\cdot\Big(K\big(T_f(h-h_1) + \mathcal{X}_0(h_1)T_s(h)\big)\nabla h_1\Big) - \nabla\cdot\big(\delta\phi K\nabla h_1\big)&\\ -\nabla\cdot\big(KT_s(h)\mathcal{X}_0(h_1)\nabla h\big)&\\ -\phi q_{Lf}(x,h_1,h) - \phi q_{Ls}(x,h_1,h) = -Q_fT_f(h-h_1) - Q_sT_s(h).& \quad (3.3)\end{aligned}$$

Notice that we do not use $h^+ = \sup(0,h)$ and $h_1^+ = \sup(0,h_1)$ in functions T_s and T_f because a maximum principle will ensure that these supremums are useless. Likewise, we have canceled the terms $\mathcal{X}_0(h)$ (resp. $\mathcal{X}_0(h_1)$) in front of $\partial_t h$ and ∇h (resp. $\partial_t h_1$). We have also used Remark 4. System (3.3) is completed by the following boundary and initial conditions:

$$h = h_D, \quad h_1 = h_{1,D} \quad \text{in } \Gamma\times(0,T), \tag{3.4}$$

$$h(0,x) = h_0(x), \quad h_1(0,x) = h_{1,0}(x) \quad \text{in } \Omega, \tag{3.5}$$

with the compatibility conditions

$$h_0(x) = h_D(0,x), \quad h_{1,0}(x) = h_{1,D}(0,x), \quad x\in\Gamma.$$

Let us now detail the mathematical assumptions. We begin with the characteristics of the porous structure. We assume the existence of two positive real numbers K_- and K_+ such that the hydraulic conductivity tensor is a bounded elliptic and uniformly positive definite tensor:

$$0 < K_-|\xi|^2 \le \sum_{i,j=1,2} K_{i,j}(x)\xi_i\xi_j \le K_+|\xi|^2 < \infty \quad x\in\Omega,\ \xi\in\mathbb{R}^2,\ \xi\ne 0.$$

We assume that porosity is constant in the aquifer. Indeed, in the field envisaged here, the effects due to variations in ϕ are negligible compared with those due to density contrasts. From a mathematical point of view, these assumptions do not change the complexity of the analysis but rather avoid cumbersome computations.

Source terms Q_f and Q_s are given functions of $L^2(0,T;H)$. Leakage term q_{Lf} and q_{Ls} are defined by (2.27). We simplify notations by setting

$$\begin{aligned} q_{Lf}(x,h_1,h) &= \big(1-\chi_0(h_1)\big)\chi_0(h-h_1)Q_{Lf}(x),\\ q_{Ls}(x,h_1,h) &= \chi_0(h_2-h)Q_{Ls}(x)\big(R_{Ls}(x)+h_1/2+h/2\big), \end{aligned} \tag{3.6}$$

where Q_{Lf}, Q_{Ls} and $Q_{Ls}R_{Ls}$ are functions of $L^2(0,T;H)$ such that

$$Q_{Lf}\geq 0,\ Q_{Ls}\geq 0,\ R_{Ls}\geq 0 \text{ a.e. in } \Omega\times(0,T). \tag{3.7}$$

Assumption $Q_{Lf}\geq 0$ a.e. in Ω_T means that the leakage through the aquitard is upwards (indeed leakage occurs from low to high piezometric head, see [8]). We also assume

$$-\big(\max(Q_f,0)+\max(Q_s,0)\big)h_2+Q_{Lf}+Q_{Ls}R_{Ls}\geq 0 \text{ a.e. in } \Omega\times(0,T). \tag{3.8}$$

This assumption which could appears rather technical is actually introduced because the aquifer's depth is at most h_2. All the source terms thus have to compensate somehow. Assumption (3.8) is the mathematical companion of the common-sense principle 'a filled box can no more be filled in'. Notice for instance that pumping of freshwater corresponds to assumption $Q_f\leq 0$ a.e. in $\Omega\times(0,T)$. Functions h_D and $h_{1,D}$ belong to the space $L^2(0,T;H^1(\Omega))\cap H^1(0,T;(H^1(\Omega))')$ while functions h_0 and $h_{1,0}$ are in $H^1(\Omega)$. Finally, we assume that the boundary and initial data satisfy physically realistic conditions on the hierarchy of interfaces depths:

$$0\leq h_{1,D}\leq h_D\leq h_2 \text{ a.e. in } \Gamma\times(0,T),\quad 0\leq h_{1,0}\leq h_0\leq h_2 \text{ a.e. in } \Omega.$$

We state and prove the following existence result.

Theorem 1. *Assume a low spatial heterogeneity for the hydraulic conductivity tensor:*

$$K_-\leq K_+\leq 2K_-.$$

Then for any $T>0$, problem (3.3)–(3.5) *admits a weak solution* (h,h_1) *satisfying* $(h-h_D,h_1-h_{1,D})\in W(0,T)\times W(0,T)$. *Futhermore the following maximum principle holds true:*

$$0\leq h_1(t,x)\leq h(t,x)\leq h_2\quad \textit{for a.e. } x\in\Omega \textit{ and for any } t\in(0,T).$$

Next section is devoted to the proof of Theorem 1. Let us sketch our strategy. First step consists in using a Schauder fixed point theorem for proving an existence result for an auxiliary regularized and truncated problem. More precisely we regularize the step function $\mathcal{X}_0$ with a parameter $\epsilon>0$ and we introduce a weight based on the velocity of the salt front in the equation of the upper free interface. Subsequent difficulty is that the mapping used for the fixed point approach has to be continuous in the strong topology of $L^2(0,T;H^1(\Omega))$. We then prove that we have sufficient control on the velocity of the salt front to ignore the latter weight. We show that the regularized solution satisfies the maximum principles announced in Theorem 1. We finally show sufficient uniform estimates to let the regularization ϵ tend to zero.

4. Proof. Without lost of generality, we can simplify the equations by taking null leakage terms $q_{Lf} = q_{Ls} = 0$ for the existence proof. The leakage terms will be re-inserted when stating the maximum principle results. Let $\epsilon > 0$ and pick a constant $M > 0$ that we will precise later. For any $x \in \mathbb{R}_+^*$, we set

$$L_M(x) = \min\Big(1, \frac{M}{x}\Big).$$

Such a truncation L_M was originally introduced in [19]. It allows to use the following point in the estimates hereafter. For any $(g, g_1) \in (L^\infty(0,T;H^1(\Omega)))^2$, setting

$$d(g, g_1) = -T_s(g) L_M\big(||\nabla g_1||_{L^2}\big)\nabla g_1,$$

we have

$$\|d(g, g_1)\|_{L^\infty(0,T;H)} = \sup_{t\in(0,T)} \|T_s(g)L_M\big(\|\nabla g_1\|_{L^2}\big)\nabla g_1\|_H \leq M h_2.$$

We also define a regularized step function for $\mathcal{X}_0$ by

$$\mathcal{X}_0(h_1) = \begin{cases} 0 & \text{if} \quad h_1 \leq 0 \\ 1 & \text{if} \quad h_1 > 0 \end{cases}, \qquad \mathcal{X}_0^\epsilon(h_1) = \begin{cases} 0 & \text{if} \quad h_1 \leq 0 \\ h_1/(h_1^2+\epsilon)^{1/2} & \text{if} \quad h_1 > 0. \end{cases}$$

Then $0 \leq \mathcal{X}_0^\epsilon \leq 1$ and $\mathcal{X}_0^\epsilon \to \mathcal{X}_0$ as $\epsilon \to 0$. Introducing the regularization $\mathcal{X}_0^\epsilon$ of $\mathcal{X}_0$, we replace system (3.3) by the following one:

$$\begin{aligned}
&\phi\partial_t h^\epsilon - \nabla\cdot(\delta\phi\nabla h^\epsilon) - \nabla\cdot\big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla h^\epsilon\big)\\
&\qquad -\nabla\cdot\big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)L_M\big(||\nabla h_1^\epsilon||_{L^2}\big)\nabla h_1^\epsilon\big) = -Q_sT_s(h^\epsilon),\\
&\phi\partial_t h_1^\epsilon - \nabla\cdot(\delta\phi\nabla h_1^\epsilon) - \nabla\cdot\Big(K\big(T_f(h^\epsilon - h_1^\epsilon) + T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\big)\nabla h_1^\epsilon\Big)\\
&\qquad -\nabla\cdot\big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla h^\epsilon\big) = -Q_fT_f(h^\epsilon - h_1^\epsilon) - Q_sT_s(h^\epsilon).
\end{aligned}$$

The proof is outlined as follows : In the first step, using Schauder theorem, we prove that for every $T > 0$ and every $\epsilon > 0$, the above regularized system completed by the initial and boundary conditions

$$h^\epsilon = h_D,\ h_1^\epsilon = h_{1,D} \text{ in } \Gamma\times(0,T), \quad h^\epsilon(0,x) = h_0(x),\ h_1^\epsilon(0,x) = h_{1,0}(x) \text{ a.e. in } \Omega,$$

has a solution $(h^\epsilon, h_1^\epsilon)$ such that $(h^\epsilon - h_D, h_1^\epsilon - h_{1,D}) \in W(0,T)\times W(0,T)$. We observe that the sequence $(h^\epsilon - h_D,\ h_1^\epsilon - h_{1,D})$ is uniformly bounded in $(L^2(0,T;V))^2$ and we show the maximum principle $0 \leq h_1^\epsilon(t,x) \leq h^\epsilon(t,x) \leq h_2$ a.e. in Ω_T for every $\epsilon > 0$. Finally we prove that any (weak) limit (h, h_1) in $(L^2(0,T;H^1(\Omega))\cap H^1(0,T;V'))^2$ of the sequence $(h^\epsilon,\ h_1^\epsilon)$ is a solution of the original problem.

4.1. Step 1: Existence for the regularized system. We now omit ϵ for the sake of simplicity in the notations. Then the weak formulation of the latter problem reads: for any $w \in V$,

$$\int_0^T \phi\langle \partial_t h, w\rangle_{V',V} dt + \int_{\Omega_T} \delta\phi \nabla h \cdot \nabla w \, dxdt + \int_{\Omega_T} KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h \cdot \nabla w \, dxdt$$

$$+ \int_{\Omega_T} KT_s(h)\mathcal{X}_0^\epsilon(h_1) L_M\big(||\nabla h_1||_{L^2}\big)\nabla h_1 \cdot \nabla w \, dxdt + \int_{\Omega_T} Q_s T_s(h) w \, dxdt = 0, \tag{4.1}$$

$$\int_0^T \phi\langle \partial_t h_1, w\rangle_{V',V} dt + \int_{\Omega_T} \delta\phi \nabla h_1 \cdot \nabla w \, dxdt$$

$$+ \int_{\Omega_T} K\Big(\big(\mathcal{X}_0^\epsilon(h_1)T_s(h) + T_f(h-h_1)\big)\nabla h_1 + T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\Big) \cdot \nabla w \, dxdt$$

$$+ \int_{\Omega_T} \big(Q_f T_f(h-h_1) + Q_s T_f(h)\big) w \, dxdt = 0. \tag{4.2}$$

For the fixed point strategy, we define the application $\mathcal{F}$ by

$$\begin{aligned} \mathcal{F} : L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)) &\longrightarrow L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)) \\ (\bar{h},\bar{h}_1) &\longmapsto \mathcal{F}(\bar{h},\bar{h}_1) = \big(\mathcal{F}_1(\bar{h},\bar{h}_1) = h, \mathcal{F}_2(\bar{h},\bar{h}_1) = h_1\big), \end{aligned}$$

where (h,h_1) is the solution of the following variational problem:

$$\int_0^T \phi\langle \partial_t h, w\rangle_{V',V} + \int_{\Omega_T} \delta\phi \nabla h \cdot \nabla w + \int_{\Omega_T} KT_s(\bar{h})\mathcal{X}_0^\epsilon(\bar{h}_1)\nabla h \cdot \nabla w$$

$$+ \int_{\Omega_T} KT_s(\bar{h}) L_M\big(||\nabla \bar{h}_1||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar{h}_1)\nabla \bar{h}_1 \cdot \nabla w + \int_{\Omega_T} Q_s T_s(\bar{h}) w = 0, \tag{4.3}$$

$$\int_0^T \phi\langle \partial_t h_1, w\rangle_{V',V} + \int_{\Omega_T} \delta\phi \nabla h_1 \cdot \nabla w$$

$$+ \int_{\Omega_T} K\big(T_s(\bar{h})\mathcal{X}_0^\epsilon(\bar{h}_1) + T_f(\bar{h}-\bar{h}_1)\big)\nabla h_1 \cdot \nabla w$$

$$+ \int_{\Omega_T} KT_s(\bar{h})\mathcal{X}_0^\epsilon(\bar{h}_1)\nabla h \cdot \nabla w + \int_{\Omega_T} \big(Q_f T_f(\bar{h}-\bar{h}_1) + Q_s T_s(\bar{h})\big) w = 0, \tag{4.4}$$

for any $w \in V$. Indeed we know from classical parabolic theory (see *e.g.* [16]) that the linear variational system (4.3)–(4.4) admits an unique solution. The end of the present subsection is devoted to the proof of a fixed point property for application $\mathcal{F}$.

Continuity of $\mathcal{F}_1$:
Let $(\bar{h^n}, \bar{h_1^n})$ be a sequence of functions of $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ and $(\bar{h},\bar{h}_1)$ be a function of $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ such that

$$(\bar{h^n}, \bar{h_1^n}) \to (\bar{h},\bar{h}_1) \text{ in } L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)).$$

We set $h_n = \mathcal{F}_1(\bar{h^n}, \bar{h_1^n})$ and $h = \mathcal{F}_1(\bar{h},\bar{h}_1)$. We aim showing that $h_n \to h$ in $L^2(0,T;H^1(\Omega))$.

For all $n \in \mathbb{N}$, h_n satisfies (4.3). Choosing $w = h_n - h_D$ in the n-dependent counterpart of (4.3) yields:

$$\int_0^T \phi\langle \partial_t(h_n - h_D), (h_n - h_D)\rangle_{V',V} dt + \int_{\Omega_T} \big(\delta\phi + KT_s(\bar{h}^n)\mathcal{X}_0^\epsilon(\bar{h}_1^n)\big)\nabla h_n \cdot \nabla h_n \, dxdt$$

$$
\begin{aligned}
&= -\int_{\Omega_T} KT_s(\bar h^n)L_M\big(||\nabla \bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla \bar h_1^n\cdot\nabla(h_n-h_D)\,dxdt\\
&+\int_{\Omega_T}\big(\delta\phi+KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla h_D\,dxdt-\int_{\Omega_T}Q_sT_s(\bar h^n)(h_n-h_D)\,dxdt\\
&-\int_0^T\phi\langle\partial_t h_D,(h_n-h_D)\rangle_{V',V}dt.
\end{aligned}\tag{4.5}
$$

Function $h_n - h_D$ belongs to $L^2(0,T;V)\cap H^1(0,T;V')$ and then to $\mathcal{C}(0,T;L^2(\Omega))$. Thus, thanks moreover to Lemma 1, we write

$$
\int_0^T \phi\langle\partial_t(h_n-h_D),(h_n-h_D)\rangle_{V',V}dt=\frac{\phi}{2}||h_n(\cdot,T)-h_D||_H^2-\frac{\phi}{2}||h_0-h_{D|t=0}||_H^2.
$$

Besides

$$
\int_{\Omega_T}\big(\delta\phi+KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla h_n\,dxdt\ge\delta\phi||\nabla h_n||^2_{L^2(0,T;H)}.
$$

Then applying Cauchy-Schwarz and Young inequalities, we get for any $\epsilon_1>0$

$$
\begin{aligned}
&\Big|\int_{\Omega_T}\big(\delta\phi+KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla h_D\Big|\\
&\qquad\le(\delta\phi+K_+h_2)||\nabla h_n||_{L^2(0,T;H)}||\nabla h_D||_{L^2(0,T;H)}\\
&\qquad\le\frac{\varepsilon_1}{2}||\nabla h_n||^2_{L^2(0,T;H)}+\frac{(\delta\phi+K_+h_2)^2}{2\varepsilon_1}||\nabla h_D||^2_{L^2(0,T;H)},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|-\int_{\Omega_T}KT_s(\bar h^n)L_M\big(||\nabla\bar h_1^n||_{H^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla\bar h_1^n\cdot\nabla h_n\Big|\\
&\qquad\le\sqrt{T}K_+||d(\bar h^n,\bar h_1^n)||_{L^\infty(0,T;H)}||\nabla h_n||_{L^2(0,T;H)}\\
&\qquad\le MK_+h_2\sqrt{T}||\nabla h_n||_{L^2(0,T;H)}\le\frac{K_+^2M^2T}{2\varepsilon_1}h_2^2+\frac{\varepsilon_1}{2}||\nabla h_n||^2_{L^2(0,T;H)}.
\end{aligned}
$$

Since it depends on h_D, the next term is simply estimated by

$$
\begin{aligned}
&\Big|\int_{\Omega_T}KT_s(\bar h^n)L_M\big(||\nabla\bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla\bar h_1^n\cdot\nabla h_D\,dxdt\Big|\\
&\qquad\le\sqrt{T}K_+||d(\bar h^n,\bar h_1^n)||_{L^\infty(0,T;H)}||h_D||_{L^2(0,T;H^1)}\le MK_+h_2\sqrt{T}||h_D||_{L^2(0,T;H^1)}.
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
&\Big|-\int_0^T\phi\langle\partial_t h_D,(h_n-h_D)\rangle_{V',V}dt\Big|\\
&\qquad\le\frac{\phi}{2\,\delta}||\partial_t h_D||^2_{L^2(0,T;(H^1(\Omega))')}+\frac{\delta\,\phi}{2}||h_n||^2_{L^2(0,T;H^1)}+\frac{\phi}{2}||h_D||^2_{L^2(0,T;H)},
\end{aligned}
$$

and

$$
\Big|-\int_{\Omega_T}Q_sT_s(\bar h^n)(h_n-h_D)\,dxdt\Big|\le\frac{\|Q_s\|_H^2}{2\phi}h_2^2+\frac{\phi}{2}||h_n-h_D||^2_{L^2(0,T;H)}.
$$

Using all the latter estimates in (4.5), we get after simplifications

$$
\begin{aligned}
&\frac{\phi}{2}||h_n(\cdot,T)-h_D||_H^2+(\frac{\delta\phi}{2}-\varepsilon_1)||\nabla h_n||^2_{L^2(0,T;H)}\\
&\quad\le\frac{\phi}{2}||h_0-h_{D|t=0}||_H^2+\Big(\frac{\|Q_s\|_H^2}{2\phi}+\frac{K_+^2M^2T}{2\varepsilon_1}\Big)h_2^2+\frac{(\delta\phi+K_+h_2)^2}{2\varepsilon_1}||h_D||^2_{L^2(0,T;H^1)}\\
&\qquad\qquad+\frac{\phi}{2\,\delta}||\partial_t h_D||^2_{L^2(0,T;(H^1(\Omega))')}+\frac{\phi}{2}\int_0^T||h_n-h_D||_H^2\,dt\\
&\qquad\qquad+\frac{\delta\phi}{2}\int_0^T||h_n||_H^2\,dt+MK_+h_2\sqrt{T}||h_D||_{L^2(0,T;H^1)}+\frac{\phi}{2}||h_D||^2_{L^2(0,T;H)}.
\end{aligned}
\tag{4.6}
$$

We choose ε_1 such that $\delta\phi/2-\varepsilon_1\ge\epsilon_0>0$ for some $\epsilon_0>0$. Relation (4.6) with Gronwall lemma enables to conclude that there exists real numbers $A_M=A_M(\phi,\delta,K,h_0,h_D,h_2,Q_s,M,T)$ and $B_M=B_M(\phi,\delta,K,h_0,h_D,h_2,Q_s,M,T)$ depending only on the data of the problem such that

$$
||h_n||_{L^\infty(0,T;H)}\le A_M,\qquad ||h_n||_{L^2(0,T;H^1)}\le B_M.
\tag{4.7}
$$

Hence sequence $(h_n)_n$ is uniformly bounded in $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;H)$. Notice that the estimate in $L^\infty(0,T;H)$ is justified by the fact that we could make the same computations replacing T by any $\tau\le T$ in the time integration. In the sequel, we set

$$C_M=\max(A_M,B_M).$$

We now prove that $(\partial_t(h_n-h_D))_n$ is bounded in $L^2(0,T;V')$. We have

$$
\begin{aligned}
&||\partial_t(h_n-h_D)||_{L^2(0,T;V')}=\sup_{||w||_{L^2(0,T;V)}\le1}\Big|\int_0^T\langle\partial_t(h_n-h_D),w\rangle_{V',V}\Big|\\
&=\sup_{||w||_{L^2(0,T;V)}\le1}\Big|\int_0^T-\langle\partial_t h_D,w\rangle_{V',V}-\frac{1}{\phi}\int_{\Omega_T}\big(\delta\phi+KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla w\\
&\qquad-\int_{\Omega_T}KT_s(\bar h^n)L_M\big(||\nabla\bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla\bar h_1^n\cdot\nabla w-\int_{\Omega_T}Q_sT_s(\bar h^n)w\Big|.
\end{aligned}
$$

Since

$$
\Big|\int_{\Omega_T}\big(\delta\phi+KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla w\Big|\le\big(\delta\phi+K_+h_2\big)||h_n||_{L^2(0,T;H^1(\Omega))}||w||_{L^2(0,T;V)},
$$

and since h_n is uniformly bounded in $L^2(0,T;H^1(\Omega))$, we write

$$
\Big|\int_{\Omega_T}\big(\delta\phi+KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla h_n\cdot\nabla w\,dxdt\Big|\le\big(\delta\phi+K_+h_2\big)C_M||w||_{L^2(0,T;V)}.
\tag{4.8}
$$

Furthermore we have

$$
\Big|\int_{\Omega_T}T_s(\bar h^n)L_M\big(||\nabla\bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla\bar h_1^n\cdot\nabla w\,dxdt\Big|\le Mh_2\sqrt{T}||w||_{L^2(0,T;V)}
\tag{4.9}
$$

and

$$
\Big|\int_{\Omega_T}Q_sT_s(\bar h^n)w\,dxdt\Big|\le\|Q_s\|_Hh_2||w||_{L^2(0,T;V)}.
\tag{4.10}
$$

Summing up (4.8)–(4.10), we conclude that

$$(4.11) \qquad ||\partial_t h_n||_{L^2(0,T;V')} \le D_M,$$

$$D_M = \frac{1}{\phi}\Big(||\partial_t h_D||^2_{L^2(0,T;(H^1(\Omega))')} + \delta\phi C_M + h_2(K_+ C_M + M\sqrt{T} + \|Q_s\|_H)\Big).$$

We have proved that $\big(h_n\big)_n$ is uniformly bounded in the space $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')$. Using Aubin's lemma, we extract a subsequence, not relabeled for convenience, $(h_n)_n$, converging strongly in $L^2(\Omega_T)$ and weakly in $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')$ to some limit denoted by ℓ. Using in particular the strong convergence in $L^2(\Omega_T)$ and thus the convergence a.e. in Ω_T, we check that ℓ is a solution of equation (4.3). The solution of (4.3) being unique, we have $\ell = h$.

It remains to prove that $(h_n)_n$ actually tends to h strongly in $L^2(0,T;H^1(\Omega))$. Subtracting the weak formulation (4.3) to its n-dependent counterpart for the test function $w = h_n - h$, we get

$$\begin{aligned}
&\int_0^T \phi\langle\partial_t(h_n-h), h_n-h\rangle_{V',V} + \int_{\Omega_T}\big(\delta\phi + KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla(h_n-h)\cdot\nabla(h_n-h)\\
&\qquad - \int_{\Omega_T} K\big(T_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n) - T_s(\bar h)\mathcal{X}_0^\epsilon(\bar h_1)\big)\nabla(h_n-h)\cdot\nabla h + \int_{\Omega_T} K\Big(T_s(\bar h^n)\\
&\qquad L_M\big(||\nabla\bar h_1^n||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1^n)\nabla\bar h_1^n - T_s(\bar h)L_M\big(||\nabla\bar h_1||_{L^2}\big)\mathcal{X}_0^\epsilon(\bar h_1)\nabla\bar h_1\Big)\cdot\nabla(h_n-h)\\
(4.12)\qquad &\qquad + \int_{\Omega_T} Q_s\big(T_s(\bar h^n) - T_s(\bar h)\big)\,(h_n-h) = 0.
\end{aligned}$$

Using assumption $(\bar h^n, \bar h_1^n) \to (\bar h, \bar h_1)$ in $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ and the above results of convergence for h_n, the limit as $n\to\infty$ in (4.12) reduces to

$$\lim_{n\to\infty}\Big(\int_{\Omega_T}\big(\delta\phi + KT_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)\big)\nabla(h_n-h)\cdot\nabla(h_n-h)\,dxdt\Big) = 0.$$

Due to the positiveness of K, we infer from the latter relation that

$$\lim_{n\to\infty}\Big(\int_{\Omega_T}\delta\phi|t\nabla(h_n-h)|^2\,dxdt + \int_{\Omega_T} K_- T_s(\bar h^n)\mathcal{X}_0^\epsilon(\bar h_1^n)|t\nabla(h_n-h)|^2\,dxdt\Big) \le 0.$$

Hence $\nabla h_n \to \nabla h$ strongly in $L^2(0,T;H)$. Continuity of $\mathcal{F}_1$ for the strong topology of $L^2(0,T;H^1(\Omega))$ is proved.

Continuity of $\mathcal{F}_2$:
Likewise, we prove the continuity of $\mathcal{F}_2$ by setting $h_{1,n} = \mathcal{F}_2(\bar h^n, \bar h_1^n)$ and $h_1 = \mathcal{F}_2(\bar h, \bar h_1)$ and showing that $h_{1,n} \to h_1$ in $L^2(0,T;H^1(\Omega))$. The key estimates are obtained using the same type of arguments than in the proof of the continuity of $\mathcal{F}_1$. We thus do not detail the computations. Let us only emphasize that we can now use the estimate (4.7) previously derived for h^n, thus the dependence with regard to C_M in the following estimates:

$$(4.13) \qquad ||h_{1,n}||_{L^\infty(H)} \le E_M = E_M\big(\phi,\delta,K,h_{1,0},h_{1,D},h_2,Q_s,Q_f,M,C_M,T\big),$$

$$(4.14) \quad ||h_{1,n}||_{L^2(0,T;H^1)} \le F_M = F_M\big(\phi,\delta,K,h_{1,0},h_{1,D},h_2,Q_s,Q_f,M,C_M,T\big).$$

We set

$$C_{1,M} = max(E_M, F_M).$$

One also computes that

$$
\begin{aligned}
(4.15) \qquad & ||\partial_t h_{1,n}||_{L^2(0,T;V')} \leq D_{1,M},\\
& D_{1,M} = \frac{1}{\phi}\Big(\delta\phi + \Big(2K_+ C_M + K_+ C_{1,M} + 2\big(\|Q_f\|_H + \|Q_s\|_H\big)\Big)h_2\Big).
\end{aligned}
$$

Conclusion

$\mathcal{F}$ is continuous in $(L^2(0,T;H^1(\Omega)))^2$ because its two components $\mathcal{F}_1$ and $\mathcal{F}_2$ are. Furthermore, let $A \in \mathbb{R}_+^*$ be the real number defined by

$$A = \max(C_M, D_M, C_{1,M}, D_{1,M}),$$

and W be the nonempty (strongly) closed convex bounded set in $(L^2(0,T;H^1(\Omega)))^2$ defined by

$$
\begin{aligned}
W = \Big\{&(g, g_1) \in \big(L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')\big)^2;\ (g(0), g_1(0)) = (h_0, h_{1,0}),\\
&(g_{|\Gamma}, g_{1|\Gamma}) = (h_D, h_{1,D}),\ ||(g,g_1)||_{(L^2(0,T;H^1(\Omega))\cap H^1(0,T;V'))^2} \leq A\Big\}.
\end{aligned}
$$

We have shown that $\mathcal{F}(W) \subset W$. It follows from Schauder theorem [23, cor. 9.7] that there exists $(h, h_1) \in W$ such that $\mathcal{F}(h, h_1) = (h, h_1)$. This fixed point for $\mathcal{F}$ is a weak solution of problem (4.1)–(4.2).

4.2. Step 2: Elimination of the auxiliary function L_M . We now claim that there exist a real number $B > 0$, not depending on ϵ neither on M, such that any weak solution $(h, h_1) \in W$ of problem (4.1)–(4.2) satisfies

$$(4.16) \qquad ||\nabla h||_{L^2(0,T;H)} \leq B \quad \text{and} \quad ||\nabla h_1||_{L^2(0,T;H)} \leq B.$$

Taking $w = h - h_D$ (resp. $w = h_1 - h_{1,D}$) in (4.1) (resp. (4.2)) leads to

$$
\begin{aligned}
&\int_0^T \phi\langle\partial_t h, h - h_D\rangle_{V',V} + \int_{\Omega_T} \delta\phi\nabla h\cdot\nabla(h - h_D) + \int_{\Omega_T} KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\cdot\nabla(h-h_D)\\
(4.17)&= -\int_{\Omega_T} KT_s(h)\mathcal{X}_0^\epsilon(h_1)L_M\big(||\nabla h_1||_{L^2}\big)\nabla h_1\cdot\nabla(h-h_D) - \int_{\Omega_T} Q_s T_s(h)(h-h_D)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \phi\langle\partial_t h_1, h_1 - h_{1,D}\rangle_{V',V} + \int_{\Omega_T} \delta\phi\nabla h_1\cdot\nabla(h_1 - h_{1,D}) + \int_{\Omega_T} K\big(T_s(h)\mathcal{X}_0^\epsilon(h_1)\\
&\qquad + T_f(h-h_1)\big)\nabla h_1\cdot\nabla(h_1 - h_{1,D}) = -\int_{\Omega_T} KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\cdot\nabla(h_1 - h_{1,D})\\
(4.18) &\quad -\int_{\Omega_T} \big(Q_f T_f(h-h_1) + Q_s T_s(h)\big)(h_1 - h_{1,D}).
\end{aligned}
$$

Summing up relations (4.17) and (4.18), and using the decomposition

$$
\begin{aligned}
&K\nabla h\cdot\nabla h + KL_M\big(||\nabla h_1||_{L^2}\big)\nabla h_1\cdot\nabla h + K\nabla h_1\cdot\nabla h_1 + K\nabla h\cdot\nabla h_1\\
&= K\nabla(h+h_1)\cdot\nabla(h+h_1) + K\big(1 - L_M\big(||\nabla h_1||_{L^2}\big)\big)\nabla h_1\cdot\nabla h_1\\
&\quad - K\big(1 - L_M\big(||\nabla h_1||_{L^2}\big)\big)\nabla h_1\cdot\nabla(h+h_1),
\end{aligned}
$$

we write

$$
\begin{aligned}
&\int_0^T \phi\big(\langle\partial_t(h-h_D), h-h_D\rangle_{V',V} + \langle\partial_t(h_1-h_{1,D}), h_1-h_{1,D}\rangle_{V',V}\big)\\
&\quad+\int_{\Omega_T}\delta\phi\big(\nabla h\cdot\nabla h+\nabla h_1\cdot\nabla h_1\big)+\int_{\Omega_T}KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla(h+h_1)\cdot\nabla(h+h_1)\\
&\quad+\int_{\Omega_T}K\Big(\big(1-L_M(||\nabla h_1||_{L^2})\big)T_s(h)\mathcal{X}_0^\epsilon(h_1)+T_f(h-h_1)\Big)\nabla h_1\cdot\nabla h_1\\
&=\int_{\Omega_T}K\big(1-L_M(||\nabla h_1||_{L^2})\big)T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h_1\cdot\nabla(h+h_1)+\int_{\Omega_T}\big(\delta\phi+KT_s(h)\\
&\mathcal{X}_0^\epsilon(h_1)\big)\nabla h\cdot\nabla h_D+\int_{\Omega_T}\big(\delta\phi+KT_s(h)\mathcal{X}_0^\epsilon(h_1)+KT_f(h-h_1)\big)\nabla h_1\cdot\nabla h_{1,D}\\
&\quad+\int_{\Omega_T}KT_s(h)L_M(||\nabla h_1||_{L^2})\mathcal{X}_0^\epsilon(h_1)\nabla h_1\cdot\nabla h_D+\int_{\Omega_T}KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\cdot\nabla h_{1,D}\\
&\qquad\qquad-\int_{\Omega_T}\big(Q_sT_s(h)(h-h_D)+\big(Q_fT_f(h-h_1)+Q_sT_s(h)\big)(h_1-h_{1,D})\big)\\
&\qquad\qquad-\int_0^T\phi\big(\langle\partial_t h_D, h-h_D\rangle_{V',V}+\langle\partial_t h_{1,D}, h_1-h_{1,D}\rangle_{V',V}\big):=\sum_{i=1}^{11}J_i
\end{aligned}
\tag{4.19}
$$

We now estimate all the integral terms 'J_i' in the latter relations. We set $u=h-h_D$ and $v=h_1-h_{1,D}$. First, we note that

$$
\begin{aligned}
|J_1| &= \frac{\phi}{2}\int_\Omega\big(u^2(T,x)-u_0^2(x)\big)\,dx+\frac{\phi}{2}\int_\Omega\big(v^2(T,x)-v_0^2(x)\big)\,dx,\\
|J_2| &= \int_{\Omega_T}\delta\phi|\nabla h|^2\,dxdt+\int_{\Omega_T}\delta\phi|\nabla h_1|^2\,dxdt,\\
|J_3| &\geq \int_{\Omega_T}K_-T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla(h+h_1)|^2\,dxdt,\\
|J_4| &\geq \int_{\Omega_T}K_-\Big(\big(1-L_M(||\nabla h_1||_{L^2})\big)T_s(h)\mathcal{X}_0^\epsilon(h_1)+T_f(h-h_1)\Big)|\nabla h_1|^2\,dxdt.
\end{aligned}
$$

Next, applying the Cauchy-Schwarz and Young inequalities, we obtain the following set of estimates for any $\varepsilon_1>0$:

$$
\begin{aligned}
|J_5| &\leq \int_{\Omega_T}\big(1-L_M\big(||\nabla h_1||_{L^2}\big)\big)T_s(h)\mathcal{X}_0^\epsilon(h_1)\Big(\frac{K_+^2}{4K_-}|\nabla h_1|^2+K_-|\nabla(h+h_1)|^2\Big),\\
|J_6| &\leq \int_{\Omega_T}\frac{\delta\phi}{4}\,|\nabla h|^2+\frac{\varepsilon_1K_+}{4}\int_{\Omega_T}T_s(h)\mathcal{X}_0^\epsilon(h_1)\,|\nabla h|^2\\
&\quad+\int_{\Omega_T}\big(\delta\phi+\frac{K_+}{\varepsilon_1}T_s(h)\mathcal{X}_0^\epsilon(h_1)\big)|\nabla h_D|^2,\\
|J_7| &\leq \int_{\Omega_T}\frac{\delta\phi}{4}\,|\nabla h_1|^2+\int_{\Omega_T}\frac{\varepsilon_1K_+}{4}T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla h_1|^2+\int_{\Omega_T}\frac{K_+}{2}T_f(h-h_1)|\nabla h_1|^2\\
&\quad+\int_{\Omega_T}\Big(\delta\phi+\frac{K_+}{2}T_f(h-h_1)\Big)|\nabla h_{1,D}|^2+\frac{1}{\varepsilon_1}\int_{\Omega_T}T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla h_{1,D}|^2,\\
|J_8| &\leq \int_{\Omega_T}\frac{\varepsilon_1K_+}{4}T_s(h)\big)\mathcal{X}_0^\epsilon(h_1)|\nabla h_1|^2+\int_{\Omega_T}\frac{K_+}{\varepsilon_1}T_s(h)L_M^2\big(||\nabla h_1||_{L^2}\big)\mathcal{X}_0^\epsilon(h_1)|\nabla h_D|^2,
\end{aligned}
$$

and

$$|J_9| \le \int_{\Omega_T} \frac{\varepsilon_1 K_+}{4} T_s(h)\mathcal{X}_0^\epsilon(h_1)|\nabla h|^2 + \int_{\Omega_T} \frac{K_+}{\varepsilon_1} T_s(h)|\nabla h_{1,D}|^2,$$

$$\begin{aligned}|J_{10}| &\le \int_{\Omega_T} T_s(h)|Q_s\, u| + \int_{\Omega_T} T_f(h-h_1)|Q_f\, v|\, dxdt + \int_{\Omega_T} T_s(h)|Q_s\, v| \\ &\le \frac{3\|Q_s\|_H^2 + 2\|Q_f\|_H^2}{2\,\phi} h_2^2 + \frac{\phi}{2}\int_{\Omega_T} |u|^2 + \frac{\phi}{2}\int_{\Omega_T} |v|^2,\end{aligned}$$

$$\begin{aligned}|J_{11}| &\le \frac{1}{4}\int_{\Omega_T} \delta\phi\, |\nabla(h-h_D)|^2 + \frac{1}{4}\int_{\Omega_T} \delta\phi\, |\nabla(h_1 - h_{1,D})|^2 + \frac{\phi}{\delta}\|\partial_t h_D\|^2_{L^2(0,T;V')} \\ &+ \frac{\phi}{\delta}\|\partial_t h_{1,D}\|^2_{L^2(0,T;V')} \le \frac{\delta\phi}{4}\int_{\Omega_T} \big(|\nabla h|^2 + |\nabla h_1|^2\big) + \frac{\delta\phi}{4}\int_{\Omega_T} \big(|\nabla h_D|^2 + |\nabla h_{1,D}|^2\big) \\ &+ \frac{\phi}{\delta}\big(\|\partial_t h_D\|^2_{L^2(0,T;V')} + \|\partial_t h_{1,D}\|^2_{L^2(0,T;V')}\big).\end{aligned}$$

Summing up all these estimates, we obtain

$$\begin{aligned}&\phi\int_\Omega u^2(T,x) + \phi\int_\Omega v^2(T,x) + \int_{\Omega_T} \underbrace{\big(\delta\phi - \varepsilon_1 K_+ T_s(h)\mathcal{X}_0^\epsilon(h_1)\big)}\big(|\nabla h|^2 + |\nabla h_1|^2\big) \\ &+ \int_{\Omega_T} 2\underbrace{\Big((K_- - \frac{K_+^2}{4K_-})\big(1 - L_M(||\nabla h_1||)\big)\mathcal{X}_0^\epsilon(h_1)T_s(h) + (K_- - \frac{K_+}{2})T_f(h-h_1)\Big)}|\nabla h_1|^2 \\ &+2\int_{\Omega_T} \big(K_- T_s(h)\mathcal{X}_0^\epsilon(h_1)L_M\big(||\nabla h_1||_{L^2}\big)\big)|\nabla(h+h_1)|^2 \le \phi\int_{\Omega_T}|u|^2 + \phi\int_{\Omega_T}|v|^2 + C,\end{aligned}$$

where $C = C(u_0, v_0, h_D, h_{1,D}, h_2, Q_s, Q_f)$. We now aim applying the Gronwall lemma in the latter relation. We thus choose $\varepsilon_1 > 0$ such as terms over the curly bracket are respectively positive and nonnegative, namely:

$$\begin{aligned}&K_+ T_s(h)\mathcal{X}_0^\epsilon(h_1)\varepsilon_1 < \delta\phi, \\ &1 - L_M(x) = 1 - \min(1, M/x) \ge 0 \text{ and } K_+ \le 2K_-.\end{aligned}$$

The first condition is fulfilled if we choose for instance ε_1 such that $\varepsilon_1 < \delta\phi/(K_+ h_2)$. The second one follows the assumption on permeability in Theorem 1.

Now we apply the Gronwall lemma and we deduce that there exists a real number B, that does not depend on ϵ nor on M, such that

$$||h||_{L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))} \le B \quad \text{and} \quad ||h_1||_{L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))} \le B.$$

In particular, $||\nabla h_1||_{L^2(0,T;H)} \le B$ and this estimate does not depend on the choice of the real number M that defines function L_M. Hence if we choose $M = B$, any weak solution of the system

$$\begin{aligned}&\phi\partial_t h - \nabla\cdot(\delta\phi\nabla h) - \nabla\cdot\big(KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\big) \\ &\qquad -\nabla\cdot\big(KT_s(h)\mathcal{X}_0^\epsilon(h_1)L_B(||\nabla h_1||_{L^2})\nabla h_1\big) = -Q_s T_s(h), \\ &\phi\partial_t h_1 - \nabla\cdot(\delta\phi\nabla h_1) - \nabla\cdot\Big(K\big(T_f(h-h_1) + T_s(h)\mathcal{X}_0^\epsilon(h_1)\big)\nabla h_1\Big) \\ &\qquad -\nabla\cdot\big(KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\big) = -Q_f T_f(h-h_1) - Q_s T_s(h)\end{aligned}$$

in Ω_T, with the initial and boundary conditions

$$h = h_D \text{ and } h_1 = h_{1,D} \text{ on } \Gamma,\ h(0,x) = h_0 \text{ and } h_1(0,x) = h_{1,0}(x) \text{ a.e. in } \Omega,$$

satisfies $L_B\big(||\nabla h_1||_{L^2}\big) = 1$. Then the term $L_B\big(||\nabla h_1||_{L^2}\big) = 1$ may be dropped.

4.3. Step 3: Maximum Principles . We are going to prove that for almost every $x \in \Omega$ and for all $t \in (0,T)$,

$$0 \leq h_1(t,x) \leq h(t,x) \leq h_2.$$

• First show that $h(t,x) \leq h_2$ a.e. $x \in \Omega$ and $\forall t \in (0,T)$. We set

$$h_m = \big(h - h_2\big)^+ = \sup(0, h - h_2) \in L^2(0,T;V).$$

It satisfies $\nabla h_m = \chi_{\{h>h_2\}} \nabla h$ and $h_m(t,x) \neq 0$ iff $h(t,x) > h_2$, where χ denotes the characteristic function. Let $\tau \in (0,T)$. Taking $w(t,x) = h_m(t,x)\chi_{(0,\tau)}(t)$ in (4.1) yields:

$$\int_0^T \phi \langle \partial_t h, h_m \chi_{(0,\tau)} \rangle_{V',V} + \int_0^\tau \int_\Omega \delta\phi \nabla h \cdot \nabla h_m + \int_0^\tau \int_\Omega K T_s(h) \mathcal{X}_0^\epsilon(h_1) \nabla h \cdot \nabla h_m$$
$$+ \int_0^\tau \int_\Omega K T_s(h) L_M\big(||\nabla h_1||_{L^2}\big) \mathcal{X}_0^\epsilon(h_1) \nabla h_1 \cdot \nabla h_m + \int_0^\tau \int_\Omega Q_s T_s(h) h_m = 0,$$

that is

$$\int_0^\tau \phi \langle \partial_t h, h_m \rangle_{V',V} + \int_0^\tau \int_\Omega \delta\phi \chi_{\{h>h_2\}} |\nabla h|^2 + \int_0^\tau \int_\Omega K T_s(h) \mathcal{X}_0^\epsilon(h_1) \chi_{\{h>h_2\}} |\nabla h|^2$$
$$+ \int_0^\tau \int_\Omega K T_s(h) L_M\big(||\nabla h_1||_{L^2}\big) \mathcal{X}_0^\epsilon(h_1) \nabla h_1 \cdot \nabla h_m(x,t)$$
$$+ \int_0^\tau \int_\Omega Q_s T_s(h) h_m(x,t) = 0. \tag{4.20}$$

In order to evaluate the first term in the lefthand side of (4.20), we apply Lemma 1 with function f defined by $f(\lambda) = \lambda - h_2$, $\lambda \in \mathbb{R}$. We write

$$\int_0^\tau \phi \langle \partial_t h, h_m \rangle_{V',V} dt = \frac{\phi}{2} \int_\Omega \Big(h_m^2(\tau,x) - h_m^2(0,x) \Big)\, dx = \frac{\phi}{2} \int_\Omega h_m^2(\tau,x)\, dx,$$

since $h_m(0,\cdot) = \big(h_0(\cdot) - h_2(\cdot)\big)^+ = 0$. Since $T_s(h)\chi_{\{h>h_2\}} = 0$ by definition of T_s, the three last terms in the lefthand side of (4.20) are null. Hence (4.20) becomes:

$$\frac{\phi}{2} \int_\Omega h_m^2(\tau,x)\, dx \leq - \int_0^\tau \int_\Omega \delta\phi \chi_{\{h>h_2\}} |\nabla h|^2\, dxdt \leq 0.$$

Then $h_m = 0$ a.e. in Ω_T. Including the leakage term q_{Ls} defined by (3.6) does not impact the result because of the factor $\chi_0(h_2 - h)$ in its definition.

• Now we claim that $h_1(t,x) \leq h(t,x)$ a.e. $x \in \Omega$ and $\forall t \in (0,T)$.
We now set

$$h_m = \big(h_1 - h\big)^+ \in L^2(0,T;V).$$

Let $\tau \in (0,T)$. We recall that $h_m(0,\cdot) = 0$ a.e. in Ω thanks to the maximum principle satisfied by the initial data h_0 and $h_{1,0}$. Moreover, $\nabla(h_1 - h) \cdot \nabla h_m = \chi_{\{h_1 - h > 0\}} |\nabla(h_1 - h)|^2$. Thus, taking $w(t,x) = h_m(x,t)\chi_{(0,\tau)}(t)$ in (4.2) – (4.1) gives:

$$\frac{\phi}{2} \int_\Omega h_m^2(\tau,x) + \int_0^\tau \int_\Omega \delta\phi \chi_{\{h_1-h>0\}} |\nabla(h_1 - h)|^2 + \int_0^\tau \int_\Omega K\big(\mathcal{X}_0^\epsilon(h_1) T_s(h) dt$$
$$+ T_f(h - h_1)\big) \nabla h_1 \cdot \nabla h_m - \int_0^\tau \int_\Omega K T_s(h) \mathcal{X}_0^\epsilon(h_1) L_M\big(||\nabla h_1||_{L^2}\big) \nabla h_1 \cdot \nabla h_m$$
$$+ \int_0^\tau \int_\Omega Q_f T_f(h - h_1) h_m = 0. \tag{4.21}$$

Since $T_f(h-h_1)\chi_{\{h_1-h>0\}}=0$ by definition of T_f and since we now have $M=B$ such that $L_B\big(||\nabla h_1||_{L^2}\big)=1$, we infer from (4.21) that

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx=-\int_0^\tau\int_\Omega \delta\phi\chi_{\{h_1-h>0\}}|\nabla(h_1-h)|^2\,dxdt\leq 0.$$

Thus $h_1(\tau,\cdot)\leq h(\tau,\cdot)$ a.e. in Ω and for any $\tau\in(0,T)$. Presence of the leakage terms defined in (3.6) does not change the picture. Indeed term q_{Ls} disappears in the computation (4.2) $-$ (4.1) and $q_{Lf}h_m=0$ because of the term $\chi_0(h-h_1)$ in the definition of q_{Lf}.

• Finally we show $0\leq h_1(t,x)$ a.e. $x\in\Omega$ and $\forall t\in(0,T)$.
We now set

$$h_m=\big(-h_1\big)^+\in L^2(0,T;V).$$

Let $\tau\in(0,T)$. For this part of the proof, we re-include the leakage terms q_{Lf} and q_{Ls} in the model because they appear in the assertion (3.8) which is used here. Taking $w(t,x)=-h_m(x,t)\chi_{(0,\tau)}(t)$ in (4.2) leads to :

$$\begin{aligned}&\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)+\int_0^\tau\int_\Omega\delta\phi\chi_{\{h_1<0\}}|\nabla h_1|^2-\int_0^\tau\int_\Omega K\big(T_s(h)\mathcal{X}_0^\epsilon(h_1)\\&+T_f(h-h_1)\big)\nabla h_1\cdot\nabla h_m-\int_0^\tau\int_\Omega KT_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h\cdot\nabla h_m\\&-\int_0^\tau\int_\Omega\big(Q_fT_f(h-h_1)+Q_sT_s(h)-q_{Lf}-q_{Ls}\big)h_m=0.\end{aligned}\tag{4.22}$$

We note that if $\nabla h_m\neq 0$ then $\mathcal{X}_0^\epsilon(h_1)=0$ because $h_1\leq 0$. We have moreover

$$\begin{aligned}-\int_0^\tau\int_\Omega &K\big(T_s(h)\mathcal{X}_0^\epsilon(h_1)+T_f(h-h_1)\big)\nabla h_1\cdot\nabla h_m\,dxdt\\&\geq\int_0^\tau\int_\Omega K_-T_f(h-h_1)\chi_{\{h_1<0\}}|\nabla h_1|^2\,dxdt,\end{aligned}$$

and, due to assumptions (3.7) and (3.8),

$$-\int_0^\tau\int_\Omega\big(Q_fT_f(h-h_1)+Q_sT_s(h)-q_{Lf}-q_{Ls}\big)h_m\,dxdt\geq 0.$$

Equation (4.22) thus gives :

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)dt\,dx\leq\int_0^\tau\int_\Omega\big(\delta\phi+K_-T_f(h-h_1)\big)\chi_{\{h_1<0\}}|\nabla h_1|^2\,dxdt\leq 0.$$

We conclude that $h_m(\tau,\cdot)=0$, that is $h_1(\tau,\cdot)\geq 0$, a.e. in Ω for any $\tau\in(0,T)$.

4.4. Step 4: Existence for the initial system. In the latter subsections, we have proved the existence of a weak solution $\big(h^\epsilon,h_1^\epsilon\big)\in\big(L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))\big)^2$ of the regularized problem

$$\phi\partial_t h^\epsilon-\nabla\cdot\big(\delta\phi\nabla h^\epsilon\big)-\nabla\cdot\big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla(h^\epsilon+h_1^\epsilon)\big)=-Q_sT_s(h^\epsilon),\tag{4.23}$$

$$\begin{aligned}&\phi\partial_t h_1^\epsilon-\nabla\cdot\big(\delta\phi\nabla h_1^\epsilon\big)-\nabla\cdot\big(K\big(T_f(h^\epsilon-h_1^\epsilon)+T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\big)\nabla h_1^\epsilon\big)\\&\qquad-\nabla\cdot\big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla h^\epsilon\big)=-Q_fT_f(h^\epsilon-h_1^\epsilon)-Q_sT_s(h^\epsilon),\end{aligned}\tag{4.24}$$

with the initial and boundary conditions

$$h^\epsilon = h_D,\ h_1^\epsilon = h_{1,D} \text{ in } \Gamma\times(0,T), \quad h^\epsilon(0,x)=h_0,\ h_1^\epsilon(0,x)=h_{1,0}(x) \text{ a.e. in } \Omega.$$

Furthermore this solution satisfies the following maximum principle :

$$\forall t\in(0,T),\ a.e.\ x\in\Omega, \quad 0\le h_1^\epsilon(t,x)\le h^\epsilon(t,x)\le h_2,$$

and the following uniform estimates (with respect to ϵ) :

$$\text{(UE)}\left\{\begin{array}{ll} \|h^\epsilon\|_{L^2(0,T;H^1(\Omega))}\le C, & \|h_1^\epsilon\|_{L^2(0,T;H^1(\Omega))}\le C,\\ \|\partial_t h^\epsilon\|_{L^2(0,T;V')}\le C, & \|\partial_t h_1^\epsilon\|_{L^2(0,T;V')}\le C. \end{array}\right.$$

We now proceed to the last step in the proof of Theorem 1, namely we let $\epsilon\to 0$. We infer from the above estimates that $(h^\epsilon - h_D)_\epsilon$ and $(h_1^\epsilon - h_{1,D})_\epsilon$ are uniformly bounded in $W(0,T)$. We deduce thanks to the compactness result of Aubin that $(h^\epsilon - h_D)_\epsilon$ and $(h_1^\epsilon - h_{1,D})_\epsilon$ are sequentially compact in $L^2(0,T;H)$. Up to the extraction of a subsequence, not relabeled for convenience, we claim that there exists functions h and h_1 such that $(h-h_D, h_1-h_{1,D})\in W(0,T)^2$ and

$$\left\{\begin{array}{ll} h^\epsilon \longrightarrow h & \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega\times(0,T),\\ h^\epsilon \rightharpoonup h & \text{weakly in } L^2(0,T;H^1(\Omega)),\\ \partial_t h^\epsilon \rightharpoonup \partial_t h & \text{weakly in } L^2(0,T;V'),\\ h_1^\epsilon \longrightarrow h_1 & \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega\times(0,T),\\ h_1^\epsilon \rightharpoonup h_1 & \text{weakly in } L^2(0,T;H^1(\Omega)),\\ \partial_t h_1^\epsilon \rightharpoonup \partial_t h_1 & \text{weakly in } L^2(0,T;V'). \end{array}\right.$$

Letting $\epsilon\to 0$ in the weak formulation of (4.23)–(4.24) and using Lebesgue Theorem (thanks to the uniform estimates (UE)), we get at once (3.2)–(3.3). The boundary and initial condition (3.4)–(3.5) holds true since the map $i\in W(0,T)\mapsto i(0)\in H$ is continuous. Furthermore (h,h_1) satisfies a maximum principle which is consistent with physical reality:

$$0\le h_1(x,t)\le h(x,t)\le h_2,\ \forall t\in(0,T),\ \text{a.e. } x\in\Omega.$$

The proof of Theorem 1 is complete.

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