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PARABOLIC AND DEGENERATE PARABOLIC MODELS FOR PRESSURE-DRIVEN TRANSPORT PROBLEMS

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We consider two models of flow and transport in porous media, the first one for consolidational flow in compressible sedimentary basins, the second one for flow in partially saturated media. Despite the differences in these physical settings, they lead to quite similar mathematical models with a strong pressure coupling. The first model is a coupled system of pde's of parabolic type. The second one involves a coupled system of pdes of degenerate parabolic−hyperbolic type. We state an existence result of weak solutions for both models.

Keywords: Consolidational flow; partially saturated medium; degenerate parabolic pde; hyperbolic pde; coupled system of pdes; fixed point; compensated compactness.

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1. Two Models of Flow and Transport in Continuum Fluid Mechanics

This paper is devoted to the mathematical analysis of some models of pressure-driven flow processes. They play an important role in numerous natural and engineered systems, especially in subsurface hydrology. Typical applications are the study of pollutants transport in the underground, infiltration from industrial waste disposal, radionuclide repositories, saltwater intrusion in coastal aquifers, geothermal energy extraction systems, diagenetic processes in sedimentary basins, etc. A large literature is devoted to the derivation of models, based on the conservation laws, for flow and transport problems in porous media. Let us quote for instance classical textbooks Refs. 5, 25, 6, 27 and 18. We refer to Ref. 20 for the heat transfer problems and to Ref. 11 for oil reservoir simulation. Most of the models involve a partial differential equation governing the pressure coupled with some other partial differential equations for basic quantities describing the composition of the fluid (concentrations for a miscible flow, saturation for an immiscible flow, temperature…). From a mathematical viewpoint, the coupling in this system of pdes is one of the main difficulties of the problem. From a physical viewpoint, the aim is to neglect as less as possible the coupling.

In the present work, we focus on some pressure-driven processes. The aforementioned coupling is thus essentially due to pressure effects. As a motivation for the reader, we focus on two examples. In the first one, the derivation of a mathematical model for a consolidation process in a deformable porous medium leads to the study of a coupled system of parabolic equations. In the second example, we consider a flow in a partially saturated medium described by a system of equations of degenerate parabolic type.

1.1. *Consolidational fluid flow*

Accurate prediction in structurally weak geologic areas requires both mechanical deformation and fluid flow modeling. A wide range of real problems has to be studied in this way. For waste disposals, one often considers that sediments are the most important barrier for preventing a release in the biosphere.[7] Even overpressurized oil reservoirs located in stable environments may undergo settling at the start of production. We aim to provide a model coupling the effects of sediment consolidation and associated fluid flow and transport.

We consider the displacement of two miscible species transported by a compressible flow in a porous deformable medium. We begin by the part of the derivation which is not influenced by the compressibility of the medium. We denote by p the pressure and by c the mass concentration of one of the two components of the mixture. The Darcy velocity is designated by $\underline{u}$. The classical Darcy law for porous media gives

$$\underline{u} = -\frac{k}{\mu}\nabla p,$$

where k is the permeability of the medium and μ is the viscosity of the fluid. We define the hydraulic conductivity κ of the fluid by

$$\kappa = \frac{k}{\mu}.$$

We neglect the gravitational terms. The porosity of the medium is denoted by θ. The conservation of mass of each component is given by the following equations:

$$\partial_t(\theta\rho_1 c) + \mathrm{div}(\rho_1 c\underline{u}) - \mathrm{div}(\rho_1\theta\mathcal{D}(\underline{u})\nabla c) = \rho_1(q_i - q_s c), \tag{1.1}$$

$$\partial_t(\theta\rho_2(1-c)) + \mathrm{div}(\rho_2(1-c)\underline{u}) - \mathrm{div}(\rho_2\theta\mathcal{D}(\underline{u})\nabla(1-c)) = -\rho_2 q_s(1-c). \tag{1.2}$$

Each component is borne along by the flow (convection). The diffusion is due to molecular agitation which carry it along the direction of its concentration gradient. The dispersion lies with the heterogenities of the macroscopic velocity. The diffusive and dispersive effects are both modeled by the tensor $\mathcal{D}(\underline{u})$. For the usual rates of Peclet's number, the displacement is essentially induced by convection. The injection and production source terms are denoted by q_i and q_s. Now we model the compressibility of the fluid. We assume that pressure p and densities ρ_i are related by the following state equations:

$$\frac{d\rho_i}{\rho_i} = z_i dp, \quad i = 1, 2,$$

with $z_1 \geq z_2 \geq 0$, each real number z_i being the compressibility coefficient of the ith component of the mixture. By choosing $z_1 = z_2 = 0$, we would consider an incompressible fluid. Using the latter relations in Eqs. (1.1)–(1.2), we get

$$\partial_t(\theta c) + \theta z_1 c \partial_t p + \operatorname{div}(c\underline{u}) - \operatorname{div}(\theta \mathcal{D}(\underline{u})\nabla c) = q_i - q_s c,$$
$$\partial_t(\theta(1-c)) + \theta z_2 (1-c)\partial_t p + \operatorname{div}((1-c)\underline{u}) - \operatorname{div}(\theta \mathcal{D}(\underline{u})\nabla(1-c)) = -q_s(1-c).$$

Here we have used the slight compressibility assumption of Ref. 16 to neglect the terms of order $\mathcal{O}(z_i|\underline{u}|^2)$ with regard to the term $\underline{u}\cdot\nabla c$. Now, summing up the two latter equations we obtain an equation for p which expresses the total mass conservation during the displacement. The flow is then governed by the following system:

$$\partial_t \theta + \theta a(c)\partial_t p + \operatorname{div}(\underline{u}) = q_i - q_s, \tag{1.3}$$

$$\theta \partial_t c + \theta b(c)\partial_t p + \underline{u}\cdot\nabla c - \operatorname{div}(\theta \mathcal{D}(\underline{u})\nabla c) = q_i(1-c), \tag{1.4}$$

where we set

$$a(c) = (z_1 - z_2)c + z_2, \quad b(c) = (z_1 - z_2)c(1-c).$$

We now include in the model the effects of the rock compressibility. When charging a water saturated medium with little permeability (especially clay mineral), almost no compressing is observed for small observation times. Indeed, at the beginning of the experience, the charge induces an increase in the water pressure which has to be drained by the porous environment. But the final compressing may be very important. This phenomenon is the consolidation process. As the changing pore space is the controlling process for consolidational fluid flow, the variations of porosity are one of the keys of the model. First attempts to account for the phenomenon were based on depth-dependent porosity models $\theta = \theta(z)$. For instance, Athy's model,[4] is $\theta(z) = \theta_o \exp(-Mz)$, $(\theta_o, M) \in \mathbb{R}_+^2$ being specified by measurements. But this relation implies that the porosity change is not controlled by pressure changes. A physically more consistent approach is to derive an equation for the porosity. Denoting by σ the total stress and by σ_s the stress within the skeleton, we have

$$\sigma = \theta p + (1-\theta)\sigma_s,$$

the effects due to pressure water being given by θp. Following Terzaghi's theory, we define the effective stress $\sigma_e = (1-\theta)(\sigma_s - p)$. It follows that $\sigma = \sigma_e + p$. Assuming that the total stress remains unchanged, we write

$$d\sigma_e = -dp. \tag{1.5}$$

If the grains of the porous rock are incompressible, the deformation is mainly produced by the rearrangement of the assembly of grains (see Ref. 15). Then, as a bulk volume V deforms, its solid part $V_s = (1-\theta)V$ remains unchanged:

$$\frac{dV_s}{d\sigma_e} = -\frac{d\theta}{d\sigma_e}V + (1-\theta)\frac{dV}{d\sigma_e} = 0.$$

Bearing in mind (1.5), we infer from the latter relation that

$$\frac{1}{V}\frac{dV}{d\sigma_e} = -\frac{1}{1-\theta}\frac{d\theta}{dp}. \tag{1.6}$$

Assuming relative small volume changes and an elastic behavior for the soil, one generally defines the soil compressibility constant $\alpha \in \mathbb{R}$ by

$$\alpha = \frac{1}{V}\frac{dV}{d\sigma_e}.$$

It then follows from Eq. (1.6) that

$$\frac{d\theta}{dp} = (1-\theta)\alpha. \tag{1.7}$$

This equation is physically wrong. Indeed it involves high porosity loss at low porosities. Moreover, as already mentioned in Ref. 8, the sediment compressibility α cannot be treated as a constant:

$$\alpha = \alpha(p). \tag{1.8}$$

Obviously it should decrease during the consolidation process. We also guess from Ref. 8 that hydraulic conductivity κ and sediment compressibility α undergo similar changes during the process. For instance, assuming a Kozeny–Carman relation for κ,[7]

$$\kappa = \kappa(\theta(p)) = \kappa_o \frac{\theta(p)^3}{(1-\theta(p))^2},$$

one considers

$$\alpha = \alpha(\theta(p)) = \alpha_o \frac{\theta(p)^3}{(1-\theta(p))^2}.$$

Finally, the model for consolidational flow reads

$$\partial_t\theta(p) + \theta(p)a(c)\partial_t p + \mathrm{div}(\underline{u}) = q_i - q_s, \quad \underline{u} = -\kappa(\theta(p))\nabla p, \tag{1.9}$$

$$\theta(p)\partial_t c + \theta(p)b(c)\partial_t p + \underline{u}\cdot\nabla c - \mathrm{div}(\theta(p)\mathcal{D}(\underline{u})\nabla c) = q_i(1-c), \tag{1.10}$$

where function θ is defined by (1.7)–(1.8), $a(c) = (z_1 - z_2)c + z_2$, $b(c) = (z_1 - z_2)c(1-c)$. Note that if Eq. (1.9) is written using (1.7) as

$$((1-\theta(p))\alpha(p) + \theta(p)a(c))\partial_t p + \mathrm{div}(\underline{u}) = q_i - q_s,$$

one recognizes the coefficient of water storage

$$S^w_{o,p} = (1-\theta)\alpha + \theta a.$$

Note that, in the present derivation, the latter storativity depends on pressure and concentration. The term $(1-\theta)\alpha$ accounts for the rock compressibility, and the term θa accounts for the fluid compressibility.

1.2. *Miscible flow in a partially saturated medium*

We now describe a model for the displacement and transport of miscible species in a partially saturated porous medium, for instance in the context of the drying of a weakly permeable material. The unsaturated zone is prone to contamination from agriculture, where many chemicals such as fertilizers and pesticides are frequently applied to the field. The unsaturated zone is also sometimes viewed as a receptacle for waste storage. We also quote the modern gold mining methods as a wide range of applications. Assume that two components are transported in a wetting phase (water) in the presence of a non-wetting fluid (air). The basis for the mathematical modeling is once again the mass conservation principle. We thus rewrite Eqs. (1.1)–(1.2):

$$\partial_t(\theta\rho_1 c) + \operatorname{div}(\rho_1 c\underline{u}) - \operatorname{div}(\rho_1\theta\mathcal{D}(\underline{u})\nabla c) = \rho_1(q_i - q_s)c, \tag{1.11}$$

$$\partial_t(\theta\rho_2(1-c)) + \operatorname{div}(\rho_2(1-c)\underline{u}) - \operatorname{div}(\rho_2\theta\mathcal{D}(\underline{u})\nabla(1-c)) = -\rho_2 q_s(1-c). \tag{1.12}$$

But now function θ is the volumetric moisture content defined by

$$\theta = \phi s,$$

where ϕ is the porosity of the medium and s is the effective degree of saturation

$$s = \frac{S - S_r}{S_s - S_r},$$

where S is the saturation, S_s and S_r are the saturation and residual water contents respectively. Assuming that the air present in the unsaturated zone has infinite mobility allows to admit Richards hypothesis. The saturation s and then function θ are thus considered as monotone functions depending on the pressure head p. This is similar to assume that the pressure is given by a capillary pressure $P_c = P_c(s)$. Following the lines of the latter subsection (assuming once again the slight compressibility of the fluid), we get

$$\partial_t\theta(p) + \theta(p)a(c)\partial_t p + \operatorname{div}(\underline{u}) = q_i - q_s, \quad \underline{u} = -\kappa(\theta(p))\nabla p, \tag{1.13}$$

$$\theta(p)\partial_t c + \theta(p)b(c)\partial_t p + \underline{u}\cdot\nabla c - \operatorname{div}(\theta(p)\mathcal{D}(\underline{u})\nabla c) = q_i(1-c), \tag{1.14}$$

where functions a and b are still defined by

$$a(c) = (z_1 - z_2)c + z_2, \quad b(c) = (z_1 - z_2)c(1-c).$$

Note that if the fluid is assumed incompressible, $z_1 = z_2 = 0$, then Eq. (1.13) is the classical Richards equation in the pressure formulation. The key of the model is a correct definition of the volumetric moisture content θ and of the mobility function κ. Let us assume that the saturation pressure P_s is zero. The fracture component of the medium is fully-saturated in the groundwater region $\{x;\ p(x,\cdot) > P_s = 0\}$, while it is partially saturated in the capillary fringe $\{x;\ P_d < p(x,\cdot) \le 0\}$. The dry part is the set $\{x;\ p(x,\cdot) \le P_d\}$. The moisture content is such that

$$\theta(p) = \begin{cases} \phi & \text{(constant porosity)} & \text{if } p > 0, \\ \theta(p) & \text{(with } 0 \le \theta(p) \le \phi \text{ and } \theta'(p) > 0) & \text{if } P_d < p \le 0, \\ 0 & \text{(dry zone)} & \text{if } p \le P_d. \end{cases} \tag{1.15}$$

The permeability of the soil remains essentially equal to the saturated coefficient of permeability until the air-entry value of the soil is reached. And at the residual water content, both the moisture content and the permeability become zero (see for instance van Genuchten *et al.*[32]). The mobility has thus the following form:

$$\kappa(\theta(p)) = \begin{cases} k/\mu & \text{(constant mobility)} & \text{if } p > 0, \\ \kappa(\theta(p)) & (0 \le (\kappa\circ\theta)(p) \le k/\mu, (\kappa\circ\theta)'(p) > 0) & \text{if } P_d < p \le 0, \\ 0 & \text{(dry zone)} & \text{if } p \le P_d. \end{cases} \tag{1.16}$$

Commonly used pairs (θ, κ) are given by the van Genuchten−Mualem model,[31,22] by the van Genuchten−Burdine model,[31,10] or by the Brooks−Corey model.[9] See Remark 1 below for an explicit example. A significant amount of work has been performed in obtaining values for the empirical scaling parameters.

2. Mathematical Setting of the Problem and Main Results

We consider a domain Ω of $\mathbb{R}^3$ with $\mathcal{C}^1$ boundary Γ. The unit normal pointing outward Ω is denoted by ν. The time interval of interest is $(0,T)$, $\Omega_T = \Omega \times (0,T)$. In view of the similarities of the models derived in Sec. 1, we consider the following system of pdes in Ω_T.

$$\partial_t\theta(p) + \theta(p)a(c)\partial_t p + \operatorname{div}(\underline{u}) = q_i - q_s, \quad \underline{u} = -\kappa(\theta(p))\nabla p, \tag{2.1}$$

$$\theta(p)\partial_t c + \theta(p)b(c)\partial_t p + \underline{u}\cdot\nabla c - \operatorname{div}(\theta(p)\mathcal{D}(\underline{u})\nabla c) = q_i(1-c). \tag{2.2}$$

Functions a and b are defined in $(0,1)$ by

$$a(x) = (z_1 - z_2)x + z_2, \quad b(x) = (z_1 - z_2)x(1-x), \quad x \in (0,1),$$

and are continuously extended to $\mathbb{R}$. The source terms q_i and q_s are some given non-negative functions of $L^2(\Omega)$. The diffusive and dispersive effects are expressed by the tensor $\mathcal{D}(\underline{u})$,

$$\mathcal{D}(\underline{u}) = |\underline{u}|(\alpha_L\mathcal{E}(\underline{u}) + \alpha_T(\mathrm{Id} - \mathcal{E}(\underline{u}))) + D_m\mathrm{Id},$$

where $\mathcal{E}(\underline{u})_{ij} = \underline{u}_i\underline{u}_j/|\underline{u}|^2$, α_L and α_T are the longitudinal and transverse dispersion constants, $\alpha_L \ge \alpha_T \ge 0$, and $D_m \ge 0$ is the molecular diffusion. Assuming $D_m > 0$ is mathematically convenient because this hypothesis ensures that the concentration equation is of parabolic type. But for the usual rates of flow, convection is the highly dominant process. One thus has to consider the degenerate case where $D_m = \alpha_L = \alpha_T = 0$ (see Theorem 2.2). Tortuosity effects could be included in the model by replacing the term $\mathrm{div}(\theta(p)\mathcal{D}(\underline{u})\nabla c)$ by $\mathrm{div}(\theta(p)^\tau\mathcal{D}(\underline{u})\nabla c)$, where τ is some positive real number. The most important coupling is induced in the system by the functions θ and κ which are pressure-dependent. We assume

$$\theta \in \mathcal{C}^1(\mathbb{R}), \quad 0 \le \theta(x) \le \theta_+, \quad \theta'(x) \ge 0 \;\; \forall\, x \in \mathbb{R}, \tag{2.3}$$

$$\kappa \in \mathcal{C}(\mathbb{R}_+), \quad 0 \le \kappa(x) \le \kappa_+ \;\; \forall\, x \in \mathbb{R}_+, \tag{2.4}$$

where θ_+ and κ_+ are given non-negative real numbers. Note that functions θ and κ characterize the mathematical type of the problem. Indeed tensor $\mathcal{D}$ satifies

$$\mathcal{D}(\underline{u})\xi \cdot \xi \ge (D_m + \alpha_T|\underline{u}|)\,|\xi|^2, \quad |\mathcal{D}(\underline{u})\xi| \le (D_m + \alpha_L|\underline{u}|)\,|\xi|, \tag{2.5}$$

for any $\xi \in \mathbb{R}^3$. System (2.1)$-$(2.2) is thus of parabolic type if θ and κ are positive functions. System (2.1)$-$(2.2) is of degenerate parabolic type if θ and κ are non-negative functions.

The problem is completed by the following initial and boundary conditions.

$$\begin{gathered} \underline{u} \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), p_{|_{\Gamma_2}} = P_2 \quad \text{in } (0,T), \\ p(x,0) = p_0(x) \quad \text{in } \Omega, \end{gathered} \tag{2.6}$$

$$\begin{gathered} \theta(p)\mathcal{D}(\underline{u})\nabla c \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T),\; c_{|_{\Gamma_2}} = C_2 \quad \text{in } (0,T), \\ c(x,0) = c_0(x) \quad \text{in } \Omega, \end{gathered} \tag{2.7}$$

where the boundary Γ of Ω is the disjoint union of Γ_1 and Γ_2. The mixed boundary conditions are chosen to insert in the model infiltration problems. Functions P_2 and C_2 belong to the space $H^1(0,T;H^{1/2}(\Gamma))$. Functions $p_0 \in H^2(\Omega)$, $c_0 \in H^2(\Omega)$ satisfy the compatibility conditions

$$\begin{cases} p_{0|_{\Gamma_2}} = P_2(x,0), \quad c_{0|_{\Gamma_2}} = C_2(x,0), \\ \kappa(\theta(p_0))\nabla p_0 \cdot \nu_1 = 0, \theta(p_0)\mathcal{D}(-\kappa(\theta(p_0))\nabla p_0)\nabla c_0 \cdot \nu_1 = 0. \end{cases}$$

We also assume that $c_0 \in L^\infty(\Omega)$ is an admissible concentration:

$$0 \le c_0(x) \le 1 \quad \text{a.e. in } \Omega. \tag{2.8}$$

The first result of this paper is devoted to the parabolic setting of the problem.

Theorem 2.1. (Consolidational flow model) *Assume that there exist two real numbers θ_- and κ_- such that*

$$\theta(x) \ge \theta_- > 0 \quad \forall\, x \in \mathbb{R}, \quad \kappa(x) \ge \kappa_- > 0 \quad \forall\, x \in \mathbb{R}_+. \tag{2.9}$$

Assume

$$D_m > 0, \quad \alpha_L \ge \alpha_T > 0. \tag{2.10}$$

Then problem (2.1)$-$(2.2), (2.6)$-$(2.7) *admits a weak solution* $(p,\, c)$ *satisfying*

(i) *the function $p \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ is solution of* (2.1), (2.6);
(ii) *the function $c \in L^\infty(\Omega_T) \cap L^2(0,T;H^1(\Omega))$ is a solution of* (2.2), (2.7), *verified in $L^2(0,T;(V)')$, where $V = \{f \in L^2(0,T;W^{1,4}(\Omega)); f_{|_{\Gamma_2}} = 0\}$. It satisfies $0 \le c(x,t) \le 1$ a.e. in Ω_T.*

Let us give some references for this type of coupled parabolic problem. Amirat *et al.* have proved in Ref. 3 the existence of weak solutions for some similar problem, but assuming a lower coupling, that is $\theta = \theta(x)$ and $\kappa = \kappa(x)$. The author have studied in Ref. 13 the case where the coupling is induced by the concentrations, that is $\theta = \theta(x)$ and $\kappa = \kappa(c)$. We also quote Daïm *et al.*[14] who obtained a weak existence result for a two-phase incompressible flow model, that is $\theta = \theta(p)$ and $\kappa = \kappa(c)$.

We now consider the fully degenerate setting:

$$\partial_t \theta(p) + \theta(p) a(c) \partial_t p + \operatorname{div}(\underline{u}) = q_i - q_s, \quad \underline{u} = -\kappa(\theta(p)) \nabla p, \tag{2.11}$$

$$\theta(p) \partial_t c + \theta(p) b(c) \partial_t p + \underline{u} \cdot \nabla c = q_i (1 - c), \tag{2.12}$$

$$\begin{aligned} \underline{u} \cdot \nu_1 = 0 \text{ on } \Gamma_1 \times (0,T), p_{|\Gamma_2} &= P_2 \quad \text{in } (0,T), \\ p(x,0) &= p_0(x) \quad \text{in } \Omega, \end{aligned} \tag{2.13}$$

$$c(x,0) = c_0(x) \quad \text{in } \Omega. \tag{2.14}$$

We assume $P_2(x,t) \geq P_d$ in $\Gamma_2 \times (0,T)$, $p_0(x) \geq P_d$ in Ω. For the latter degenerate parabolic−hyperbolic system, we claim and prove the following result.

Theorem 2.2. (Partially saturated model) *Assume that functions θ and κ satisfy* (2.3)−(2.4). *Assume, moreover,*

$$\text{there exists } \varepsilon_0 > 0 \text{ such that } \kappa \text{ is increasing in } (0, \varepsilon_0), \tag{2.15}$$

$$\lim_{p \to P_d} \theta(p) = 0, \quad \theta(p) > 0 \quad \text{if } p > P_d, \quad \lim_{s \to 0} \kappa(s) = 0, \quad \kappa(s) > 0 \quad \text{if } s > 0, \tag{2.16}$$

$$\exists\, q > 0; \begin{cases} \lim\limits_{p \to P_d} ((\kappa \circ \theta)(p))^{q-3} \theta'(p) (\kappa' \circ \theta)(p) < \infty, \\ \lim\limits_{p \to P_d} \dfrac{((\kappa \circ \theta)(p))^{2q-1}}{\theta(p)^3} < \infty, \\ \lim\limits_{p \to P_d} \dfrac{((\kappa \circ \theta)(p))^{2q-1}}{\theta(p)^3} < \infty. \end{cases} \tag{2.17}$$

Then problem (2.11)−(2.14) *admits a weak solution* $(p,\, c)$ *such that*

(i) *Equation* (2.1) *is satisfied in* $L^2(0,T; H^{-1}(\Omega))$; *the pressure function* p *belongs to* $L^2(\Omega_T)$ *and is such that* $\underline{u} \in (L^2(\Omega_T))^3$;
(ii) *the function* c *belongs to* $L^\infty(\Omega_T)$ *and satisfies* $0 \leq c(x,t) \leq 1$ *a.e. in* Ω_T.

Remark 2.1. Assumptions (2.15)−(2.17) could appear as rather technical and then completely utopian. We thus detail a classical model for the pair moisture content-mobility and check that it satisfies (2.15)−(2.17). The expression of the mobility proposed by van Genuchten,[31] is

$$\kappa(\theta) = \frac{k_o}{\mu} \left(\frac{\theta}{\phi}\right)^{\lambda} \left(1 - \left(1 - \left(\frac{\theta}{\phi}\right)^{1/m}\right)^m\right)^2,$$

where λ is the pore connectivity parameter and $m = 1 - 1/n$. The real numbers λ and n are empirical parameters. One often considers that $\lambda = 1/2$. The van

Genuchten parameter n in Ref. 31 is $n = 1.28$. Many *scenarios* use $n = 3$. The moisture content is expressed as

$$\theta = \phi(1 + |\alpha p|^n)^{-m},$$

the number α being once again an empirical parameter. For the sake of clarity, we have written this model setting $P_d = -\infty$. One easily checks that assumptions (2.15)–(2.17) are satisfied by this model. Note that assumption (2.17) is not as severe as it seems since we may choose q sufficiently large to satisfy it.

There is a huge literature concerning the Richards equation. We quote especially the fundamental works Refs. 2 and 1, and the papers Refs. 19, 12, 28 devoted to the study of the degenerate in time equation

$$\partial_t \theta(u) - \Delta u = 0.$$

In the one-dimensional case, Yin states in Ref. 33 the existence of weak solutions for the fully degenerate equation

$$\partial_t \theta(u) - \partial_x(\kappa(u)\partial_x u) = 0.$$

Yin assumed $\theta' > 0$, $\kappa' > 0$, $\lim_{p\to P_d} \theta'(p) = \lim_{p\to P_d} \kappa'(p) = \lim_{p\to P_d} \kappa(p)/\theta'(p) = 0$. The latter assumption is comparable to (2.17) despite our existence proof is completely different from Yin's one. To our knowledge the analysis of the latter pressure equation coupled with a hyperbolic one was never performed. Daïm *et al.*[14] only consider the parabolic setting of Theorem 2.1.

The paper is organized as follows. Section 3 is devoted to the proof of Theorem 2.1. The main difficulty lies in the strong nonlinear couplings. Defining an adapted Kirchhoff's transform and using a double fixed point approach, we prove that classical results for parabolic equations,[21] apply. In Sec. 4, we prove the existence result for the fully degenerate setting of Theorem 2.2. Theorem 2.1 gives an existence result for a parabolic regularization of the problem. Due to the degeneration of functions θ and κ, we then have to extract subsequences of convenient truncated solutions to state enough compactness results to pass to the limit in the nonlinearities. We use especially compensated compactness arguments.

3. Proof of Theorem 2.1

The problem is characterized by the coupling between pressure and concentration and the strong nonlinearities in the pressure equation (2.1). We adopt two strategies to overcome these difficulties. Fixed point approach is now classical for the study of strongly coupled problems (see Ref. 17). In the present paper, we cannot follow the lines of the fixed point approach of Refs. 3 and 13. Indeed, the nonlinearities in the pressure equation (2.1) do not allow one to state a uniqueness result for the pressure solution. We thus construct some "double fixed point" approach. The second key of this proof is the use of a Kirchhoff's transform to linearize the divergence part of the pressure equation. Indeed, assumptions (2.3)–(2.4) are sufficient to define the function F by

$$F(p) = \int^p \kappa(\theta(s))ds. \tag{3.1}$$

By (2.9), F is a bijective application and the existence of P such that

$$P = F(p)$$

is equivalent to the existence of p solution of the original pressure problem. The Kirchhoff's transform of Eq. (2.1) being

$$((\theta' \circ F^{-1})(P) + a(c)(\theta \circ F^{-1}(P))(F^{-1})'(P)\partial_t P - \Delta P = q_i - q_s,$$

we are led to consider the following problem in Ω_T:

$$\sigma(c,P)\partial_t P - \Delta P = q_i - q_s, \tag{3.2}$$

$$\begin{aligned} &\zeta(P)\partial_t c + \zeta(P)(F^{-1})'(P)b(c)\partial_t P - \nabla P \cdot \nabla c \\ &\qquad - \operatorname{div}(\zeta(P)\mathcal{D}(\nabla P)\nabla c) = q_i(1-c), \end{aligned} \tag{3.3}$$

$$\begin{gathered} \nabla P \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), P_{|\Gamma_2} = F(P_2) \quad \text{in } (0,T), \\ P(x,0) = F(p_0(x)) \quad \text{in } \Omega, \end{gathered} \tag{3.4}$$

$$\begin{gathered} \zeta(P)\mathcal{D}(\nabla P)\nabla c \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad c_{|\Gamma_2} = C_2 \quad \text{in } (0,T), \\ c(x,0) = c_0(x) \quad \text{in } \Omega. \end{gathered} \tag{3.5}$$

We set

$$\begin{aligned} \sigma(c,P) &= ((\theta' \circ F^{-1})(P) + a(c)(\theta \circ F^{-1})(P))(F^{-1})'(P), \\ \zeta(P) &= (\theta \circ F^{-1})(P). \end{aligned}$$

Note that there is no more nonlinearity in the space derivatives of the pressure equation (3.2). We also have

$$0 < \sigma_- = \frac{z_2\theta_-}{\kappa_+} \le \sigma(c,P) \le \frac{z_1\theta_+}{\kappa_-}.$$

We now construct a Schauder fixed point approach (see Ref. 26). Let us define two closed convex subsets K_c and K_p of $L^2(\Omega_T)$ by

$$\begin{aligned} K_c &= \{f \in L^2(\Omega_T);\ 0 \le f(x,t) \le 1 \text{ a.e. in } \Omega_T\}, \\ K_p &= \left\{ f \in L^2(\Omega_T);\ \left\| f - \frac{1}{|\Omega|}\int_\Omega f \right\|_{L^2(\Omega_T)} \le M_p \right\}, \end{aligned}$$

the constant M_p being defined in Lemma 3.1 below. Let $(\bar{c}, \bar{P})$ such that

$$\bar{c} \in K_c, \quad \bar{P} \in K_p.$$

We begin by considering the unique solution P of the following problem:

$$\sigma(\bar{c}, \bar{P})\partial_t P - \Delta P = q_i - q_s \quad \text{in } \Omega_T, \tag{3.6}$$

$$
\begin{aligned}
\nabla P \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad P_{|\Gamma_2} = F(P_2) \quad \text{in } (0,T),\\
P(x,0) = F(p_0(x)) \quad \text{in } \Omega.
\end{aligned} \tag{3.7}
$$

We prove the following result.

Lemma 3.1. *For any fixed* $(\bar{c}, \bar{P})$ *in* $K_c \times K_p$, *there is a unique function* $P \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ *solution of* (3.6)–(3.7). *It satisfies*

$$
\|P\|_{L^\infty(0,T;H^1(\Omega))} + \|\sqrt{\sigma_-}P\|_{H^1(0,T;L^2(\Omega))} \le C, \tag{3.8}
$$

$$
\left\| P - \frac{1}{|\Omega|}\int_\Omega P \right\|_{L^2(\Omega_T)} \le M_p, \tag{3.9}
$$

where C *and* M_p *only depend on the data of the original problem* (2.1), (2.6).

Proof. We begin by some *a priori* estimates. Assume P is a solution of (3.6)–(3.7). We multiply Eq. (3.6) by $\partial_t P$ and integrate by parts over Ω. We obtain

$$
\begin{aligned}
&\int_\Omega \sigma(\bar{c},\bar{P})|\partial_t P|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla P|^2 dx\\
&\qquad - \int_{\Gamma_2} \partial_t F(P_2)(\nabla P \cdot \nu_2) ds = \int_\Omega (q_i - q_s)\partial_t P dx.
\end{aligned} \tag{3.10}
$$

Since $P_2 \in H^1(0,T;H^{1/2}(\Gamma_2))$ and Γ is smooth, there exists a function $\tilde{P}_2 \in H^1(\Omega_T)$ such that $\tilde{P}_{2|\Gamma_2} = P_2$ (see for instance Ref. 24). We thus write

$$
\begin{aligned}
-\int_{\Gamma_2} \partial_t F(P_2)(\nabla P \cdot \nu_2) ds &= -\int_\Omega \operatorname{div}(\partial_t F(\tilde{P}_2)\nabla P) dx\\
&= \int_\Omega \partial_t F(\tilde{P}_2)(-\sigma(\bar{c},\bar{P})\partial_t P + q_i - q_s) dx\\
&\quad - \int_\Omega \nabla P \cdot \nabla(\partial_t F(\tilde{P}_2)) dx.
\end{aligned}
$$

Using the Cauchy–Schwarz and Young inequalities, we get

$$
\begin{aligned}
\left| \int_{\Gamma_2} \partial_t F(P_2)(\nabla P \cdot \nu_2) ds \right| &\le \left(\frac{C}{\delta} + 1\right)\|P_2\|^2_{H^1(0,T;H^{1/2}(\Gamma_2))}\\
&\quad + \delta \int_\Omega |\partial_t P|^2 dx + C\int_\Omega |\nabla P|^2 dx\\
&\quad + \|P_2\|_{H^1(0,TH^{1/2}(\Gamma_2))}(\|q_i\|^2_{L^2(\Omega)} + \|q_s\|^2_{L^2(\Omega)}).
\end{aligned} \tag{3.11}
$$

We also have

$$
\left| \int_\Omega (q_i - q_s)\partial_t P dx \right| \le \delta \int_\Omega |\partial_t P|^2 \, dx + \frac{C}{\delta}(\|q_i\|^2_{L^2(\Omega)} + \|q_s\|^2_{L^2(\Omega)}). \tag{3.12}
$$

We choose $\delta = \sigma_-/4$ so that $\sigma(\bar{c},\bar{P}) - 2\delta \ge \sigma_-/2 > 0$. Using (3.10)–(3.12) and the Gronwall lemma, we get the estimates announced in Lemma 3.1. Estimate (3.9) follows from the Poincaré–Wirtinger inequality. These estimates are sufficient to assert the existence of a solution P of problem (3.6)–(3.7) (see Ref. 21). The uniqueness of the solution is obvious since Eq. (3.6) is linear. Indeed, if $P^\sharp$ and $P^\flat$ are two solutions of (3.6)–(3.7), then $P = P^\sharp - P^\flat$ fulfills

$$\sigma(\bar{c}, \bar{P})\partial_t P - \Delta P = 0 \quad \text{in } \Omega_T,$$
$$\nabla P \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad P_{|\Gamma_2} = 0 \quad \text{in } (0,T), \quad P(x,0) = 0 \quad \text{in } \Omega.$$

Following the previous lines, we infer from the Gronwall lemma that $\nabla P = 0$ a.e. in Ω_T. Since $P_{|\Gamma_2} = 0$, it follows that $P(x,t) = P^\sharp(x,t) - P^\flat(x,t) = 0$ a.e. in Ω_T. □

We now regularize P by convolution in space and time. Let $\psi \in C^\infty(\mathbb{R}^4)$, $\psi \geq 0$, with support in the unit ball such that $\int_{\mathbb{R}^4} \psi(x,t)\, dxdt = 1$. For $\eta > 0$ small enough, we set $\psi_\eta(x,t) = \psi(x/\eta, t/\eta)/\eta^4$. We extend P outside Ω_T, keeping the same notations for convenience: we now have $P \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \cap H^1(\mathbb{R}; L^2(\mathbb{R}^3))$. We then define $\tilde{P}$ by

$$\tilde{P} = \psi_\eta * P.$$

We denote in the same way its restriction to Ω_T. It satisfies $\tilde{P} \in C^\infty(\overline{\Omega_T})$, and as η tends to zero

$$\tilde{P} \to P \quad \text{strongly in } H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

In Eqs. (3.3), (3.5), we replace P by $\tilde{P}$. As in Ref. 17, we also regularize the mechanical dispersion term. The tensor $\mathcal{D}(\nabla P)$ is replaced by $\mathcal{D}(\nabla\tilde{P}/(1+\eta|\nabla\tilde{P}|))$. The components of this tensor belong to $L^\infty(\Omega_T)$. The dominated convergence theorem gives

$$\mathcal{D}(\nabla\tilde{P}/(1+\eta|\nabla\tilde{P}|)) \to \mathcal{D}(\nabla P) \quad \text{strongly in } (L^2(\Omega_T))^{3\times 3} \quad \text{as } \eta \to 0.$$

We then consider the following regularized problem in Ω_T.

$$\begin{aligned}\partial_t(\zeta(\tilde{P})c^\eta) - \zeta'(\tilde{P})c^\eta\partial_t\tilde{P} + \zeta(P)(F^{-1})'(P)b(c^\eta)\partial_t\tilde{P} - \nabla\tilde{P}\cdot\nabla c^\eta \\ - \operatorname{div}(\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta) = q_i(1-c^\eta),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta\cdot\nu_1 = 0 \quad \text{on } \Gamma_1\times(0,T), \ c^\eta_{|\Gamma_2} = C_2 \quad \text{in } (0,T), \\ c^\eta(x,0) = c_0(x) \quad \text{in } \Omega.\end{aligned} \tag{3.14}$$

We recognize a slight modification of (3.3), (3.5). We simply used the decomposition $\zeta(P)\partial_t c = \partial_t(\zeta(P)c) - \zeta'(P)c\partial_t P$ in order to get the first estimates for c despite the nonlinearity in front of the time derivative. We claim the following result.

Lemma 3.2. *For any $\eta > 0$, there exists a unique function c^η in $L^\infty(\Omega_T) \cap L^2(0,T; H^1(\Omega))$ solution of (3.13)–(3.14). It satisfies the following uniform estimates:*

$$0 \leq c^\eta(x,t) \leq 1 \quad \text{a.e. in } \Omega_T, \tag{3.15}$$

$$\|\zeta(\tilde{P})c^\eta\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{3.16}$$

$$\|\theta_-^{1/2}c^\eta\|_{L^2(0,T;H^1(\Omega))} \leq C, \quad \|\theta_-^{1/2}|\nabla\tilde{P}|^{1/2}\nabla c^\eta\|_{(L^2(\Omega_T))^3} \leq C. \tag{3.17}$$

Proof. For the existence of a solution c^η to the parabolic problem with smooth coefficients (3.13)−(3.14) we refer to the classical textbook Ref. 21, pp. 178−179. In the present proof, we just present the key *a priori* estimates. An essential step is stating the maximum principle (3.15) for the concentration solution c^η. In the proof below, the regularity of $\tilde{P}$ is essential. But since the maximum principle (3.15) does not depend on η, it will remain true after the suppression of the regularization. Let us show that $c^\eta(c,t) \geq 0$ almost everywhere in Ω_T. We set $c_-^\eta = \sup(0, -c^\eta)$. We multiply Eq. (3.13) by $\zeta(\tilde{P})c_-^\eta$ and integrate by parts over Ω. Bearing in mind that $c_-^\eta|_{\Gamma_2} = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\zeta(\tilde{P})c_-^\eta|^2 dx - \int_\Omega \zeta'(\tilde{P})\zeta(\tilde{P})\partial_t\tilde{P}|c_-^\eta|^2 dx - \int_\Omega \zeta(\tilde{P})c_-^\eta(\nabla\tilde{P}\cdot\nabla c_-^\eta)dx \\
&\quad + \int_\Omega \zeta(P)\zeta(\tilde{P})\mathcal{D}(\nabla\tilde{P})\nabla c_-^\eta\cdot\nabla c_-^\eta\,dx + \int_\Omega \zeta(P)\zeta'(\tilde{P})c_-^\eta\mathcal{D}(\nabla\tilde{P})\nabla c_-^\eta\cdot\nabla\tilde{P}dx \\
&= -\int_\Omega q_i c_-^\eta\,dx + \int_\Omega q_i|c_-^\eta|^2 dx.
\end{aligned} \tag{3.18}$$

Note that the term containing b is omitted because b is extended by zero in $\mathbb{R}_-$. Using Cauchy−Schwarz and Young inequalities, we write the following set of estimates

$$\begin{aligned}
\left|\int_\Omega \zeta'(\tilde{P})\zeta(\tilde{P})\partial_t\tilde{P}|c_-^\eta|^2\,dx\right| &\leq C\|\tilde{P}\|_{\mathcal{C}^1(\Omega_T)}\int_\Omega |c_-^\eta|^2\,dx, \\
\left|\int_\Omega \zeta(\tilde{P})c_-^\eta(\nabla\tilde{P}\cdot\nabla c_-^\eta)\,dx\right| &\leq \delta\int_\Omega |\nabla\tilde{P}||\nabla c_-^\eta|^2\,dx + \frac{C}{\delta}\int_\Omega |\nabla\tilde{P}||c_-^\eta|^2\,dx \\
&\leq \delta\int_\Omega |\nabla\tilde{P}||\nabla c_-^\eta|^2\,dx \\
&\quad + \frac{C}{\delta}\|\tilde{P}\|_{\mathcal{C}^1(\Omega_T)}\int_\Omega |c_-^\eta|^2\,dx, \\
\int_\Omega \zeta(P)\zeta(\tilde{P})\mathcal{D}(\nabla\tilde{P})\nabla c_-^\eta\cdot\nabla c_-^\eta\,dx &\geq \theta_-^2\int_\Omega (D_m + \alpha_T|\nabla\tilde{P}|)\,|\nabla c_-^\eta|^2\,dx, \\
\left|\int_\Omega \zeta(P)\zeta'(\tilde{P})c_-^\eta\mathcal{D}(\nabla\tilde{P})\nabla c_-^\eta\cdot\nabla\tilde{P}\,dx\right| &\leq \delta\int_\Omega |\nabla\tilde{P}||\nabla c_-^\eta|^2\,dx \\
&\quad + \frac{C\theta_+^2\alpha_T}{\delta}\int_\Omega |\nabla\tilde{P}|^2\,|c_-^\eta|^2\,dx \\
&\leq \delta\int_\Omega |\nabla\tilde{P}|\,|\nabla c_-^\eta|^2\,dx \\
&\quad + \frac{C}{\delta}\|\tilde{P}\|^2_{\mathcal{C}^1(\Omega_T)}\int_\Omega |c_-^\eta|^2\,dx, \\
\int_\Omega q_i c_-^\eta\,dx &\geq 0,
\end{aligned}$$

for any $\delta > 0$. We thus infer from (3.18) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\zeta(\tilde{P})c_-^\eta|^2\,dx + \int_\Omega \big((\theta_-^2\alpha_T - 2\delta) + \theta_-^2 D_m\big)|\nabla\tilde{P}|\,|\nabla c_-^\eta|^2\,dx$$
$$\le C(\|\tilde{P}\|_{\mathcal{C}^1(\Omega_T)})\int_\Omega |c_-^\eta|^2\,dx \le \frac{1}{\theta_-^2}C(\|\tilde{P}\|_{\mathcal{C}^1(\Omega_T)})\int_\Omega |\zeta(\tilde{P})c_-^\eta|^2\,dx.$$

We choose $\delta > 0$ such that $\theta_-^2\alpha_T - 2\delta \ge \delta_o > 0$. We then apply the Gronwall lemma. Since $c_-^\eta(x,0) = 0$, it gives $(\zeta(\tilde{P})c_-^\eta)(x,t) = 0$ almost everywhere in Ω_T. Bearing in mind that ζ is a non-negative function, we conclude that $c_-^\eta(x,t) = 0$ and thus $c^\eta(x,t) \ge 0$ almost everywhere in Ω_T. Noting that Eq. (3.13) can be rewritten as

$$\partial_t\big(\zeta(\tilde{P})(c^\eta - 1)\big) - \zeta'(\tilde{P})(c^\eta - 1)\partial_t\tilde{P} + \zeta(P)(F^{-1})'(P)b(c^\eta)\partial_t\tilde{P}$$
$$- \nabla\tilde{P}\cdot\nabla(c^\eta - 1) - \operatorname{div}(\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla(c^\eta - 1)) = q_i(1 - c^\eta),$$

we prove that $1 - c^\eta(x,t) \ge 0$ almost everywhere in Ω_T by similar computations. The L^∞ estimate (3.15) is established.

Knowing (3.15), the proof for (3.16)–(3.17) is easier. Assume c^η is a solution of (3.13)–(3.14). We multiply Eq. (3.13) by c^η and we integrate over Ω. Integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \zeta(\tilde{P})|c^\eta|^2\,dx - \frac{1}{2}\int_\Omega \zeta'(\tilde{P})\partial_t\tilde{P}|c^\eta|^2\,dx + \int_\Omega b(c^\eta)\,c^\eta\,\zeta(P)(F^{-1})'(P)\,\partial_t\tilde{P}$$
$$- \int_\Omega (\nabla\tilde{P}\cdot\nabla c^\eta)\,c^\eta\,dx + \int_\Omega \zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta\cdot\nabla c^\eta\,dx$$
$$- \int_{\Gamma_2} C_2(\zeta(F(P_2))\mathcal{D}(\nabla\tilde{P})\nabla c^\eta\cdot\nu_2)\,ds = \int_\Omega q_i(1 - c^\eta)c^\eta\,dx. \tag{3.19}$$

We already know that c^η is uniformly bounded in $L^\infty(\Omega_T)$ and $\tilde{P}$ is uniformly bounded in $H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$. So most of the terms in (3.19) are straightforward estimated using the Cauchy–Schwarz inequality. We only detail the computations for the Γ_2-boundary term. Since $C_2 \in L^2(0,T;H^{1/2}(\Gamma_2))$, there exists a function $\tilde{C}_2 \in L^2(0,T;H^1(\Omega))$ such that $\tilde{C}_{2|_{\Gamma_2}} = C_2$. We thus write

$$\int_{\Gamma_2} C_2\,(\zeta(F(P_2))\mathcal{D}(\nabla\tilde{P})\nabla c^\eta\cdot\nu_2)ds = \int_\Omega \operatorname{div}(\tilde{C}_2\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta)\,dx$$
$$= \int_\Omega \tilde{C}_2\operatorname{div}(\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta)dx + \int_\Omega \zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta\cdot\nabla\tilde{C}_2 dx. \tag{3.20}$$

We now estimate the terms on the right-hand side of Eq. (3.20). On the one hand, using the Cauchy–Schwarz inequality, we write

$$\left|\int_\Omega \zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta\cdot\nabla\tilde{C}_2\,dx\right|$$
$$\le C\||\nabla\tilde{P}|^{1/2}\nabla c^\eta\|_{(L^2(\Omega))^3}\||\nabla\tilde{P}|^{1/2}\nabla\tilde{C}_2\|_{(L^2(\Omega))^3}.$$

On the right-hand side of the latter relation the first term is controlled by the dispersive term in (3.20). The $L^2(0,T)$-norm of the second one is uniformly bounded by a constant because $|\nabla\tilde{P}|^{1/2}$ is uniformly bounded in $L^\infty(0,T;L^4(\Omega))$ and $\nabla\tilde{C}_2 \in L^2(0,T;L^2(\Omega)) \subset L^2(0,T;L^{4/3}(\Omega))$. On the other hand, we use Eq (3.13) and write

$$\begin{aligned}\int_\Omega \tilde{C}_2 \mathrm{div}(\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c^\eta)dx &= \int_\Omega \tilde{C}_2\zeta(\tilde{P})\partial_t c^\eta dx \\ &+ \int_\Omega \tilde{C}_2\,\zeta(P)(F^{-1})'(P)b(c^\eta)\partial_t\tilde{P}dx - \int_\Omega \tilde{C}_2\,(\nabla\tilde{P}\cdot\nabla c^\eta)dx \\ &- \int_\Omega \tilde{C}_2 q_i(1-c^\eta)dx.\end{aligned}$$

The estimates of these terms are obvious, with in particular

$$\int_\Omega \tilde{C}_2\zeta(\tilde{P})\partial_t c^\eta dx = \int_\Omega \partial_t(\tilde{C}_2\zeta(\tilde{P})c^\eta)dx - \int_\Omega c^\eta\partial_t(\tilde{C}_2\zeta(\tilde{P}))dx,$$

with $\tilde{C}_2 \in H^1(0,T;L^2(\Omega))$ and $(c^\eta,\tilde{P})$ uniformly bounded in $(L^\infty(0,T;L^2(\Omega)))^2$. Then, using the properties of the diffusion tensor $\mathcal{D}$ and the Gronwall lemma, we infer from (3.19) that ∇c^η and $|\nabla\tilde{P}|^{1/2}\nabla c^\eta$ are uniformly bounded in $(L^2(\Omega_T))^3$. We finally consider briefly the question of the uniqueness of the solution of (3.13)–(3.14). If we assume that there exists two solutions $c^\sharp$ and $c^\flat$, the function $c = c^\sharp - c^\flat$ is such that

$$\begin{aligned}\partial_t(\zeta(\tilde{P})c) - \zeta'(\tilde{P})c\partial_t\tilde{P} + \zeta(P)(F^{-1})'(P)(b(c^\sharp) - b(c^\flat))\partial_t\tilde{P} - \nabla\tilde{P}\cdot\nabla c \\ - \mathrm{div}(\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c) = -q_i c,\end{aligned} \tag{3.21}$$

$$\begin{gathered}\zeta(P)\mathcal{D}(\nabla\tilde{P})\nabla c\cdot\nu_1 = 0 \quad \text{on } \Gamma_1\times(0,T), \quad c_{|_{\Gamma_2}} = 0 \quad \text{in } (0,T), \\ c(x,0) = 0 \quad \text{in } \Omega.\end{gathered} \tag{3.22}$$

The function b being Lipschitz, we prove that $c = 0$ by multiplying (3.21) by $\zeta(\tilde{P})c$, by integrating by parts over Ω and by using the Gronwall lemma as in the proof of the maximum principle for c^η. □

Lemma 3.3. (i) *The sequence* (c^η) *is sequentially compact in* $L^2(\Omega_T)$.
(ii) *The solution* P *of problem* (3.6)–(3.7) *lies in a compact subset of* K_p.

Proof. Let V be defined by $V = \{f \in W^{1,4}(\Omega);\ f_{|_{\Gamma_2}} = 0\}$. We multiply Eq. (3.13) by $\psi \in L^2(0,T;V)$ and integrate over Ω_T. Some computations yield to

$$\left|\langle\partial_t(\zeta(\tilde{P})c^\eta),\psi\rangle_{L^2(0,T;V'),L^2(0,T;V)}\right| \le C\|\psi\|_{L^2(0,T;V)}.$$

So the sequence $(\partial_t(\zeta(\tilde{P})c^\eta))$ is uniformly bounded in $L^2(0,T;V')$. We note that $H^1(\Omega) \subset L^2(\Omega) = (L^2(\Omega))' \subset V'$, the first embedding being compact. We conclude with the estimates of Lemmas 3.1 and 3.2 and an argument of Aubin's type (Ref. 29, Corollary 4) that the sequence $(\zeta(\tilde{P})c^\eta)$ is sequentially compact in $\mathcal{C}(0,T;L^2(\Omega))$.

On the other hand, (P) is uniformly bounded in $H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$. Function (P) and then $(\tilde{P})$ thus lie in a compact subset of K_p. Since furthermore $\zeta(\tilde{P}) \geq \theta_- > 0$, we conclude that (c^η) is sequentially compact in $L^2(\Omega_T)$. The proof is done. $\square$

Thanks to the existence and uniqueness results of Lemmas 3.1 and 3.2, we define the mapping $\mathcal{T} : K_c \times K_p \to K_c \times K_p$ by

$$\mathcal{T}(\bar{c}, \bar{P}) = (c^\eta, P).$$

Lemma 3.4. *The mapping $\mathcal{T}$ admits a fixed point denoted (c^η, P^η).*

Proof. We deduce from Lemma 3.3 that the image $\mathcal{T}(K_c \times K_p)$ of the closed convex subset $K_c \times K_p$ is compact in $K_c \times K_p$. It remains to state the continuity of the mapping $\mathcal{T}$. For this purpose, let $(\bar{c}_m, \bar{P}_m)$ be a sequence of $K_c \times K_p$ converging strongly in $(L^2(\Omega_T))^2$ to $(\bar{c}, \bar{P})$. We define P_m solution of

$$\begin{gathered}\sigma(\bar{c}_m, \bar{P}_m)\partial_t P_m - \Delta P_m = q_i - q_s \quad \text{in } \Omega_T,\\ \nabla P_m \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad P_{m|_{\Gamma_2}} = F(P_2) \quad \text{in } (0,T),\\ P_m(x,0) = F(p_0(x)) \quad \text{in } \Omega.\end{gathered}$$

In view of Lemmas 3.1 and 3.3, there is a subsequence, not relabeled for convenience, and a function $P \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that

$$P_m \rightharpoonup P \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \quad \text{and a.e. in } \Omega_T,$$

and P is a solution of

$$\begin{gathered}\sigma(\bar{c}, \bar{P})\partial_t P - \Delta P = q_i - q_s \quad \text{in } \Omega_T,\\ \nabla P \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad P_{|_{\Gamma_2}} = F(P_2) \quad \text{in } (0,T), \quad P(x,0) = F(p_0(x)) \quad \text{in } \Omega.\end{gathered}$$

Due to the uniqueness of the solution of this problem, the whole sequence (P_m) converges to P. We use a compensated compactness argument to get a strong convergence result for the pressure gradient, in view to pass to the limit in the nonlinearities of the concentration equation. Let $A_m = (\partial_t P_m, \nabla P_m)$ and $B_m = (0, \nabla P_m)$. These vectors are both uniformly bounded in $(L^2(\Omega_T))^4$. We also have uniform bounds in $L^2(\Omega_T)$ and then compactness results in $(H^1(\Omega_T))'$ for curl $A_m = 0$ and div $B_m = \sigma(\bar{c}_m, \bar{P}_m)\partial_t P_m - q_i + q_s$. By the div$-$curl lemma,[23,30] we thus assert that

$$A_m B_m \rightharpoonup AB \quad \text{in } \mathcal{D}'(\Omega_T),$$

where $A_m \rightharpoonup A = (\partial_t P, \nabla P)$ and $B_m \rightharpoonup B = (0, \nabla P)$ weakly in $L^2(\Omega_T)$. We conclude that

$$\nabla P_m \to \nabla P \quad \text{in } (L^2(\Omega_T))^3. \tag{3.23}$$

It also follows that $\nabla\tilde{P}_m$ converges to ∇P in $(L^2(\Omega_T))^3$. Let then c_m^η be the solution in Ω_T of

$$\begin{gathered}\partial_t(\zeta(\tilde{P}_m)c_m^\eta) - \zeta'(\tilde{P}_m)c_m^\eta\partial_t\tilde{P}_m + \zeta(P_m)(F^{-1})'(P_m)b(c_m^\eta)\partial_t\tilde{P}_m - \nabla\tilde{P}_m\cdot\nabla c_m^\eta \\ - \operatorname{div}(\zeta(P_m)\mathcal{D}(\nabla\tilde{P}_m)\nabla c_m^\eta) = q_i(1-c_m^\eta), \\ \zeta(P_m)\mathcal{D}(\nabla\tilde{P}_m)\nabla c_m^\eta\cdot\nu_1 = 0 \quad \text{on } \Gamma_1\times(0,T), \quad c_m^\eta|_{\Gamma_2} = C_2 \quad \text{in } (0,T), \\ c_m^\eta(x,0) = c_0(x) \quad \text{in } \Omega.\end{gathered}$$

In view of Lemmas 3.2 and 3.3, we can extract a subsequence, not relabeled for convenience, (c_m^η) converging strongly in $L^2(\Omega_T)$ and weakly in $L^2(0,T;H^1(\Omega))$ to a function c^η. Using in particular the strong convergence (3.23), we check that c^η is solution of problem (3.13)–(3.14). Furthermore, due to the uniqueness of the solution of (3.13)–(3.14), we ensure that the whole sequence (c_m^η) converges to c^η as $m\to+\infty$. The mapping $\mathcal{T}$ is continuous, and this completes the proof. □

We collect the results obtained in the previous lines. We can associate with any real number $\eta>0$ the fixed point $(c^\eta,P^\eta)\in K_c\times K_p$ of the mapping $\mathcal{T}$. It is a solution of the following system in Ω_T:

$$\sigma(c^\eta,P^\eta)\partial_t P^\eta - \Delta P^\eta = q_i - q_s \quad \text{in } \Omega_T, \tag{3.24}$$

$$\begin{gathered}\nabla P^\eta\cdot\nu_1 = 0 \quad \text{on } \Gamma_1\times(0,T), \quad P^\eta_{|\Gamma_2} = F(P_2) \quad \text{in } (0,T), \\ P^\eta(x,0) = F(p_0(x)) \quad \text{in } \Omega,\end{gathered} \tag{3.25}$$

$$\begin{gathered}\partial_t(\zeta(\tilde{P}^\eta)c^\eta) - \zeta'(\tilde{P}^\eta)c^\eta\partial_t\tilde{P}^\eta + \zeta(P^\eta)(F^{-1})'(P^\eta)b(c^\eta)\partial_t\tilde{P}^\eta \\ - \nabla\tilde{P}^\eta\cdot\nabla c^\eta - \operatorname{div}(\zeta(P^\eta)\mathcal{D}(\nabla\tilde{P}^\eta)\nabla c^\eta) = q_i(1-c^\eta),\end{gathered} \tag{3.26}$$

$$\begin{gathered}\zeta(P^\eta)\mathcal{D}(\nabla\tilde{P}^\eta)\nabla c^\eta\cdot\nu_1 = 0 \quad \text{on } \Gamma_1\times(0,T), \quad c^\eta_{|\Gamma_2} = C_2 \quad \text{in } (0,T), \\ c^\eta(x,0) = c_0(x) \quad \text{in } \Omega.\end{gathered} \tag{3.27}$$

We recall that $\tilde{P}^\eta = \psi_\eta * P^\eta$. We can get similar uniform estimates for (c^η,P^η) than the ones derived in Lemmas 3.1 and 3.2. In particular, we recall that by construction $0\le c^\eta(x,t)\le 1$ almost everywhere in Ω_T. The estimates of Lemma 3.2 are thus straightforward. We thus assert the existence of limit functions $P\in L^\infty(0,T;L^2(\Omega))\cap H^1(0,T;L^2(\Omega))$ and $c\in L^\infty(\Omega_T)\cap L^2(0,T;H^1(\Omega))$ such that (for extracted subsequences)

$$\begin{aligned}&P^\eta,\tilde{P}^\eta \rightharpoonup P \quad \text{weakly in } L^2(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega)) \quad \text{and} \quad \text{a.e. in } \Omega_T, \\ &\nabla P^\eta,\nabla\tilde{P}^\eta \to \nabla P \quad \text{in } (L^2(\Omega_T))^3 \quad \text{and a.e. in } \Omega_T, \\ &c^\eta \rightharpoonup c \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \text{ strongly in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T.\end{aligned}$$

The strong convergence of the pressure gradient ∇P^η is proved in the same way as the convergence (3.23) of ∇P_m in Lemma 3.4. Letting $\eta\to 0$ in (3.21)–(3.25), we state the existence of a weak solution (P,c) of problem (3.2)–(3.5). We end the proof by considering the inverse Kirchhoff's transform to turn back to problem (2.1)–(2.2), (2.6)–(2.7). The proof of Theorem 2.1 is achieved.

4. Proof of Theorem 2.2

We now aim to prove the existence result for the fully degenerate problem of Theorem 2.2. From Theorem 2.1, we can assert that there exists a weak solution $(p^\varepsilon, c^\varepsilon)$ of the following parabolic regularization of the problem, for any $\varepsilon > 0$:

$$\partial_t\theta^\varepsilon(p^\varepsilon) + \theta^\varepsilon(p^\varepsilon)a(c^\varepsilon)\partial_t p^\varepsilon + \operatorname{div}(\underline{u}^\varepsilon) = q_i - q_s, \quad \underline{u}^\varepsilon = -\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))\nabla p^\varepsilon, \tag{4.1}$$

$$\begin{aligned}\theta^\varepsilon(p^\varepsilon)\partial_t c^\varepsilon + \theta^\varepsilon(p^\varepsilon)b(c^\varepsilon)\partial_t p^\varepsilon + \underline{u}^\varepsilon \cdot \nabla c^\varepsilon - \varepsilon^{\bar{c}}\operatorname{div}(\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon)\\ = q_i(1 - c^\varepsilon),\end{aligned} \tag{4.2}$$

$$\underline{u}^\varepsilon \cdot \nu_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad p^\varepsilon_{|\Gamma_2} = P_2 \quad \text{in } (0,T), \quad p^\varepsilon(x,0) = p_0(x) \quad \text{in } \Omega, \tag{4.3}$$

$$\varepsilon^{\bar{c}}\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon \cdot \nu = 0 \quad \text{on } \Gamma \times (0,T), \quad c^\varepsilon(x,0) = c_0(x) \quad \text{in } \Omega, \tag{4.4}$$

where

$$\theta^\varepsilon = \theta + \varepsilon^\lambda, \quad \kappa^\varepsilon = \kappa + \varepsilon^\lambda. \tag{4.5}$$

The parameters $\bar{c}$ and λ are chosen such that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\theta(P_d+\varepsilon)}{\varepsilon^\lambda} = 0, \quad \lim_{\varepsilon\to 0}\frac{(\kappa\circ\theta)(P_d+\varepsilon)}{\varepsilon^\lambda} = 0,\\ 0 < \lambda < \bar{c}.\end{aligned} \tag{4.6}$$

Assumption (4.6) is a precision for (2.16).

We prove in this section that an extracted subsequence of solution of problem (4.1)–(4.4) weakly converges in some sense to a solution of problem (2.11)–(2.14) under the assumptions of Theorem 2.2.

We begin with some uniform estimates. We choose $\varepsilon < \varepsilon_0$ so that, due to (2.15), κ is an increasing function in $(0, \varepsilon)$. Then, for the pressure, we derived the following uniform estimates in Sec. 3 (see Lemma 3.1):

$$\|(\sqrt{(\theta(p^\varepsilon) + \theta'(p^\varepsilon) + \varepsilon^\lambda)(\kappa(\theta^\varepsilon(p^\varepsilon)) + \varepsilon^\lambda)})\partial_t p^\varepsilon\|_{L^2(\Omega_T)} \leq C, \tag{4.7}$$

$$\|(\kappa(\theta(p^\varepsilon)) + \varepsilon^\lambda)\nabla p^\varepsilon\|_{(L^\infty(0,T;L^2(\Omega)))^3} \leq C. \tag{4.8}$$

By construction, function c^ε is a physically admissible concentration, that is

$$0 \leq c^\varepsilon(x,t) \leq 1 \quad \text{a.e. in } \Omega_T. \tag{4.9}$$

We define the limit concentration $c \in L^\infty(\Omega_T)$ such that $0 \leq c(x,t) \leq 1$ a.e. in Ω_T by

$$c^\varepsilon \rightharpoonup c \quad \text{weakly} \star \text{ in } L^\infty(\Omega_T). \tag{4.10}$$

We then multiply Eq. (4.2) by $\varepsilon^\lambda c^\varepsilon$ and integrate by parts. It is similar to the computation done in (3.19). Using estimates (4.7)–(4.8), one checks that

$$\|\sqrt{\varepsilon^{\lambda+c}\theta^\varepsilon(p^\varepsilon)(D_m + \alpha_T|\underline{u}^\varepsilon|)}\nabla c^\varepsilon\|_{(L^2(\Omega_T))^3} \leq C. \tag{4.11}$$

Assumption (4.6) ensures that $(\lambda + \bar{c})/2 < \bar{c}$. Thus estimate (4.11) proves that the divergence part $\operatorname{div}(\varepsilon^{\bar{c}}\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon)$ of the concentration equation (4.2) disappears when passing to the limit $\varepsilon \to 0$.

Functions θ, θ' and $\kappa \circ \theta$ are null in $(-\infty, P_d)$ and we cannot ensure that $p^\varepsilon(x,t) > P_d$ a.e. in Ω_T. Thus estimates (4.7)–(4.8) are completely useless in a potentially important part of the domain. To exploit (4.7)–(4.8) we introduce a convenient truncature function. First let $\mathcal{G}$ be a primitive of the function $\sqrt{\theta}(\kappa \circ \theta)$:

$$\mathcal{G}(p^\varepsilon) = \int^{p^\varepsilon} \sqrt{\theta(s)}(\kappa \circ \theta)(s)\, ds. \tag{4.12}$$

In view of (4.7)–(4.8), function $\mathcal{G}(p^\varepsilon)$ is uniformly bounded in $H^1(\Omega_T)$. We thus define the limit function $\bar{\mathcal{G}}$ such that

$$\mathcal{G}(p^\varepsilon) \to \bar{\mathcal{G}} \quad \text{in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T.$$

We then define the truncature function T_{P_d} by

$$T_{P_d}(x) = \begin{cases} x & \text{if } x \geq P_d, \\ P_d & \text{if } x < P_d. \end{cases}$$

Let

$$t^\varepsilon = T_{P_d}(p^\varepsilon). \tag{4.13}$$

By definition of θ, $\mathcal{G}$ and T_{P_d}, we note that

$$\mathcal{G}(p^\varepsilon) = \mathcal{G}(T_{P_d}(p^\varepsilon)) = \mathcal{G}(t^\varepsilon)$$

and the latter convergence result reads

$$\mathcal{G}(t^\varepsilon) \to \bar{\mathcal{G}} \quad \text{in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T. \tag{4.14}$$

Function $\mathcal{G}$ is of course not bijective in $(-\infty, P_d)$. We thus consider a bijective continuous extension $\tilde{\mathcal{G}}$ of $\mathcal{G}_{|_{(P_d,\infty)}}$ to $\mathbb{R}$. Setting

$$\mathtt{t} = \tilde{\mathcal{G}}^{-1}(\bar{\mathcal{G}}), \tag{4.15}$$

we have $\tilde{\mathcal{G}}(t^\varepsilon) = \mathcal{G}(t^\varepsilon) \to \bar{\mathcal{G}} = \tilde{\mathcal{G}}(\mathtt{t})$ in $L^2(\Omega_T)$ and a.e. in Ω_T. Function $\tilde{\mathcal{G}}$ being continuous and bijective, we conclude that

$$t^\varepsilon = T_{P_d}(p^\varepsilon) \to \mathtt{t} \quad \text{in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T. \tag{4.16}$$

Since $\theta^\varepsilon(p^\varepsilon) = \theta^\varepsilon(T_{P_d}(p^\varepsilon)) = \theta^\varepsilon(t^\varepsilon) = \theta(t^\varepsilon) + \varepsilon^\lambda$ and $\theta'(p^\varepsilon) = \theta'(t^\varepsilon)$, with $\theta \in \mathcal{C}^1(\mathbb{R})$, we also have

$$\theta^\varepsilon(p^\varepsilon) \to \theta(\mathtt{t}) \quad \text{in } L^p(\Omega_T)\ \forall\, p < \infty, \quad \text{and a.e. in } \Omega_T, \tag{4.17}$$

$$\theta'(p^\varepsilon) \to \theta'(\mathtt{t}) \quad \text{in } L^p(\Omega_T)\ \forall\, p < \infty, \quad \text{and a.e. in } \Omega_T, \tag{4.18}$$

$$\theta'(p^\varepsilon)\partial_t p^\varepsilon = \partial_t(\theta(p^\varepsilon)) \rightharpoonup \theta'(\mathtt{t})\partial_t \mathtt{t} \quad \text{weakly in } L^2(0,T; H^{-1}(\Omega)). \tag{4.19}$$

For the limit behavior of the Darcy velocity, we write $(\kappa \circ \theta)(p^\varepsilon)\nabla p^\varepsilon = \nabla(\mathcal{F}(t^\varepsilon)) \rightharpoonup \nabla(\mathcal{F}(\mathtt{t})) = \mathcal{F}'(\mathtt{t})\nabla \mathtt{t}$, the continuity of the function $\mathcal{F}$ deriving from the one of θ and κ. It means

$$\underline{u}^\varepsilon = -(\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon)\nabla p^\varepsilon \rightharpoonup \underline{u} = -(\kappa \circ \theta)(\mathtt{t})\nabla \mathtt{t} \quad \text{weakly in } (L^2(\Omega_T))^3. \tag{4.20}$$

Now, we state some compactness result for a weighted Darcy velocity. Let $Q=\max(q,1)$, q being defined by assumption (2.17). Let $\mathcal{H}^\varepsilon=\int^{p^\varepsilon}((\kappa^\varepsilon\circ\theta^\varepsilon)(s))^Q ds$. We have

$$\operatorname{curl}(\nabla\mathcal{H}^\varepsilon)=0.$$

Using Eq. (4.1) and estimate (4.7), the first line of assumption (2.17) and estimate (4.8), we easily check that

$$\begin{aligned}\operatorname{div}(\nabla\mathcal{H}^\varepsilon)=&-((\kappa^\varepsilon\circ\theta^\varepsilon)(p^\varepsilon))^{Q-1}\operatorname{div}(\underline{u}^\varepsilon)\\&+(Q-1)((\kappa^\varepsilon\circ\theta^\varepsilon)(p^\varepsilon))^{Q-3}\theta'(p^\varepsilon)(\kappa'\circ\theta^\varepsilon)(p^\varepsilon)\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))\nabla p^\varepsilon\\&\cdot\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))\nabla p^\varepsilon\end{aligned}$$

is uniformly bounded in $L^2(0,T;L^1(\Omega))$. Function $\nabla\mathcal{H}^\varepsilon=((\kappa^\varepsilon\circ\theta^\varepsilon)(p^\varepsilon))^Q\nabla p^\varepsilon$ is thus uniformly bounded in $L^2(0,T;W^{1,1}(\Omega))$. Differentiating in space equation (4.1), we obtain

$$\begin{aligned}\partial_t(((\kappa^\varepsilon\circ\theta^\varepsilon)(p^\varepsilon))^Q\nabla p^\varepsilon)&+\nabla\left(\frac{((\kappa^\varepsilon\circ\theta^\varepsilon)(p^\varepsilon))^Q}{\theta'+a(c^\varepsilon)\theta^\varepsilon}\operatorname{div}\underline{u}^\varepsilon\right)\\&=\nabla\left(\frac{((\kappa^\varepsilon\circ\theta^\varepsilon)(p^\varepsilon))^Q}{\theta'+a(c^\varepsilon)\theta^\varepsilon}(q_i-q_s)\right).\end{aligned}$$

The boundary conditions completing this equation are easily derived from (4.3). Using once again assumption (2.17), we check that the latter relation implies that $\partial_t\nabla\mathcal{H}^\varepsilon$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$. By a compactness argument of Aubin's type, we conclude that $\nabla\mathcal{H}^\varepsilon$ is sequentially compact in $(L^{4/3}(\Omega_T))^3$ and then in $(L^2(\Omega_T))^3$. We write

$$((\kappa^\varepsilon\circ\theta^\varepsilon)(p^\varepsilon))^Q\nabla p^\varepsilon\to((\kappa\circ\theta)(\mathtt{t}))^Q\nabla\mathtt{t}\quad\text{in }(L^2(\Omega_T))^3.\tag{4.21}$$

We now have to differentiate the proof between two cases: the case where $z_1\neq z_2$ and the one where $z_1=z_2$.

Assume $z_1\neq z_2$. We can rewrite Eq. (4.2) as

$$\begin{aligned}\partial_t(\theta^\varepsilon(p^\varepsilon)c^\varepsilon)+z_1\theta^\varepsilon(p^\varepsilon)c^\varepsilon\partial_t p^\varepsilon+\operatorname{div}(\underline{u}^\varepsilon c^\varepsilon)\\-\varepsilon^{\bar{c}}\operatorname{div}(\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon)=q_i-c^\varepsilon q_s,\end{aligned}\tag{4.22}$$

and if $z_1\neq z_2$, we rewrite it as

$$\begin{aligned}\partial_t(\theta^\varepsilon(p^\varepsilon)c^\varepsilon)-\frac{1}{z_1-z_2}(z_1z_2\theta^\varepsilon(p^\varepsilon)+z_1\theta^\varepsilon(p^\varepsilon)')\partial_t p^\varepsilon-\frac{z_1}{z_1-z_2}\operatorname{div}(\underline{u}^\varepsilon)\\+\operatorname{div}(\underline{u}^\varepsilon c^\varepsilon)-\varepsilon^{\bar{c}}\operatorname{div}(\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon)=q_i-c^\varepsilon q_s-z_1\frac{q_i-q_s}{z_1-z_2}.\end{aligned}\tag{4.23}$$

We apply the div$-$curl lemma to the vectors

$$\begin{aligned}A^\varepsilon=\bigg(&\theta^\varepsilon(p^\varepsilon)c^\varepsilon-\frac{1}{z_1-z_2}(z_1z_2\mathcal{O}^\varepsilon(p^\varepsilon)+z_1\theta^\varepsilon(p^\varepsilon)),\\&-\frac{z_1}{z_1-z_2}\underline{u}^\varepsilon+\underline{u}^\varepsilon c^\varepsilon-\varepsilon^{\bar{c}}\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon\bigg),\end{aligned}$$

where $\mathcal{O}^\varepsilon$ is a primitive of function θ^ε, and

$$B^\varepsilon = (((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \partial_t p^\varepsilon, ((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \nabla p^\varepsilon).$$

These vectors are uniformly bounded in $(L^2(\Omega_T))^4$. Moreover, $\operatorname{div} A^\varepsilon = q_i - c^\varepsilon q_s - z_1(q_i - q_s)/(z_1 - z_2)$ (because of (4.23)) and $\operatorname{curl}(B^\varepsilon) = 0$ are uniformly bounded in $L^2(\Omega_T)$ and thus compact in $H^{-1}(\Omega_T)$. Thus we can pass to the limit in the product $\langle A^\varepsilon, B^\varepsilon \rangle$. We get

$$\begin{aligned}
&\theta^\varepsilon(p^\varepsilon)c^\varepsilon((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \partial_t p^\varepsilon - \frac{1}{z_1 - z_2}(z_1 z_2 \mathcal{O}^\varepsilon(p^\varepsilon) + z_1 \theta^\varepsilon(p^\varepsilon))((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \partial_t p^\varepsilon \\
&\quad + \left(-\frac{z_1}{z_1 - z_2}\underline{u}^\varepsilon + \underline{u}^\varepsilon c^\varepsilon - \varepsilon^{\bar{c}}\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon\right)((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \nabla p^\varepsilon \\
&\quad \rightharpoonup \theta(\mathtt{t})c((\kappa \circ \theta)(\mathtt{t}))^Q \partial_t \mathtt{t} - \frac{1}{z_1 - z_2}(z_1 z_2 \mathcal{O}(\mathtt{t}) + z_1 \theta(\mathtt{t}))((\kappa \circ \theta)(\mathtt{t}))^Q \partial_t \mathtt{t} \\
&\quad + \left(-\frac{z_1}{z_1 - z_2}\underline{u} + \lim_{\varepsilon \to 0}(\underline{u}^\varepsilon c^\varepsilon)\right)((\kappa \circ \theta)(\mathtt{t}))^Q \nabla \mathtt{t} \qquad (4.24)
\end{aligned}$$

in $\mathcal{D}'(\Omega_T)$. We have denoted by $\lim_{\varepsilon \to 0}(\underline{u}^\varepsilon c^\varepsilon)$ an adherence value of $(\underline{u}^\varepsilon c^\varepsilon)$ in weak-$L^2(\Omega_T)$. Using (4.16), we know that $\mathcal{O}^\varepsilon$ (resp. θ^ε) converges a.e. to $\mathcal{O}$ (resp. θ) in Ω_T. Thus, we have

$$\begin{aligned}
&-\frac{1}{z_1 - z_2}(z_1 z_2 \mathcal{O}^\varepsilon(p^\varepsilon) + z_1 \theta^\varepsilon(p^\varepsilon))((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \partial_t p^\varepsilon \\
&\qquad \rightharpoonup -\frac{1}{z_1 - z_2}(z_1 z_2 \mathcal{O}(\mathtt{t}) + z_1 \theta(\mathtt{t}))((\kappa \circ \theta)(\mathtt{t}))^Q \partial_t \mathtt{t}
\end{aligned}$$

in $L^2(\Omega_T)$. Using (4.21), we know that

$$\begin{aligned}
&\left(-\frac{z_1}{z_1 - z_2}\underline{u}^\varepsilon + \underline{u}^\varepsilon c^\varepsilon - \varepsilon^{\bar{c}}\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon\right)((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \nabla p^\varepsilon \\
&\qquad \rightharpoonup \left(-\frac{z_1}{z_1 - z_2}\underline{u} + \lim_{\varepsilon \to 0}(\underline{u}^\varepsilon c^\varepsilon)\right)((\kappa \circ \theta)(\mathtt{t}))^Q \nabla \mathtt{t}
\end{aligned}$$

in $L^1(\Omega_T)$. We thus infer from (4.24) that

$$\theta^\varepsilon(p^\varepsilon)c^\varepsilon((\kappa^\varepsilon \circ \theta^\varepsilon)(p^\varepsilon))^Q \partial_t p^\varepsilon \rightharpoonup \theta(\mathtt{t})c((\kappa \circ \theta)(\mathtt{t}))^Q \partial_t \mathtt{t} \quad \text{weakly in } L^2(\Omega_T). \qquad (4.25)$$

Let ξ be the weak limit in $L^2(0, T; H^{-1}(\Omega)$ of $\theta^\varepsilon(p^\varepsilon)c^\varepsilon \partial_t p^\varepsilon$. We aim to prove that $\xi = \theta(\mathtt{t})c\partial_t \mathtt{t}$. With (4.25), we assert that

$$((\kappa \circ \theta)(\mathtt{t}))^Q \, \xi = ((\kappa \circ \theta)(\mathtt{t}))^Q \, \theta(\mathtt{t})c\partial_t \mathtt{t}. \qquad (4.26)$$

It already means that $\xi = \theta(\mathtt{t})c\partial_t \mathtt{t}$ if $\mathtt{t} > P_d$. We also know that

$$|\chi_{\{p^\varepsilon \le P_d + \varepsilon\}}\theta^\varepsilon(p^\varepsilon)c^\varepsilon \partial_t p^\varepsilon| \le |\theta(P_d + \varepsilon)c^\varepsilon \partial_t p^\varepsilon| = \varepsilon^\lambda |c^\varepsilon \partial_t p^\varepsilon| \frac{\theta(P_d + \varepsilon)}{\varepsilon^\lambda},$$

where $\varepsilon^\lambda |c^\varepsilon \partial_t p^\varepsilon|$ is uniformly bounded in $L^2(\Omega_T)$ (see (4.7)). Thus by assumption (4.6)

$$\chi_{\{p^\varepsilon \le P_d + \varepsilon\}}\theta^\varepsilon(p^\varepsilon)c^\varepsilon \partial_t p^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega_T). \qquad (4.27)$$

Using (4.26)–(4.27), we conclude that $\xi = \theta(\mathtt{t})c\partial_t\mathtt{t}$, that is

$$\theta^\varepsilon(p^\varepsilon)c^\varepsilon\partial_t p^\varepsilon \rightharpoonup \theta(\mathtt{t})c\partial_t\mathtt{t} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{4.28}$$

Similar computations let us prove that

$$\underline{u}^\varepsilon c^\varepsilon \rightharpoonup \underline{u}c \quad \text{weakly in } (L^2(\Omega_T))^3. \tag{4.29}$$

We have sufficient results to pass to the limit $\varepsilon \to 0$ in Eqs. (4.1) and (4.23).

Assume $z_1 = z_2$. Here the pressure equation is somewhat less coupled with the concentration's one. Indeed, it writes

$$(\theta'(p^\varepsilon)' + z_1\theta^\varepsilon(p^\varepsilon))\partial_t p^\varepsilon - \operatorname{div}(\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))\nabla p^\varepsilon) = q_i - q_s.$$

With (4.16)–(4.20), we can pass to the limit in the latter equation. Then, we write the weak formulation of the latter equation for the test function $\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))\partial_t p^\varepsilon$, and the weak formulation of the corresponding limit equation for the test function $-\kappa(\theta(\mathtt{t}))\partial_t\mathtt{t}$. Summing up these relations, we conclude that

$$\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))^{1/2}\theta'(p^\varepsilon)^{1/2}\partial_t p^\varepsilon \to \kappa(\theta(\mathtt{t}))^{1/2}\theta'(\mathtt{t})^{1/2}\partial_t\mathtt{t} \quad \text{in } L^2(\Omega_T), \tag{4.30}$$

$$\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))^{1/2}\theta^\varepsilon(p^\varepsilon)^{1/2}\partial_t p^\varepsilon \to \kappa(\theta(\mathtt{t}))^{1/2}\theta(\mathtt{t})^{1/2}\partial_t\mathtt{t} \quad \text{in } L^2(\Omega_T), \tag{4.31}$$

$$\underline{u}^\varepsilon = -\kappa^\varepsilon(\theta^\varepsilon(p^\varepsilon))\nabla p^\varepsilon \to \underline{u} = -\kappa(\theta(\mathtt{t}))\nabla\mathtt{t} \quad \text{in } (L^2(\Omega_T))^3. \tag{4.32}$$

We then aim to pass to the limit in the formulation (4.22) of the concentration equation recalled below:

$$\partial_t(\theta^\varepsilon(p^\varepsilon)c^\varepsilon) + z_1\theta^\varepsilon(p^\varepsilon)c^\varepsilon\partial_t p^\varepsilon + \operatorname{div}(\underline{u}^\varepsilon c^\varepsilon) - \varepsilon^{\bar{c}}\operatorname{div}(\theta^\varepsilon(p^\varepsilon)\mathcal{D}(\underline{u}^\varepsilon)\nabla c^\varepsilon) = q_i.$$

The difficulty is once again to pass to the limit in the term $\theta^\varepsilon(p^\varepsilon)c^\varepsilon\partial_t p^\varepsilon$. Let $\xi^\flat$ be the weak limit in $L^2(0,T;H^{-1}(\Omega))$ of $\theta^\varepsilon(p^\varepsilon)c^\varepsilon\partial_t p^\varepsilon$. By (4.17), (4.31) we claim that

$$\sqrt{(\kappa\circ\theta)(\mathtt{t})}\xi^\flat = \sqrt{(\kappa\circ\theta)(\mathtt{t})}\theta(\mathtt{t})c\partial_t\mathtt{t}.$$

The latter relation is similar to (4.26). Moreover, (4.27) remains true. We thus recover (4.28).

Finally, whenever $z_1 \neq z_2$ or $z_1 = z_2$, we have proved that the pair $(\mathtt{t}, c)$ defined by

$$\begin{aligned} c^\varepsilon &\rightharpoonup c \quad \text{weakly} \star \quad \text{in } L^\infty(\Omega_T),\\ T_{P_d}(p^\varepsilon) &\to \mathtt{t} \qquad\qquad\qquad \text{in } L^2(\Omega_T), \end{aligned}$$

satisfies

$$\begin{gathered} \partial_t\theta(\mathtt{t}) + \theta(\mathtt{t})a(c)\partial_t\mathtt{t} + \operatorname{div}(\underline{u}) = q_i - q_s, \quad \underline{u} = -\kappa(\theta(\mathtt{t}))\nabla\mathtt{t},\\ \theta(\mathtt{t})\partial_t c + \theta(\mathtt{t})b(c)\partial_t\mathtt{t} + \underline{u}\cdot\nabla c = q_i(1-c),\\ \underline{u}\cdot\nu_1 = 0 \quad \text{on } \Gamma_1\times(0,T), \quad \mathtt{t}_{|_{\Gamma_2}} = T_{P_d}(P_2) \quad \text{in } (0,T),\\ \mathtt{t}(x,0) = T_{P_d}(p_0)(x) \quad \text{in } \Omega,\\ c(x,0) = c_0(x) \quad \text{in } \Omega. \end{gathered}$$

Theorem 2.2 is proved.

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References

1. H. W. Alt and E. Di Benedetto, Nonsteady flow of water and oil through inhomogeneous porous media, *Ann. Scuola Norm. Sup. Pisa* **12** (1985) 335–392.
2. H. W. Alt and S. Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.* **183** (1983) 311–341.
3. Y. Amirat, K. Hamdache and A. Ziani, Mathematical analysis for compressible miscible displacement models in porous media, *Math. Models Meth. Appl. Sci.* **6** (1996) 729–747.
4. L. F. Athy, Density, porosity and compaction of sedimentary rocks, *Bull. AAPG*, **14** (1930) 1–24.
5. J. Bear, *Dynamics of Fluids in Porous Media* (Elsevier, 1972).
6. R. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena* (Wiley, 1960).
7. K. Bitzer, Subseabed disposal of radioactive waste: Effects of consolidational fluid flow, *Environ. Geol.* **27** (1996) 300–308.
8. T. D. Bredhoeft and B. B. Hanshaw, On the maintenance of anomalous pressures: I. Thick sedimentary sequences, *Bull. Geol. Soc. Am.* **79** (1968) 1097–1106.
9. R. H. Brooks and A. T. Corey, Properties of porous media affecting fluid flow, *J. Irrigation Drainage Div. ASCE* **IR2** (1966) 61–88.
10. N. T. Burdine, Relative permeability calculations from pore size distribution data, *Trans. AIME* **198** (1953) 71–77.
11. G. Chavent and J. Jaffré, *Mathematical Methods and Finite Elements for Reservoir Simulation*, *Studies in Mathematics and its Applications*, Vol. 17 (North-Holland, 1986).
12. X. Chen, A. Friedman and T. Kimura, Nonstationary filtration in partially saturated porous medium, *Euro. J. Appl. Math.* **5** (1994) 405–429.
13. C. Choquet, Transport of heat and mass in a fluid with vanishing mobility, *Quart. Appl. Math.* **66** (2008) 771–779.
14. F. Z. Daïm, R. Eymard and D. Hilhorst, Existence of a solution for two-phase flow in porous media: The case that the porosity depends on the pressure, *J. Math. Anal. Appl.* **326** (2007) 332–351.
15. G. de Marsily, *Hydrogéologie Quantitative* (Masson, 1981).
16. J. Douglas, Jr. and J. E. Roberts, Numerical methods for a model for compressible miscible displacement in porous media, *Math. Comput.* **41** (1983) 441–459.
17. P. Fabrie and M. Langlais, Mathematical analysis of miscible displacement in porous medium, *SIAM J. Math. Anal.* **23** (1992) 1375–1392.
18. V. Giovangigli, *Multicomponent Flow Modeling*, MESST Series (Birkhäuser, 1999).
19. J. Hulshof and N. Wolanski, Monotone flows in N-dimensional partially saturated porous media: Lipschitz continuity of the interface, *Arch. Rat. Mech. Anal.* **102** (1988) 287–305.
20. M. Kaviany, *Principles of Heat Transfer in Porous Media* (Springer, 1999).
21. O. A. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type* (Amer. Math. Soc., 1968).
22. Y. Mualem, Hydraulic conductivity of unsaturated soils: Prediction and formulas. Methods of soil analysis, in *Physical and Mineralogical Methods*, 2nd edn. (Agronomy, 1986), pp. 799–823.
23. F. Murat, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa* **5** (1978) 489–507.
24. J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques* (Masson, 1967).
25. D. W. Peaceman, *Fundamentals of Numerical Reservoir Simulation* (Elsevier, 1977).

26. J. Schauder, Der Fixpunktsatz in Funktionalraümen, *Studia Math.* **2** (1930) 171–180.
27. A. E. Scheidegger, *The Physics of Flow through Porous Media* (Univ. Toronto Press, 1974).
28. R. E. Showalter and N. Su, Partially saturated flow in a poroelastic medium, *Disc. Cont. Dynam. Syst. Ser. B* (2001) 403–420.
29. J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Math. Pura Appl.* **146** (1987) 65–96.
30. L. Tartar, Compensated compactness and applications to P.D.E., in *Non Linear Analysis and Mechanics, Heriot-Watt Symposium*, ed. R. J. Knopps, Research Notes in Math., Vol. 4 (Pitman Press, 1979), pp. 136–212.
31. M. Th. van Genuchten, A closed form equation for predicting the hydraulic conductivity of unsaturated soils, *Soil Sci. Soc. Amer. J.* **44** (1980) 892–898.
32. M. Th. van Genuchten, F. J. Leij and S. R. Yates, The RETC code for quantifying the hydraulic functions of unsaturated soils, in *Proc. of the 51th Canadian Geotechnical Conference*, Edmonton, Alberta, Oct. 4–7, 1991.
33. H. M. Yin, A singular-degenerate free boundary problem arising from the moisture evaporation in a partially saturated porous medium, *Ann. Mat. Pura Appl.* **161** (1992) 379–397.