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UNIQUENESS FOR CROSS-DIFFUSION SYSTEMS ISSUING FROM SEAWATER INTRUSION PROBLEMS

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Abstract. We consider a model mixing sharp and diffuse interface approaches for seawater intrusion phenomenons in confined and unconfined aquifers. More precisely, a phase field model is introduced in the boundary conditions on the virtual sharp interfaces. We thus include in the model the existence of diffuse transition zones but we preserve the simplified structure allowing front tracking. The three-dimensional problem then reduces to a two-dimensional model involving a strongly coupled system of partial differential equations of parabolic and elliptic type describing the evolution of the depth of the interface between salt- and freshwater and the evolution of the freshwater hydraulic head. Assuming a low hydraulic conductivity inside the aquifer, we prove the uniqueness of a weak solution for the model completed with initial and boundary conditions. Thanks to a generalization of a Meyer's regularity result, we establish that the gradient of the solution belongs to the space L^r, $r > 2$. This additional regularity combined with the Gagliardo-Nirenberg inequality for $r = 4$ allows to handle the nonlinearity of the system in the proof of uniqueness.

1. Introduction

Seawater intrusion in coastal aquifers is a major problem for water supply. The study of efficient and accurate models to simulate the displacement of a saltwater front in unsaturated porous media is motivated by the need of efficient tools for the optimal exploitation of fresh groundwater.

Observations show that, near the shoreline, fresh and salty underground water tend to separate into two distinct layers. It was the motivation for the derivation of seawater intrusion models treating salt- and freshwater as immiscible fluids. Points where the salty phase disappears may be viewed as a sharp interface. Nevertheless the explicit tracking of the interfaces remains unworkable to implement without further assumptions. An additional assumption, the so-called Dupuit approximation, consists in considering that the hydraulic head is constant along each vertical direction. It allows to assume the existence of a smooth sharp interface. Classical sharp interface models are then obtained by vertical integration based on the assumption that no mass transfer occurs between the fresh and the salty area (see [4, 11] and even the Ghyben-Herzberg static approximation). This class of models allows direct tracking of the salt front. Nevertheless the conservative form of the

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equations is perturbed by the upscaling procedure. In particular the maximum principle does not apply. Of course fresh and salty water are two miscible fluids.

Following [6], we can mix the latter abrupt interface approach with a phase field approach (here an Allen–Cahn type model in fluid-fluid context see [1, 2, 5]) for re-including the existence of a diffuse interface between fresh and salt water where mass exchanges occur. We thus combine the advantage of respecting the physics of the problem and that of the computational efficiency.

From a theoretical point of view, an advantage resulting from the addition of the diffuse area compared to the sharp interface approximation is that the system now has a parabolic structure, so it is not necessary to introduce viscous terms in a preliminary fixed point for treating degeneracy as in the case of the sharp interface approach. Another important point is that we can demonstrate a more efficient and logical maximum principle from the point of view of physics, which is not possible in the case of classical sharp interface approximation. But the main point is that we can now show the uniqueness of the solution thanks to the parabolic structure of the system that yields more regularity for the solution.

This article is devoted to the study of the wellposedness of the sharp-diffuse interface seawater intrusion model. We focus on confined aquifers. As already mentioned, the problem consists in a coupled system of quasi-linear parabolic-elliptic equations. It belongs to the wide class of cross-diffusion systems for which the equations are coupled in the highest derivatives terms and there is no general theory for such a kind of problem. For dealing with the nonlinearity in the uniqueness proof, we first prove a L^r, $r > 2$, regularity result for the gradient of the unknowns. More precisely we generalize to the quasilinear case, the regularity result given by Meyers [10] in the elliptic case and extended to the parabolic case by Bensoussan, Lions and Papanicolaou, for any elliptic operator $A = -\sum_{i,j=1}^{n} \partial_j a_{ij}(x)\partial_i$ (see [3]). The results assume that the operator A satisfies an uniform ellipticity assumption and that its coefficients are L^∞ functions. The hypothesis on A ensure the existence of an exponent $r(A) > 2$ such that the gradient of the solution of the elliptic equation (resp. of the parabolic equation) belongs to the space L^r with respect to space (resp. L^r with respect to time and space). This additional regularity combined with the Gagliardo-Nirenberg inequality let us handle the nonlinearity of the system in the proof of uniqueness.

This article is organized as follows: First, in Section 2, we detail all the mathematical notations and we present some auxiliary results. In Section 3, we present a new proof of global in time existence for the problem. Sections 4 and 5 are devoted to the proofs of the regularity and uniqueness results.

2. Auxiliary results

We consider an open bounded domain Ω of $\mathbb{R}^2$ describing the projection of the aquifer on the horizontal plane. The boundary of Ω, assumed $\mathcal{C}^1$, is denoted by Γ. The time interval of interest is $(0, T)$, T being any nonnegative real number, and we set $\Omega_T = (0, T) \times \Omega$.

For the sake of brevity we shall write $H^1(\Omega) = W^{1,2}(\Omega)$ and

$$V = H_0^1(\Omega), \quad V' = H^{-1}(\Omega), \quad H = L^2(\Omega).$$

The embeddings $V \subset H = H' \subset V'$ are dense and compact. For any $T > 0$, let $W(0,T)$ denote the space

$$W(0,T) := \big\{\omega \in L^2(0,T;V),\ \partial_t\omega \in L^2(0,T;V')\big\}$$

endowed with the Hilbertian norm $\|\cdot\|_{W(0,T)} = \big(\|\cdot\|^2_{L^2(0,T;V)} + \|\partial_t\cdot\|^2_{L^2(0,T;V')}\big)^{1/2}$. The following embeddings are continuous [9, Prop. 2.1 and Thm 3.1, chapter 1]

$$W(0,T) \subset \mathcal{C}([0,T];[V,V']_{1/2}) = \mathcal{C}([0,T];H)$$

while the embedding

$$W(0,T) \subset L^2(0,T;H) \tag{2.1}$$

is compact (Aubin's Lemma, see [12]). The following result by Mignot (see [8]) is used in the sequel.

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function such that $\limsup_{|\lambda|\to+\infty} |f(\lambda)/\lambda| < +\infty$. Let $\omega \in L^2(0,T;H)$ be such that $\partial_t\omega \in L^2(0,T;V')$ and $f(\omega) \in L^2(0,T;V)$. Then*

$$\langle \partial_t\omega, f(\omega)\rangle_{V',V} = \frac{d}{dt}\int_\Omega \Big(\int_0^{\omega(\cdot,y)} f(r)\,dr\Big)\,dy \ \text{ in } \mathcal{D}'(0,T).$$

Hence for all $0 \le t_1 < t_2 \le T$,

$$\int_{t_1}^{t_2} \langle \partial_t\omega, f(\omega)\rangle_{V',V}\,dt = \int_\Omega \Big(\int_{\omega(t_1,y)}^{\omega(t_2,y)} f(r)\,dr\Big)\,dy.$$

We now present two preliminary lemma, which are consequences of the Meyers regularity results [10], first for an elliptic equation, then for a parabolic one. The adaptation of these results will be crucial for proving the $L^r(0,T;W^{1,r}(\Omega))$, $r > 2$, regularity of the solutions.

• **Elliptic case.** We recall the following result (see Lions and Magenes [9]):

$$\forall p : 1 < p < \infty, -\Delta \text{ is an isomorphism from } W_0^{1,p}(\Omega) \text{ to } W^{-1,p}(\Omega).$$

We set $G = (-\Delta)^{-1}$ and $g(p) = \|G\|_{\mathcal{L}(W^{-1,p}(\Omega);W_0^{1,p}(\Omega))}$. We notice that g(2)=1.

Lemma 2.2. *Let $A \in (L^\infty(\Omega))^n$ be a symmetric tensor such that there exists $\alpha > 0$ satisfying*

$$\sum_{i,j=1}^n A_{i,j}(x)\xi_i\xi_j \ge \alpha|\xi|^2, \quad \forall x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

We set $\beta = \max_{1\le i,j\le n} \|A_{i,j}\|_{L^\infty(\Omega)}$. There exist $r(\alpha,\beta) > 2$, such that, for any $f \in W^{-1,r}(\Omega)$ and for any $g_0 \in W^{1,r}(\Omega)$, the unique solution u of the problem

$$\begin{gathered}\nabla\cdot(A\nabla u) = f, \forall x \in \Omega\\ u \in H_0^1(\Omega) + g_0,\end{gathered}$$

belongs to $W^{1,r}(\Omega)$. In addition, the following estimate holds:

$$\|u\|_{W^{1,r}(\Omega)} \le C(\alpha,\beta,r)\|f - \nabla\cdot(A\nabla g_0)\|_{W^{-1,r}(\Omega)}, \tag{2.2}$$

where $C(\alpha,\beta,r)$ is a constant depending only on r and on constants α and β characterizing the operator A.

Remark 2.3. The proof given in [3] allows us to precise the constant $C(\alpha,\beta,r)$. Let c be a positive real number. We set

$$\mu = \frac{\alpha+c}{\beta+c}, \quad \nu = \frac{c}{\beta+c}, \tag{2.3}$$

where c is introduced in order to ensure $\nu < \mu$. Since $g(2) = 1$ and $0 < 1-\mu+\nu < 1$, by using the properties of the map g, we can find $r > 2$ such that

$$0 < k(r) := g(r)(1-\mu+\nu) < 1. \tag{2.4}$$

Then, the smaller $(1-\mu+\nu)$ is, the bigger r can be. The determination of r depends on the constants α and β characterizing the elliptic operator A.

Let us emphasize that Lemma 2.2 holds for all p such that $2 \le p \le r(\alpha,\beta)$ thanks to the classical interpolation inequalities.

The limit case corresponds to the settings where the operator A is proportional to the Laplacian: then $\mu = 1$, $\nu = 0$ and (2.4) is satisfied for all $r \ge 2$. Taking into account the previous estimates, we can give an upper bound of $C(\alpha,\beta,r)$ as follows

$$C(\alpha,\beta,r) \le (1-g(r)(1-\mu+\nu))^{-1}\frac{g(r)}{\beta+c} = \frac{g(r)}{(1-k(r))(\beta+c)}. \tag{2.5}$$

Parabolic case. Let us give now a lemma for the parabolic context. We define $X_p = L^p(0,T;W_0^{1,p}(\Omega))$, endowed with the norm

$$\|\nabla v\|_{(L^p(\Omega_T))^n} = \Big(\int_0^T \|v(t)\|^p_{W_0^{1,p}(\Omega)}dt\Big)^{1/p}.$$

We introduce $Y_p = L^p(0,T;W^{-1,p}(\Omega))$ and we point out that the application $v \to \operatorname{div}_x v$ sends $(L^p(\Omega_T))^n$ into $L^p(0,T;W^{-1,p}(\Omega))$. We endow Y_p with the norm $\|f\|_{Y_p} = \inf_{\operatorname{div}_x g=f}\|g\|_{(L^p(\Omega_T))^n}$. We can state the following Lemma (cf. [3]).

Lemma 2.4. *Let $A \in (L^\infty(\Omega))^n$ be a symmetric tensor defined as in Lemma 2.2. Let $f \in L^2(0,T,H^{-1}(\Omega))$ and $u^0 \in H$, there exists $u \in L^2(0,T;H_0^1(\Omega))$ solution of*

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in } \Omega_T, u(0) = u^0.$$

Then, assuming that Γ is sufficiently regular, there exists $r > 2$, depending on α,β and Ω such that if $f \in L^r(0,T;W^{-1,r}(\Omega))$ and $u^0 \in W_0^{1,r}(\Omega)$ then $u \in L^r(0,T;W_0^{1,r}(\Omega))$. Furthermore, there exists $\hat{C}(\alpha,\beta,r) > 0$ such that

$$\|u\|_{W_0^{1,r}(\Omega)} \le \hat{C}(\alpha,\beta,r)(\|f\|_{L^r(0,T;W^{-1,r}(\Omega))} + \|u^0\|_{W_0^{1,r}(\Omega)}). \tag{2.6}$$

Remark 2.5. As for lemma 2.2, it is possible to precise $\hat{C}(\alpha,\beta,r)$ (cf. [3]). We set $P = \frac{\partial}{\partial t} - \Delta$, the operator associated with the homogeneous Dirichlet boundary conditions. We know that, being given $F \in Y_p$, there is a unique solution $u \in X_p$ such that

$$Pu = F \quad \text{in } \Omega_T,\ u(0) = u^0.$$

We set $\hat{g}(p) = \|P^{-1}\|_{\mathcal{L}(Y_p;X_p)}$, we recall that $\hat{g}(2) = 1$. Using the properties of the map $\hat{g}(\cdot)$, we claim that there exists $r > 2$ such that

$$0 < \hat{k}(r) := \hat{g}(r)(1-\hat{\mu}+\hat{\nu}) < 1, \tag{2.7}$$

where the constants $\hat{\mu}$, $\hat{\nu}$ are defined by

$$\hat{\mu} = \frac{\alpha + \hat{c}}{\beta + \hat{c}} \quad \text{and} \quad \hat{\nu} = \frac{\hat{c}}{\beta + \hat{c}}, \quad \forall \hat{c} > 0. \tag{2.8}$$

Since $\hat{c} > 0$, we have $\hat{\nu} < \hat{\mu}$. Again, the smaller $(1 - \hat{\mu} + \hat{\nu})$, the bigger r, and the determination of r will depend on the constants α, β characterizing the elliptic operator A. The following condition is satisfied by the constant $\hat{C}(\alpha, \beta, r)$

$$\hat{C}(\alpha, \beta, r) \le (1 - \hat{g}(r)(1 - \hat{\mu} + \hat{\nu}))^{-1} \frac{\hat{g}(r)}{\beta + \hat{c}} = \frac{\hat{g}(r)}{(1 - \hat{k}(r))(\beta + \hat{c})}. \tag{2.9}$$

3. Global in time existence result

Mathematical setting. We consider that the confined aquifer is bounded by two layers, the lower surface corresponds to $z = h_2$ and the upper surface $z = h_1$. Quantity $h_2 - h_1$ is the thickness of the aquifer. We assume that depths h_1, h_2 are constant, such that $h_2 > \delta_1 > 0$ and without lost of generality we can set $h_1 = 0$. We introduce functions T_s and T_f defined by

$$T_s(u) = h_2 - u \quad \forall u \in (\delta_1, h_2) \quad \text{and} \quad T_f(u) = u \quad \forall u \in (\delta_1, h_2).$$

Functions T_s and T_f are extended continuously and constantly outside (δ_1, h_2). $T_s(h)$ represents the thickness of the salt water zone in the reservoir, the previous extension of T_s for $h \le \delta_1$ enables us to ensure a thickness of freshwater zone always greater than δ_1 in the aquifer. We also emphasize that the function T_f only acts on the source term Q_f for avoiding the pumping when the thickness of freshwater zone is smaller than δ_1.

In the case of confined aquifer, the well adapted unknowns are the interface depth h and the freshwater hydraulic head f. The model reads (see [6]):

$$\phi\partial_t h - \nabla \cdot \Big(KT_s(h)\nabla h\Big) - \nabla \cdot \Big(\delta \nabla h\Big) + \nabla \cdot \Big(KT_s(h)\nabla f\Big) = -Q_s T_s(h), \tag{3.1}$$

$$-\nabla \cdot \Big(h_2 K \nabla f\Big) + \nabla \cdot \Big(KT_s(h)\nabla h\Big) = Q_f T_f(h) + Q_s T_s(h). \tag{3.2}$$

The above system is complemented by the boundary and initial conditions

$$h = h_D, \quad f = f_D \quad \text{in } \Gamma \times (0, T), \tag{3.3}$$

$$h(0, x) = h_0(x), \quad \text{in } \Omega, \tag{3.4}$$

with the compatibility condition

$$h_0(x) = h_D(0, x), \quad x \in \Gamma.$$

Let us now detail the mathematical assumptions. We begin with the characteristics of the porous structure. We assume the existence of two positive real numbers K_- and K_+ such that the hydraulic conductivity K is a bounded symmetric elliptic and uniformly positive definite tensor

$$0 < K_-|\xi|^2 \le \sum_{i,j=1,2} K_{i,j}(x)\xi_i\xi_j \le K_+|\xi|^2 < \infty \quad x \in \Omega, \ \xi \in \mathbb{R}^2, \ \xi \ne 0.$$

We assume that porosity ϕ is constant in the aquifer. Indeed, in the field envisaged here, the effects due to variations in ϕ are negligible compared with those due to density contrasts. From a mathematical point of view, these assumptions do not change the complexity of the analysis but rather avoid cumbersome computations. The parameter δ represents the thickness of the diffuse interface. The source terms

Q_f and Q_s are given functions of $L^2(0,T;H)$ and we assume that $Q_f \geq 0$ and $Q_s \leq 0$.

The functions h_D and f_D belong to $\big(L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')\big) \times L^2(0,T;H^1(\Omega))$ while function h_0 belongs to $H^1(\Omega)$. Finally, we assume that the boundary and initial data satisfy conditions on the hierarchy of interfaces depths:

$$0 < \delta_1 \leq h_D \leq h_2 \text{ a.e. in } \Gamma \times (0,T), \quad 0 < \delta_1 \leq h_0 \leq h_2 \text{ a.e. in } \Omega.$$

Theorem 3.1 (Existence theorem)**.** *Assume a low spatial heterogeneity for the hydraulic conductivity tensor*

$$K_+ < \frac{h_2}{h_2 - \delta_1} \inf\Big(\sqrt{\frac{\delta K_-}{3h_2}}, K_-\Big). \tag{3.5}$$

Then for any $T > 0$, problem (3.1)-(3.4) *admits a weak solution (h,f) satisfying*

$$(h - h_D, f - f_D) \in W(0,T) \times L^2(0,T;H_0^1(\Omega)).$$

Furthermore the following maximum principle holds true :

$$0 < \delta_1 \leq h(t,x) \leq h_2 \quad \textit{for a.e. } x \in \Omega \textit{ and for any } t \in (0,T).$$

Remark 3.2. Assumption (3.5) (so as (4.4)) makes only sense when considering low values for K. For the present application, this point is not restrictive since the soil permeability typically ranges from 10^{-8} to 10^{-3} m/s.

With the additional diffuse interface, the system has a parabolic structure, it is thus no longer necessary to introduce viscous terms in a preliminary fixed point step for avoiding degeneracy. But we still need to impose a minimal freshwater thickness strictly positive inside the aquifer to prove an uniform estimate in $L^2(\Omega_T)$ of the gradient of f since the presence of the diffuse interface does not allow us to get this estimate. Let us briefly sketch the strategy of the proof. First step consists in using a Schauder fixed point theorem for proving an existence result for the problem.[1] Then we establish that the solution satisfies the maximum principles announced in Theorem 3.1: First, we show that $h \geq h_2$ a.e. in Ω_T; finally we prove that $\delta_1 \leq h$ a.e. in Ω_T under assumption $Q_f \geq 0$.

Step 1: Existence for the truncated system:
Definition of the map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$. For the fixed point strategy, we define an application $\mathcal{F} : (W(0,T) + h_D) \times (L^2(0,T;H_0^1(\Omega)) + f_D) \to (W(0,T) + h_D) \times (L^2(0,T;H_0^1(\Omega)) + f_D)$ by

$$\mathcal{F}(\bar{h}, \bar{f}) = \big(\mathcal{F}_1(\bar{h}, \bar{f}), \mathcal{F}_2(\bar{h}, \bar{f})\big) = (h, f),$$

[1] More precisely, the present proof is based on the classical version of the Schauders fixed point theorem applied to the initial problem. In [8], this fixed point theorem is applied to an auxiliary truncated problem. The truncation is introduced to control the H^1-norm of f. We thus had to check that the map $\mathcal{F}$ defined below is sequentially continuous in $L^2(0,T;H^1(\Omega))$. Here we rather choose working with the strong topology $L^2(0,T;L^2(\Omega))$, which is possible since the truncation term has been dropped.

where the couple (h, f) is the solution of the following initial boundary value problem, for all $w \in L^2(0,T;V)$:

$$\begin{aligned}&\int_0^T \phi\langle \partial_t h, w\rangle_{V,V'}\,dt + \int_{\Omega_T} (\delta + T_s(\bar{h})K)\nabla h\cdot\nabla w\,dx\,dt\\&+\int_{\Omega_T} Q_s T_s(\bar{h}) w\,dx\,dt - \int_{\Omega_T} T_s(\bar{h})K\nabla\bar{f}\cdot\nabla w\,dx\,dt = 0,\end{aligned}\tag{3.6}$$

$$\begin{aligned}&\int_{\Omega_T} h_2 K\nabla f\cdot\nabla w\,dx\,dt - \int_{\Omega_T} T_s(\bar{h})K\nabla\bar{h}\cdot\nabla w\,dx\,dt\\&-\int_{\Omega_T}(Q_s T_s(\bar{h}) + Q_f T_f(\bar{h}))w\,dx\,dt = 0.\end{aligned}\tag{3.7}$$

We know from the classical theory of linear parabolic PDE's that this variational linear system has a unique solution. The end of the present subsection is devoted to the proof of the existence of a fixed point of $\mathcal{F}$ in some appropriate subset.

Sequential continuity of $\mathcal{F}_1$ in $L^2(0,T;H)$ when $\mathcal{F}$ is restricted to any bounded subset of $W(0,T) \times L^2(0,T;H^1(\Omega))$. Assume that given a bounded sequence $(\bar{h}^n, \bar{f}^n)$ in $(W(0,T)+h_D)\times(L^2(0,T;H_0^1(\Omega))+f_D)$ and a function $(\bar{h},\bar{f}) \in (W(0,T)+h_D)\times(L^2(0,T;H_0^1(\Omega))+f_D)$ such that

$$(h_n, f_n) \to (h, f) \text{ in } (L^2(0,T;H))^2.$$

We thus have

$$(\bar{h}^n, \bar{f}^n) \rightharpoonup (\bar{h}, \bar{f}) \quad \text{weakly in } W(0,T)\times L^2(0,T;H^1(\Omega));$$

that is, $\bar{h}^n \rightharpoonup \bar{h}$ weakly in $L^2(0,T,V)$ (the same for $\bar{f}^n$ and $\bar{f}$) and $\partial_t h^n \rightharpoonup \partial_t h$ weakly in $L^2(0,T,V')$.

Set $h_n = \mathcal{F}_1(\bar{h}^n, \bar{f}^n)$ and $h = \mathcal{F}_1(\bar{h}, \bar{f})$. We first intend to show that $h_n \to h$ weakly in $W(0,T)$ and thus strongly in $L^2(0,T;H)$ thanks to a classical result of Aubin.

Pick a constant $M > 0$, that we will precise later on, such that

$$\|\nabla\bar{h}_n\|_{(L^2(0,T;H))^2} \le M \quad\text{and}\quad \|\nabla\bar{f}_n\|_{(L^2(0,T;H))^2} \le M. \tag{3.8}$$

For all $n\in\mathbb{N}$, h_n satisfies (3.6). Pick any $\tau\in[0,T]$ and take $w = (h_n - h_D)\chi_{(0,\tau)}(t)$ in (3.6). It yields

$$\begin{aligned}&\phi\int_0^\tau \langle\partial_t(h_n-h_D), h_n-h_D\rangle_{V',V}\,dt + \int_{\Omega_\tau}(\delta + KT_s(\bar{h}^n))\nabla h_n\cdot\nabla h_n\,dx\,dt\\&+\int_{\Omega_T} Q_s T_s(\bar{h}_n)(h_n-h_D)\,dx\,dt - \int_{\Omega_\tau} KT_s(\bar{h}^n)\nabla\bar{f}^n\cdot\nabla(h_n-h_D)\,dx\,dt\\&=\int_{\Omega_\tau}(\delta+KT_s(\bar{h}^n))\nabla h_n\cdot\nabla h_D\,dx\,dt - \phi\int_0^\tau\langle\partial_t h_D, h_n-h_D\rangle_{V',V}\,dt.\end{aligned}\tag{3.9}$$

The functions $h_n - h_D$ belong to $W(0,T)$ and hence to $\mathcal{C}([0,T];L^2(\Omega))$. Thanks to Lemma 2.1, we can write

$$\int_0^\tau\langle\partial_t(h_n-h_D), h_n-h_D\rangle_{V',V}\,dt = \frac{1}{2}\|h_n(\cdot,\tau)-h_D\|_H^2 - \frac{1}{2}\|h_0 - h_D(\cdot,0)\|_H^2.$$

On the other hand, we have

$$\int_{\Omega_\tau}\big(\delta + KT_s(\bar{h}^n)\big)\nabla h_n\cdot\nabla h_n\,dx\,dt \ge \delta\|\nabla h_n\|^2_{L^2(0,\tau;H)^2}.$$

The real number $M > 0$ is such that $\sup_{n\geq 0} \|\nabla \bar{f}^n\|_{(L^2(0,T,H))^2} \leq M$. Using Cauchy-Schwarz and Young inequalities, we obtain that, for any $\eta_1 > 0$,

$$\begin{aligned}|\int_{\Omega_\tau} KT_s(\bar{h}^n)\nabla \bar{f}^n \cdot \nabla h_n\,dx\,dt| &\leq MK_+ l\|\nabla h_n\|_{(L^2(0,\tau;H))^2}\\ &\leq \frac{K_+^2M^2}{\eta_1}l^2 + \frac{\eta_1}{4}\|\nabla h_n\|^2_{(L^2(0,\tau;H))^2},\end{aligned}$$

and

$$\begin{aligned}&|\int_{\Omega_T} \big(\delta + KT_s(\bar{h}^n)\big)\nabla h_n \cdot \nabla h_D\,dx\,dt|\\ &\leq \frac{\eta_1}{4}\|\nabla h_n\|^2_{(L^2(0,T;H))^2} + \frac{(\delta + K_+h_2)^2}{\eta_1}\|\nabla h_D\|^2_{(L^2(0,T;H))^2}.\end{aligned}$$

Since it depends on h_D, the next term is simply estimated by

$$|\int_{\Omega_T} KT_s(\bar{h}^n)\nabla \bar{f}^n \cdot \nabla h_D\,dx\,dt| \leq MK_+h_2\|h_D\|_{L^2(0,T;H^1)}.$$

Finally we have

$$\begin{aligned}&|-\int_0^T \phi\langle\partial_t h_D, (h_n - h_D)\rangle_{V',V}\,dt|\\ &\leq \frac{\phi^2}{2\delta}\|\partial_t h_D\|^2_{L^2(0,T;(H^1(\Omega))')} + \frac{\delta}{2}\|h_n\|^2_{L^2(0,T;H^1)} + \frac{\delta}{2}\|h_D\|^2_{L^2(0,T;H^1)},\end{aligned}$$

and

$$|-\int_{\Omega_T} Q_sT_s(\bar{h}^n)(h_n - h_D)\,dx\,dt| \leq \frac{\|Q_s\|^2_{L^2(0,T;H)}}{\phi}h_2^2 + \frac{\phi}{4}\|h_n - h_D\|^2_{L^2(0,T;H)}.$$

We choose η_1 such that $\delta - \eta_1 \geq \eta_0 > 0$ for some $\eta_0 > 0$. Using the above estimates in (3.9), we obtain for all $\tau \in [0,T]$

$$\begin{aligned}&\frac{\phi}{4}\|(h_n - h_D)(\cdot,\tau)\|^2_H + \frac{1}{2}(\delta - \eta_1)\|\nabla u_{1,n}\|^2_{(L^2(0,\tau;H))^2}\\ &\leq \frac{K_+^2M^2}{\eta_1}l^2 + \frac{\phi}{2}\|h_0 - h_D(\cdot,0)\|^2_H + \frac{(\delta + K_+h_2)^2}{\eta_1}\|\nabla h_D\|^2_{(L^2(0,T;H))^2}\\ &\quad + MK_+h_2\|h_D\|_{L^2(0,T;H^1)} + \frac{\phi^2}{2\delta}\|\partial_t h_D\|^2_{L^2(0,T;(H^1(\Omega))')}\\ &\quad + \frac{\delta}{2}\|h_D\|^2_{L^2(0,T;H^1)} + \frac{\|Q_s\|^2_{L^2(0,T;H)}}{\phi}h_2^2.\end{aligned} \tag{3.10}$$

We infer from (3.10) that there exist real numbers $A_M = A_M(\delta, K, h_0, h_2, h_D, l, M)$ and $B_M = B_M(\delta, K, h_0, h_2, h_D, l, M)$ depending only on the data of the problem such that

$$\|h_n\|_{L^\infty(0,T;H)} \leq A_M, \quad \|h_n\|_{L^2(0,T;V)} \leq B_M. \tag{3.11}$$

Thus the sequence $(h_n)_n$ is uniformly bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Set

$$C_M = \max(A_M, B_M).$$

We now prove that $(\partial_t(h_n - h_D))_n$ is bounded in $L^2(0,T;V')$. Due to the assumption $h_D \in H^1(0,T;(H^1(\Omega))')$, it will follow that $(h_n)_n$ is uniformly bounded in $H^1(0,T;V')$. We have

$$\|\partial_t(h_n - h_D)\|_{L^2(0,T;V')}$$

$$
\begin{aligned}
&= \sup_{\|w\|_{L^2(0,T;V)}\le 1} \Big| \int_0^T \langle \partial_t(h_n - h_D), w\rangle_{V',V}\, dt \Big| \\
&= \sup_{\|w\|_{L^2(0,T;V)}\le 1} \Big| \int_0^T -\langle \partial_t h_D, w\rangle_{V',V}\, dt - \frac{1}{\phi}\Big(\int_{\Omega_T} \big(\delta + KT_s(\bar{h}^n)\big)\nabla h_n \cdot \nabla w \,dx\,dt \\
&\quad + \int_{\Omega_T} KT_s(\bar{h}^n)\nabla \bar{f}^n \cdot \nabla w\,dx\,dt - \int_{\Omega_T} Q_s T_s(\bar{h}^n) w\Big)\,dx\,dt\Big|.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big| \int_{\Omega_T} \big(\delta + KT_s(\bar{h}^n)\big)\nabla h_n \cdot \nabla w\,dx\,dt\Big| \\
&\le \big(\delta + K_+ h_2\big)\|h_n\|_{L^2(0,T;H^1(\Omega))}\|w\|_{L^2(0,T;V)},
\end{aligned}
$$

and since h_n is uniformly bounded in $L^2(0,T;H^1(\Omega))$, we write

$$
\Big| \int_{\Omega_T} \big(\delta + KT_s(\bar{h}^n)\big)\nabla h_n \cdot \nabla w\,dx\,dt\Big| \le \big(\delta + K_+ h_2\big) C_M \|w\|_{L^2(0,T;V)}. \tag{3.12}
$$

Furthermore we have

$$
\Big| \int_{\Omega_T} T_s(\bar{h}^n))\nabla \bar{f}^n \cdot \nabla w\,dx\,dt\Big| \le M h_2 \|w\|_{L^2(0,T;V)}, \tag{3.13}
$$

$$
\Big| \int_{\Omega_T} Q_s T_s(\bar{h}^n) w\,dx\,dt\Big| \le \|Q_s\|_{L^2(0,T;H)} h_2 \|w\|_{L^2(0,T;V)}. \tag{3.14}
$$

Summing (3.12)–(3.14), we conclude that

$$
\|\partial_t(h_n - h_D)\|_{L^2(0,T;V')} \le D_M, \tag{3.15}
$$

where

$$
D_M = \|\partial_t h_D\|^2_{L^2(0,T;(H^1(\Omega))')} + \delta C_M + \frac{h_2}{\phi}(K_+ C_M + M + \|Q_s\|_{L^2(0,T;H)}).
$$

We have proved that the sequence $\big(h_n\big)_n$ is uniformly bounded in the space $W(0,T)$. Using Aubin-Lions' lemma, we can extract a subsequence $(h_{n_k})_k$, converging strongly in $L^2(\Omega_T)$, almost everywhere in $(0,T)\times\Omega$, and weakly in $W(0,T)$ to some limit denoted by v. From the a.e. convergence in Ω_T, we see that for all $w \in W(0,T)$, $T_l(\bar{h}^n)\nabla w \to T_l(\bar{h})\nabla w$ strongly in $L^2(\Omega_T)$ by dominated convergence. It follows that v solves (3.6) and (3.3)-(3.4). By uniqueness of the solution of that system, we conclude that $v = h$ and that the whole sequence $h_n \to h$ weakly in $W(0,T)$ and strongly in $L^2(0,T;H)$.

The sequential continuity of $\mathcal{F}_1$ in $L^2(0,T;H)$ is established.

Sequential continuity of $\mathcal{F}_2$ in $L^2(0,T;H)$ when $\mathcal{F}$ is restricted to any bounded subset of $W(0,T)\times L^2(0,T;H^1(\Omega))$ As above, we study the sequential continuity of $\mathcal{F}_2$ by setting $f_n := \mathcal{F}_2(\bar{h}^n, \bar{f}^n)$, $f := \mathcal{F}_2(\bar{h}, \bar{f})$, and showing first that $f_n \to f$ in $L^2(0,T;H^1(\Omega))$ weakly. The key estimates are obtained using the same type of arguments than those in the proof of the sequential continuity of $\mathcal{F}_1$. The details are omitted. We only point out that we can use the estimate (3.11) previously derived for h_n to obtain the following estimates for f_n:

$$
\|f_n\|_{L^\infty(0,T;H)} \le E_M = E_M\big(\delta_2, K, f_D, h_2, l, M, C_M\big), \tag{3.16}
$$

$$
\|f_n\|_{L^2(0,T;V)} \le F_M = F_M\big(\delta_2, K, f_D, h_2, l, M, C_M\big). \tag{3.17}
$$

For proving the sequential compactness of f_n in $L^2(0,T;H)$, we need some further work since we can not use a Aubin's compactness criterium in the elliptic context characterizing f_n. We actually get a stronger result: we claim and prove that $h_2 f_n - \mathcal{T}_s(\bar h_n)$ converges in $L^2(0,T;H^1(\Omega))$, where $\mathcal{T}_s$ is any function such that $\mathcal{T}_s' = T_s$. Indeed, we recall that the variational formulations defining respectively f_n and f are, for any $w \in L^2(0,T;V)$,

$$\begin{aligned} &\int_{\Omega_T} h_2 K \nabla f_n \cdot \nabla w \,dx\,dt - \int_{\Omega_T} K T_s(\bar h_n) \nabla \bar h_n \cdot \nabla w \,dx\,dt \\ &- \int_{\Omega_T} (Q_s T_s(\bar h_n) + Q_f T_f(\bar h_n)) w \,dxdt = 0, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\int_{\Omega_T} h_2 K \nabla f \cdot \nabla w \,dx\,dt - \int_{\Omega_T} K T_s(\bar h) \nabla \bar h \cdot \nabla w \,dx\,dt \\ &- \int_{\Omega_T} (Q_s T_s(\bar h) + Q_f T_f(\bar h)) w \,dxdt = 0. \end{aligned} \tag{3.19}$$

Choosing $w = h_2 f_n - \mathcal{T}_s(\bar h_n) - h_2 f_D + \mathcal{T}_s(h_D)$ in (3.18) we let $n \to \infty$. The already known convergence results let us pass to the limit in

$$\begin{aligned} &\lim_{n\to\infty} \int_{\Omega_T} (Q_s T_s(\bar h_n) + Q_f T_f(\bar h_n))\big(h_2 f_n - \mathcal{T}_s(\bar h_n) - h_2 f_D + \mathcal{T}_s(h_D)\big)\,dxdt \\ &= \int_{\Omega_T} (Q_s T_s(\bar h) + Q_f T_f(\bar h))\big(h_2 f - \mathcal{T}_s(\bar h) - h_2 f_D + \mathcal{T}_s(h_D)\big)\,dx\,dt. \end{aligned}$$

Using moreover (3.19) for the test function $w = h_2 f - \mathcal{T}_s(\bar h) - h_2 f_D + \mathcal{T}_s(h_D)$, we conclude that

$$\begin{aligned} &\lim_{n\to\infty} \int_{\Omega_T} K \nabla\big(h_2 f_n - \mathcal{T}_s(\bar h_n) - h_2 f_D + \mathcal{T}_s(h_D)\big) \\ &\cdot \nabla\big(h_2 f_n - \mathcal{T}_s(\bar h_n) - h_2 f_D + \mathcal{T}_s(h_D)\big)\,dx\,dt \\ &= \int_{\Omega_T} K \nabla\big(h_2 f - \mathcal{T}_s(\bar h) - h_2 f_D + \mathcal{T}_s(h_D)\big) \\ &\quad \cdot \nabla\big(h_2 f - \mathcal{T}_s(\bar h) - h_2 f_D + \mathcal{T}_s(h_D)\big)\,dx\,dt. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \int_{\Omega_T} K \nabla (F_n - F) \cdot \nabla (F_n - F)\,dx\,dt = 0$$

if $F_n = h_2 f_n - \mathcal{T}_s(\bar h_n) - h_2 f_D + \mathcal{T}_s(h_D)$ and $F = h_2 f - \mathcal{T}_s(\bar h) - h_2 f_D + \mathcal{T}_s(h_D)$. Since $K\xi \cdot \xi \ge K_-|\xi|^2$ for any $\xi \in \mathbb{R}^2$ with $K_- > 0$, the latter result and the Poincaré inequality let us ensure that $F_n \to F$ in $L^2(0,T;V)$. Since $\mathcal{T}_s(\bar h_n) \to \mathcal{T}_s(\bar h)$ almost everywhere in Ω_T and $h_2 > 0$, it follows in particular that $f_n \to f$ in $L^2(0,T;H)$.

Existence of $\mathcal{C} \subset W(0,T) \times L^2(0,T;(H^1(\Omega))$ such that $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$. We aim now to prove that there exists a nonempty bounded closed convex set of $W(0,T) \times L^2(0,T;H^1(\Omega))$, denoted by $\mathcal{C}$, such that $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$. We notice that this result will imply that there exists a real number $M > 0$, depending only on the initial data, such that for $(h,f) = \mathcal{F}(\bar h, \bar f) \in W$, we have

$$\|\nabla h\|_{(L^2(0,T;H))^2} \le M \quad \text{and} \quad \|\nabla f\|_{(L^2(0,T;H))^2} \le M. \tag{3.20}$$

Taking $w = h - h_D \in L^2(0,T;V)$ (resp. $w = f - f_D \in L^2(0,T;V)$) in (3.6) (resp. (3.7)) leads to

$$\begin{aligned}
&\phi \int_0^T \langle \partial_t h, h - h_D\rangle_{V',V} dt + \int_{\Omega_T} \delta \nabla h \cdot \nabla (h - h_D)\,dx\,dt \\
&+ \int_{\Omega_T} K T_s(\bar h) \nabla h \cdot \nabla (h - h_D)\,dx\,dt + \int_{\Omega_T} Q_s T_s(\bar h)(h - h_D)\,dx\,dt \\
&- \int_{\Omega_T} K T_s(\bar h) \nabla \bar f \cdot \nabla (h - h_D)\,dx\,dt = 0,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\int_{\Omega_T} h_2 K \nabla f \cdot \nabla (f - f_D)\,dx\,dt - \int_{\Omega_T} K T_s(\bar h) \nabla \bar h \cdot \nabla (f - f_D)\,dx\,dt \\
&- \int_{\Omega_T} (Q_s T_s(\bar h) + Q_f T_f(\bar h))(f - f_D)\,dxdt = 0.
\end{aligned} \tag{3.22}$$

We apply Lemma 2.1 to the function f defined by $f(u) = u$ for $u \in \mathbb{R}$ to compute the first terms of (3.21). We obtain

$$\int_0^T \langle \partial_t (h - h_D), (h - h_D)\rangle_{V',V} dt = \frac{1}{2}\int_\Omega (h - h_D)^2(T,x)\,dx - \frac{1}{2}\int_\Omega (h - h_D)^2(0,x)\,dx.$$

Summing equations (3.21) and (3.22), we obtain

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega (h - h_D)(T,x)^2\,dx + \int_{\Omega_T} \delta \nabla (h - h_D) \cdot \nabla (h - h_D)\,dx\,dt \\
&+ \int_{\Omega_T} h_2 K \nabla (f - f_D) \cdot \nabla (f - f_D)\,dx\,dt \\
&+ \int_{\Omega_T} T_s(\bar h) K \nabla (h - h_D) \cdot \nabla (h - h_D)\,dx\,dt \\
&= \underbrace{\int_{\Omega_T} T_s(\bar h) K \nabla (\bar h - h_D) \cdot \nabla (f - f_D)\Big)\,dx\,dt}_{(1)} \\
&\quad + \underbrace{\int_{\Omega_T} K T_s(\bar h) \nabla (\bar f - f_D) \cdot \nabla (h - h_D)\,dx\,dt}_{(2)} \\
&\quad + \frac{\phi}{2}\int_\Omega (h - h_D)(0,x)^2\,dx - \int_{\Omega_T} \delta \nabla h_D \cdot \nabla (h - h_D)\,dx\,dt \\
&\quad - \int_{\Omega_T} h_2 K \nabla f_D \cdot \nabla (f - f_D)\,dx\,dt - \int_{\Omega_T} T_s(\bar h) K \nabla h_D \cdot \nabla (h - h_D)\,dx\,dt \\
&\quad + \int_{\Omega_T} T_s(\bar h) K \nabla h_D \cdot \nabla (f - f_D)\,dx\,dt + \int_{\Omega_T} K T_s(\bar h) \nabla f_D \cdot \nabla (h - h_D)\,dx\,dt \\
&\quad - \int_{\Omega_T} Q_s T_s(\bar h)(h - h_D)\,dx\,dt + \int_{\Omega_T} \big(Q_s T_s(\bar h) + Q_f T_f(\bar h)\big)(f - f_D)\,dx\,dt.
\end{aligned}$$

We have

$$|(1)| \le \int_{\Omega_T} \frac{\delta}{3} |\nabla (\bar h - h_D)|^2\,dx\,dt + \frac{3(h_2 - \delta_1)^2 K_+^2}{4\delta} \int_{\Omega_T} |\nabla (f - f_D)|^2\,dx\,dt,$$

$$
\begin{aligned}
|(2)| \le &\int_{\Omega_T} \frac{K_- h_2}{3} |\nabla(\bar f - f_D)|^2 \,dx\,dt \\
&+ \frac{3K_+^2(h_2-\delta_1)}{4K_- h_2} \int_{\Omega_T} T_s(\bar h) |\nabla(h-h_D)|^2 \,dx\,dt.
\end{aligned}
$$

Also

$$
\begin{gathered}
\Big|\int_{\Omega_T} \delta \nabla h_D \cdot \nabla(h-h_D)\,dx\,dt\Big| \le \frac{\delta}{2}\int_{\Omega_T} |\nabla(h-h_D)|^2\,dx\,dt + \frac{\delta}{2}\int_{\Omega_T} |\nabla h_D|^2\,dx\,dt, \\
\Big|\int_{\Omega_T} h_2 K \nabla f_D \cdot \nabla(f-f_D)\,dx\,dt\Big| \\
\le \frac{h_2K_-}{12}\int_{\Omega_T} |\nabla(f-f_D)|^2\,dx\,dt + \frac{3K_+^2 h_2}{K_-}\int_{\Omega_T} |\nabla f_D|^2\,dx\,dt, \\
\Big|\int_{\Omega_T} T_s(\bar h) K \nabla h_D \cdot \nabla(h-h_D)\,dx\,dt\Big| \\
\le \frac{K_-}{8}\int_{\Omega_T} T_s(\bar h)|\nabla(h-h_D)|^2\,dx\,dt + \frac{2K_+^2 h_2}{K_-}\int_{\Omega_T} |\nabla h_D|^2\,dx\,dt, \\
\Big|\int_{\Omega_T} T_s(\bar h) K \nabla h_D \cdot \nabla(f-f_D)\,dx\,dt\Big| \\
\le \frac{h_2K_-}{12}\int_{\Omega_T} |\nabla(f-f_D)|^2\,dx\,dt + \frac{3K_+^2 h_2}{K_-}\int_{\Omega_T} |\nabla h_D|^2\,dx\,dt, \\
\Big|\int_{\Omega_T} T_s(\bar h) K \nabla f_D \cdot \nabla(h-h_D)\,dx\,dt\Big| \\
\le \frac{K_-}{8}\int_{\Omega_T} T_s(\bar h)|\nabla(h-h_D)|^2\,dx\,dt + \frac{2K_+^2 h_2}{K_-}\int_{\Omega_T} |\nabla f_D|^2\,dx\,dt, \\
\Big|\int_{\Omega_T} Q_s T_s(\bar h)(h-h_D)\,dx\,dt\Big| \\
\le \frac{\delta}{8}\int_{\Omega_T} T_s(\bar h)|\nabla(h-h_D)|^2\,dx\,dt + \frac{2C_P^2}{\delta}\int_{\Omega_T} Q_s^2 T_s^2(\bar h)\,dx\,dt, \\
\Big|\int_{\Omega_T} \big(Q_s T_s(\bar h) + Q_f T_f(\bar h)\big)(f-f_D)\,dxdt\Big| \\
\le \frac{h_2K_-}{12}\int_{\Omega_T} |\nabla(f-f_D)|^2\,dx\,dt + \frac{3C_P^2}{h_2K_-}\int_{\Omega_T} \big(Q_s T_s(\bar h) + Q_f T_f(\bar h)\big)^2\,dx\,dt,
\end{gathered}
$$

where the Poincaré's constant is denoted by C_P. Gathering these estimates, we conclude that

$$
\begin{aligned}
&\frac{\phi}{2}\int_{\Omega}(h-h_D)(T,x)^2\,dx + \frac{\delta}{2}\int_{\Omega_T}|\nabla(h-h_D)|^2\,dx\,dt + \frac{h_2K_-}{2}\int_{\Omega_T}|\nabla(f-f_D)|^2\,dx\,dt \\
&+ \Big(\frac{h_2K_-}{4} - \frac{3(h_2-\delta_1)^2K_+^2}{4\delta}\Big)\int_{\Omega_T}|\nabla(f-f_D)|^2\,dx\,dt \\
&+ \Big(\frac{3K_-}{4} - \frac{3K_+^2(h_2-\delta_1)}{4K_-h_2}\Big)\int_{\Omega_T} T_s(\bar h)|\nabla(h-h_D)|^2\,dx\,dt \\
&\le \frac{\delta}{3}\int_{\Omega_T}|\nabla(\bar h - h_D)|^2\,dx\,dt + \frac{K_-h_2}{3}\int_{\Omega_T}|\nabla(\bar f - f_D)|^2\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\phi}{2}\int_{\Omega}(h-h_D)(0,x)^2\,dx+\frac{\delta}{2}\int_{\Omega_T}|\nabla h_D|^2\,dx\,dt\\
&+\frac{5K_+^2h_2}{K_-}\int_{\Omega_T}|\nabla f_D|^2\,dx\,dt+\frac{5K_+^2h_2}{K_-}\int_{\Omega_T}|\nabla h_D|^2\,dx\,dt\\
&+\frac{1}{2\phi}\int_{\Omega_T}Q_s^2T_s^2(\bar h)\,dx\,dt+\frac{3C_P^2}{h_2K_-}\int_{\Omega_T}\big(Q_sT_s(\bar h)+Q_fT_f(\bar h)\big)^2\,dx\,dt. \qquad (3.23)
\end{aligned}
$$

Introducing the constant

$$
\begin{aligned}
C_0 :=6\Big(&\frac{\phi}{2}\int_{\Omega}(h-h_D)(0,x)^2\,dx+\frac{\delta}{2}\int_{\Omega_T}|\nabla h_D|^2\,dx\,dt\\
&+\frac{5K_+^2h_2}{K_-}\int_{\Omega_T}|\nabla f_D|^2\,dx\,dt+\frac{5K_+^2h_2}{K_-}\int_{\Omega_T}|\nabla h_D|^2\,dx\,dt\\
&+\frac{1}{2\phi}\int_{\Omega_T}Q_s^2T_s^2(\bar h)\,dx\,dt+\frac{3C_P^2}{h_2K_-}\int_{\Omega_T}\big(Q_sT_s(\bar h)+Q_fT_f(\bar h)\big)^2\,dx\,dt\Big)
\end{aligned} \qquad (3.24)
$$

and recalling that the parameters satisfy (3.5), we infer that

$$
\delta\|\nabla(h-h_D)\|^2_{(L^2(\Omega_T))^2}+h_2K_-\|\nabla(f-f_D)\|^2_{(L^2(\Omega_T))^2}\le C_0,
$$

and

$$
\delta\|\nabla(\bar h-h_D)\|^2_{(L^2(\Omega_T))^2}+h_2K_-\|\nabla(\bar f-f_D)\|^2_{(L^2(\Omega_T))^2}\le C_0.
$$

Note that (3.23) yields

$$
\|\nabla(h-h_D)\|_{L^2(\Omega_T)}\le\sqrt{C_0/\delta},\quad \|\nabla(f-f_D)\|_{L^2(\Omega_T)}\le\sqrt{C_0/h_2K_-}
$$

and

$$
\frac{1}{2}\int_{\Omega}(h-h_D)(\tau,x)^2\,dx\le C_0,\quad \text{for all } \tau\le T.
$$

Conclusion. We introduce the set

$$
\begin{aligned}
\mathcal{C} := \big\{&(h-h_D,f-f_D)\in W(0,T)\times L^2(0,T;(H^1(\Omega)):(h(0,\cdot),\\
&f(0,\cdot))=(h_0,f_0),\ \delta\|\nabla(h-h_D)\|^2_{L^2(\Omega_T)}\\
&+h_2K_-\|\nabla(f-f_D)\|^2_{L^2(\Omega_T)}\le C_0,\ \|\partial_th\|_{L^2(0,T,V')}\le D_M.\big\}
\end{aligned} \qquad (3.25)
$$

where C_0 is defined by (3.24) and $M:=\max(\sqrt{C_0/\delta},\sqrt{C_0/h_2K_-})$. Then $\mathcal{C}$ is a nonempty, closed, convex, bounded set in $(L^2(0,T;H))^2$, defined such that $\mathcal{F}(\mathcal{C})\subset\mathcal{C}$. Indeed, let us check that $\mathcal{C}$ is closed. Let $(h_n,f_n)_n$ be a sequence in $\mathcal{C}$ such that $(h_n,f_n)\to(h,f)$ in $L^2(\Omega_T)$. Since the sequence $\big(h_n\big)_n$ is uniformly bounded in the space $W(0,T)$, we can extract a subsequence $(h_{n_k})_k$ converging weakly in $W(0,T)$ to some limit denoted by $\bar h$. Then $h=\bar h\in W(0,T)$ and $\|h\|_{W(0,T)}\le\liminf_{k\to\infty}\|h_{n_k}\|_{W(0,T)}$. Similarly, since the sequence $\big(f_n\big)_n$ is uniformly bounded in the space $L^2(0,T;H^1(\Omega))$, there exists a subsequence such that $\nabla f_{n_k}\rightharpoonup\nabla f$ weakly in $L^2(\Omega_T)$ and $\|f\|_{L^2(0,T;H^1(\Omega))}\le\liminf_{k\to\infty}\|f_{n_k}\|_{L^2(0,T;H^1(\Omega))}$. Since $\mathcal{C}$ is also a bounded set in $W(0,T))\times L^2(0,T;H^1(\Omega))$, we also proved that $\mathcal{F}$ restricted to $\mathcal{C}$ is sequentially continuous in $(L^2(0,T;H))^2$. For the fixed point strategy, it remains to show the compactness of $\mathcal{F}(\mathcal{C})$. Since we work in metric spaces, proving its sequential compactness is sufficient. The compactness of $\mathcal{F}_1(\mathcal{C})$ is straightforward due to the Aubin's theorem. Let us further detail the proof for $\mathcal{F}_2(\mathcal{C})$. Let $\{f_n\}$ be a sequence in $\mathcal{F}_2(\mathcal{C})$. It is associated with a sequence $\{(\bar h_n,\bar f_n\}$ in $\mathcal{C}$. The Aubin's compactness theorem let us ensure that there exists a subsequence, not renamed for convenience, and $\bar{h} \in W(0,T) + h_D$ such that $\bar{h}_n \to \bar{h}$ in $L^2(0,T;H)$ and almost everywhere in Ω_T. Thus we can follow the lines beginning just after (3.17) for proving that $f_n \to f$ in $L^2(0,T;H)$. The sequential compactness of $\mathcal{F}_2(\mathcal{C})$ in $L^2(0,T;H)$ is proven.

We now have the tools for using the Schauder's fixed point theorem [13, Corollary 3.6]. There exists $(h-h_D, f-f_D) \in \mathcal{C}$ such that $\mathcal{F}(h,f) = (h,f)$. Then (h,f) is a weak solution of problem (3.1)–(3.4).

Step 2: Maximum Principles. We are going to prove that for almost every $x \in \Omega$ and for all $t \in (0,T)$,

$$\delta_1 \le h(t,x) \le h_2.$$

First show that $h(t,x) \le h_2$ a.e. $x \in \Omega$ and for all $t \in (0,T)$. We set

$$h_m = \big(h - h_2\big)^+ = \sup(0, h-h_2) \in L^2(0,T;V).$$

It satisfies $\nabla h_m = \chi_{\{h>h_2\}} \nabla h$ and $h_m(t,x) \neq 0$ if and only if $h(t,x) > h_2$, where χ denotes the characteristic function. Let $\tau \in (0,T)$. Taking $w(t,x) = h_m(t,x)\chi_{(0,\tau)}(t)$ in (3.1) yields:

$$\begin{aligned}
&\int_0^\tau \phi\langle \partial_t h, h_m\chi_{(0,\tau)}\rangle_{V',V}\,dt + \int_0^\tau\int_\Omega \delta\nabla h\cdot\nabla h_m\,dx\,dt\\
&+\int_0^\tau\int_\Omega KT_s(h)\nabla h\cdot\nabla h_m\,dx\,dt + \int_0^\tau\int_\Omega KT_s(h)\nabla f\cdot\nabla h_m\,dx\,dt\\
&+\int_0^\tau\int_\Omega Q_sT_s(h)h_m\,dx\,dt = 0;
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_0^\tau \phi\langle \partial_t h, h_m\rangle_{V',V}\,dt + \int_0^\tau\int_\Omega \delta\chi_{\{h>h_2\}}|\nabla h|^2\,dx\,dt\\
&+\int_0^\tau\int_\Omega KT_s(h)\chi_{\{h>h_2\}}|\nabla h|^2\,dx\,dt + \int_0^\tau\int_\Omega KT_s(h)\nabla f\cdot\nabla h_m(x,t)\,dx\,dt\\
&+\int_0^\tau\int_\Omega Q_sT_s(h)h_m(x,t)\,dx\,dt = 0.
\end{aligned} \tag{3.26}$$

To evaluate the first term in the left-hand side of above equation, we apply Lemma 2.1 with function f defined by $f(\lambda) = \lambda - h_2$, $\lambda \in \mathbb{R}$. We set $W_1(0,T) := W(0,T) \times L^2(0,T;H^1(\Omega))$. We write

$$\int_0^\tau \phi\langle \partial_t h, h_m\rangle_{V',V}\,dt = \frac{\phi}{2}\int_\Omega \Big(h_m^2(\tau,x) - h_m^2(0,x)\Big)\,dx = \frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx,$$

since $h_m(0,\cdot) = \big(h_0(\cdot) - h_2(\cdot)\big)^+ = 0$. Moreover since $T_s(h)\chi_{\{h>h_2\}} = 0$ by definition of T_s, the three last terms in the left-hand side of (3.26) are null. Hence (3.26) becomes

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx \le -\int_0^\tau\int_\Omega \delta\chi_{\{h>h_2\}}|\nabla h|^2\,dx\,dt \le 0.$$

Then $h_m = 0$ a.e. in Ω_T.

Now we claim that $\delta_1 \le h(t,x)$ a.e. $x \in \Omega$ and for all $t \in (0,T)$. We set

$$h_m = \big(h - \delta_1\big)^- \in L^2(0,T;V).$$

Let $\tau \in (0,T)$. We recall that $h_m(0,\cdot) = 0$ a.e. in Ω by the maximum principle satisfied by the initial data h_0. Moreover, $\nabla(h-\delta_1)\cdot\nabla h_m = \chi_{\{\delta_1 - h>0\}}|\nabla(h-\delta_1)|^2$. Thus, taking $w(t,x) = h_m(x,t)\chi_{(0,\tau)}(t)$ in (3.1) and $w(t,x) = \frac{h_2-\delta_1}{h_2} h_m(x,t)\chi_{(0,\tau)}(t)$ in (3.2) and adding the two equations gives

$$
\begin{aligned}
&\int_0^\tau \phi\langle \partial_t h, h_m(x,t)\rangle_{V',V} dt + \int_{\Omega_\tau} (\delta + KT_s(h))\nabla h\cdot\nabla h_m \,dx\,dt \\
&- \int_{\Omega_\tau} KT_s(h)\nabla f\cdot\nabla h_m\,dx\,dt + \int_{\Omega_T} (h_2-\delta_1)K\nabla f\cdot\nabla h_m\,dx\,dt \\
&- \int_{\Omega_\tau} T_s(h)\frac{h_2-\delta_1}{h_2}K\nabla h\cdot\nabla h_m\,dx\,dt \\
&+ \int_{\Omega_\tau} \Big(Q_sT_s(h)\big(1-\frac{h_2-\delta_1}{h_2}\big)h_m - Q_fT_f(h)\frac{h_2-\delta_1}{h_2}h_m\Big)\,dx\,dt = 0.
\end{aligned}
$$

By definition of $T_s(h)$, $T_s(h)\chi_{\{h<\delta_1\}} = h_2-\delta_1$, we can simplify the above equation as follows

$$
\begin{aligned}
&\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)dx + \int_{\Omega_\tau} \chi_{\{h<\delta_1\}}\delta\nabla h\cdot\nabla h\,dx\,dt \\
&+ \int_{\Omega_\tau} Q_fT_f(h)\chi_{\{h<\delta_1\}}(\delta_1-h)\frac{h_2-\delta_1}{h_2}\,dx\,dt \\
&+ \int_{\Omega_\tau} (h_2-\delta_1)\big(1-\frac{h_2-\delta_1}{h_2}\big)\chi_{\{h<\delta_1\}}K\nabla h\cdot\nabla h\,dx\,dt \\
&+ \int_{\Omega_\tau} \chi_{\{h<\delta_1\}}(h-\delta_1)Q_s(h_2-\delta_1)\big(1-\frac{h_2-\delta_1}{h_2}\big)\,dx\,dt = 0.
\end{aligned}
$$

We first note that $1-(h_2-\delta_1)/h_2 \geq 0$. Since moreover $Q_f \geq 0$ and $Q_s \leq 0$, the previous equation leads to

$$
\frac{1}{2}\int_\Omega h_m^2(\tau,x)dx \leq 0
$$

and then $h_m = 0$ a.e. in Ω_T.

4. Regularity result

Thanks to a generalization of the Meyer's regularity result given in section 2, we establish that the gradient of the solution belongs to the space $(L^r(\Omega_T))^2$, for some $r>2$. We remind that the exponent r depends only on coefficients (α,β) determined by the elliptic operator A. We are going to precise this dependency with respect to the physical parameters. In our particular case, the tensor A defined in Lemma 2.2 is equal to K. Then it is symmetric and $\alpha = K_-, \beta = K_+$ and $g(r) = \|(\Delta)^{-1}\|_{L(W^{-1,r}(\Omega),W_0^{1,r}(\Omega))}$. Thus we have, for any real number $c>0$,

$$
\mu = \frac{K_-+c}{K_++c} \quad\text{and}\quad \nu = \frac{c}{(K_++c)}, \tag{4.1}
$$

the positivity of c ensuring $\nu<\mu$. If $r>2$ is such that $k(r) := g(r)(1-\mu+\nu) < 1$, then the exponent r is appropriate. Conversely, being given $r>2$, we can always adjust K_- and K_+ so that $k(r) = g(r)(1-\mu+\nu)<1$ (since tensor K is assumed to be symmetric). Let us detail this part. We take $c>0$ and we fix $r>2$. We choose the physical parameters (K_-,K_+) is in the following way:

$$
g(r)\big(1-\frac{K_-}{K_++c}\big) < 1,\ \forall c>0 \iff \big(K_++c\big) < \frac{g(r)}{g(r)-1}K_-,\ \forall c>0.
$$

Letting $c \to 0$, we obtain

$$K_+ < \frac{g(r)}{g(r)-1} K_-. \tag{4.2}$$

The condition (4.2) implies a low spatial heterogeneity for the hydraulic conductivity tensor, so as the assumption (3.5).

Concerning the parabolic equation, the tensor A defined in Lemma 2.4 is equal to $(\delta \mathcal{I}d + T_l(\bar{h})K)$, then it is symmetric. Thereby, we have $\alpha = \frac{\delta}{\phi}$, $\beta = \frac{\delta}{\phi} + K_+ \frac{(h_2-\delta_1)}{\phi}$ and $\hat{g}(r) = \|P^{-1}\|_{L(Y_r,X_r)}$. It follows that

$$\begin{gathered}
\hat{\mu} = \frac{\alpha + \hat{c}}{\beta + \hat{c}} = \frac{\delta + \phi\hat{c}}{\delta + \phi\hat{c} + K_+(h_2 - \delta_1)} \quad \text{and} \\
\hat{\nu} = \frac{\hat{c}}{(\beta + \hat{c})} = \frac{\phi\hat{c}}{\delta + \phi\hat{c} + K_+(h_2 - \delta_1)}, \quad \forall \hat{c} > 0.
\end{gathered} \tag{4.3}$$

Since $\hat{c} > 0$, we obtain $\hat{\nu} < \hat{\mu}$. If $r \geq 2$ is such that $\hat{k}(r) := \hat{g}(r)(1 - \hat{\mu} + \hat{\nu}) < 1$, the exponent r is appropriate.

As previously, let $r > 2$, we can always adjust h_2, δ_1, K_+ and δ such that $\hat{k}(r) = \hat{g}(r)(1 - \hat{\mu} + \hat{\nu}) < 1$. Namely, we impose

$$\begin{gathered}
\big(\delta + K_+(h_2 - \delta_1) + \phi\hat{c}\big) < \frac{\hat{g}(r)}{\hat{g}(r) - 1}\delta, \ \forall \hat{c} > 0 \\
\iff K_+ < \frac{1}{\hat{g}(r) - 1} \times \frac{\delta}{h_2 - \delta_1} - \frac{\phi\hat{c}}{h_2 - \delta_1}, \ \forall \hat{c} > 0.
\end{gathered}$$

Letting $\hat{c} \to 0$, we obtain the following limitation for hydraulic conductivity inside the aquifer,

$$\frac{K_+(h_2 - \delta_1)}{\delta} < \frac{1}{\hat{g}(r) - 1}. \tag{4.4}$$

Let $r_1(K_-, K_+) > 2$ be the biggest real number such that $g(r_1)(1 - \mu - \nu) < 1$ where μ and ν are defined by (4.1) and we denote by $r_2(\delta, \delta_1, h_2, K_+) > 2$ the biggest real number such that $\hat{g}(r_2)(1 - \hat{\mu} - \hat{\nu}) < 1$ where $\hat{\mu}$ and $\hat{\nu}$ are defined by (4.3). We set

$$r(\delta, \delta_1, h_2, K_-, K_+) = Inf(r_1(K_-, K_+), r_2(\delta, \delta_1, h_2, K_+)). \tag{4.5}$$

Proposition 4.1. *Let (h, f) be a solution of (3.1)–(3.4) and $r(\delta, \delta_1, h_2, K_-, K_+) > 2$ be the real number determined by (4.5). Furthermore we assume that there exists $\gamma, 0 < \gamma < 1$, such that the physical parameters satisfy (3.5) and*

$$\frac{K_+(h_2 - \delta_1)}{\delta} \leq (1 - \gamma) \times \frac{(1 - k(r))}{g(r)} \times \frac{(1 - \hat{k}(r))}{\hat{g}(r)} \times \frac{h_2}{h_2 - \delta_1}. \tag{4.6}$$

If $(h_D, f_D) \in L^r(0, T; W^{1,r}(\Omega))^2$, $\partial_t h_D \in L^r(0, T; W^{-1,r}(\Omega))$, $h_0 \in W^{-1,2}(\Omega)$ and $(Q_s, Q_f) \in L^r(\Omega_T)^2$, then ∇h and ∇h_1 are in $(L^r(\Omega_T))^2$. Moreover, we have

$$\|\nabla h\|_{(L^r(\Omega_T))^2} \leq C_1(\phi, h_2, h_0, h_D, f_D, Q_s, Q_f, K_-, K_+, \delta, \delta_1), \tag{4.7}$$

$$\|\nabla f\|_{(L^r(\Omega_T))^2} \leq C_2(\phi, h_2, h_0, h_D, f_D, Q_s, Q_f, K_-, K_+, \delta, \delta_1). \tag{4.8}$$

Proof. We turn back to the construction of the solution in Step 1 of Theorem 3.1. We recall that this solution appears as the fixed point of an application. In the following lines, we thus give the tools for incorporating the $L^r(\Omega_T)$, $r > 2$, regularity result in this construction process.

We set $W_1(0,T) := X(0,T) \times L^2(0,T;H^1(\Omega))$. Let (M', M'') be two strictly positive real numbers that we will define later. We set

$$\begin{aligned}\widetilde{W} = \Big\{&(g,g_1) \in \big(W_1(0,T) \cap \big(L^r(0,T;W^{1,r}(\Omega))\big)^2 : g(0) = h_0,\\ &(g|_\Gamma, g_1|_\Gamma) = (h_D, f_D),\ \|(g;g_1)\|_{W_1(0,T)} \le K_M,\\ &\|\nabla g_1\|_{(L^r(\Omega_T))^2} \le M'; \|\nabla g\|_{(L^r(\Omega_T))^2} \le M''\Big\},\end{aligned} \tag{4.9}$$

where K_M depends on the constants C_0 and D_M defined in (3.25). Our goal is to check that the application $\mathcal{F}$ defined in the first step of Theorem 3.1 satisfies $\mathcal{F}(\widetilde{W}) \subset \widetilde{W}$. Applying Lemma 2.4 to (3.6), we deduce that

$$\begin{aligned}&\|\nabla h\|_{(L^r(\Omega_T))^2}\\ &\le \frac{\hat{g}(r)}{(1-\hat{k}(r))(\hat{\beta}+\hat{c})}\Big\{\frac{(h_2-\delta_1)}{\phi}\big(K_+\|\nabla \bar{f}\|_{(L^r(\Omega_T))^2} + \|Q_s\|_{L^r(\Omega_T)}\big)\\ &\quad + \|\partial_t h_D\|_{L^r(0,T;W^{-1,r}(\Omega))}\\ &\quad + \frac{1}{\phi}\|h_0\|_{W^{-1,2}(\Omega)} + \frac{\delta + K_+(h_2-\delta_1)}{\phi}\|\nabla h_D\|_{(L^r(\Omega_T))^2}\Big\}.\end{aligned} \tag{4.10}$$

In the same way, applying Lemma 2.2 to (3.7), we obtain

$$\begin{aligned}&h_2\|\nabla f\|_{(L^r(\Omega_T))^2}\\ &\le \frac{g(r)}{(1-k(r))(\beta+c)}\Big((h_2-\delta_1)\big(K_+\|\nabla h\|_{(L^r(\Omega_T))^2} + \|Q_s\|_{L^r(\Omega_T)}\big)\\ &\quad + h_2\|Q_f\|_{L^r(\Omega_T)} + h_2\|\nabla f_D\|_{(L^r(\Omega_T))^2}\Big).\end{aligned} \tag{4.11}$$

So, taking into account (4.10), we infer from (4.11) that

$$\begin{aligned}&\|\nabla f\|_{(L^r(\Omega_T))^2}\\ &\le \frac{g(r)}{(1-k(r))(\beta+c)} \times \frac{\hat{g}(r)}{(1-\hat{k}(r))(\hat{\beta}+\hat{c})} \times \frac{(h_2-\delta_1)^2K_+^2}{\phi h_2}\|\nabla \bar{f}\|_{(L^r(\Omega_T))^2}\\ &\quad + \frac{g(r)}{h_2(1-k(r))(\beta+c)}\Big\{\Big(\frac{(h_2-\delta_1)^2 \times \hat{g}(r)K_+}{\phi(1-\hat{k}(r))(\hat{\beta}+\hat{c})} + (h_2-\delta_1)\Big)\|Q_s\|_{L^r(\Omega_T)}\\ &\quad + \frac{(h_2-\delta_1)\hat{g}(r)K_+}{(1-\hat{k}(r))(\hat{\beta}+\hat{c})}\Big(\frac{1}{\phi}\|h_0\|_{W^{1,r}(\Omega)} + \frac{\delta+K_+(h_2-\delta_1)}{\phi}\|\nabla h_D\|_{(L^r(\Omega_T))^2}\\ &\quad + \|\partial_t h_D\|_{L^r(0,T;W^{-1,r}(\Omega))}\Big)\\ &\quad + \Big(h_2\|Q_f\|_{L^{r'}(\Omega_T)} + \|\nabla f_D\|_{(L^r(\Omega_T))^2}\Big)\Big\}.\end{aligned} \tag{4.12}$$

Let γ be such that $0 < \gamma < 1$ and assume that ϕ, h_2, K_-, K_+, δ and δ_1 satisfy, for any positive real numbers c and $\hat{c}$,

$$\frac{g(r)}{(1-k(r))(\beta+c)} \times \frac{\hat{g}(r)}{(1-\hat{k}(r))(\hat{\beta}+\hat{c})}\frac{(h_2-\delta_1)^2K_+^2}{\phi h_2} \le 1-\gamma. \tag{4.13}$$

Using $\beta = K_+$ and $\hat\beta = \frac{\delta}{\phi} + K_+\frac{(h_2-\delta_1)}{\phi}$, it is easy to check that assumption (4.6) implies (4.13), indeed (4.6) holds if and only if

$$K_+ \le \underbrace{(1-\gamma)\times\frac{(1-k(r))}{g(r)}\times\frac{(1-\hat k(r))}{\hat g(r)}}_{\eta<1}\times\frac{h_2}{h_2-\delta_1}\times\frac{\delta}{(h_2-\delta_1)}$$

which in turn implies

$$\begin{aligned}
&K_+ \le (1-\gamma)\times\frac{(1-k(r))}{g(r)}\times\frac{(1-\hat k(r))}{\hat g(r)}\times\frac{h_2}{h_2-\delta_1-\eta h_2}\times\frac{\delta}{(h_2-\delta_1)}\\
&\Longrightarrow K_+\big(1-\eta\frac{h_2}{h_2-\delta 1}\big) \le (1-\gamma)\times\frac{(1-k(r))}{g(r)}\times\frac{(1-\hat k(r))}{\hat g(r)}\times\frac{\delta h_2}{(h_2-\delta_1)^2}\\
&\Longrightarrow K_+ \le (1-\gamma)\times\frac{(1-k(r))}{g(r)}\times\frac{(1-\hat k(r))}{\hat g(r)}\times\frac{h_2\big(\delta+K_+(h_2-\delta_1)\big)}{(h_2-\delta_1)^2} \Longrightarrow (4.13).
\end{aligned}$$

The constant M' is chosen such that the initial and boundary conditions and the source terms satisfy

$$\begin{aligned}
&\frac{1}{\gamma}\times\frac{g(r)}{h_2(1-k(r))(\beta+c)}\Big\{\frac{(h_2-\delta_1)\hat g(r)K_+}{(1-\hat k(r))(\hat\beta+\hat c)}\Big(\frac{(h_2-\delta_1)}{\phi}\|Q_s\|_{L^r(\Omega_T)}\\
&+\frac{1}{\phi}\|h_0\|_{W^{1,r}(\Omega)}+\frac{\delta+K_+(h_2-\delta_1)}{\phi}\|\nabla h_D\|_{(L^r(\Omega_T))^2}\\
&+\|\partial_t h_D\|_{L^r(0,T;W^{-1,r}(\Omega))}\Big)\\
&+\Big((h_2-\delta_1)\|Q_s\|_{L^r(\Omega_T)}+h_2\|Q_f\|_{L^r(\Omega_T)}+h_2\|\nabla f_D\|_{L^r(\Omega_T)^2}\Big)\Big\}\le M'.
\end{aligned}\tag{4.14}$$

Considering (4.12), (4.13) and (4.14), we deduce that

$$\|\nabla f\|_{(L^r(\Omega_T))^2}\le M',$$

and

$$\begin{aligned}
&\|\nabla h\|_{(L^r(\Omega_T))^2}\\
&\le M'' := \frac{\hat g(r)}{(1-\hat k(r))(\hat\beta-\hat c)}\Big(\frac{(h_2-\delta_1)}{\phi}\big(K_+M'+\|Q_s\|_{L^r(\Omega_T)}\\
&\quad+\|\partial_t h_D\|_{L^r(0,T;W^{-1,r}(\Omega))}\big)+\frac{1}{\phi}\|h_0\|_{W^{1,r}(\Omega)}\\
&\quad+\frac{\delta+K_+(h_2-\delta_1)}{\phi}\|\nabla h_D\|_{(L^r(\Omega_T))^2}\Big).
\end{aligned}\tag{4.15}$$

Let $\widetilde W$ be the bounded closed convex defined by (4.9), (4.14) and (4.15). Indeed, let us check that $\mathcal D$ is closed. Let $(h_n,f_n)_n$ be a sequence of $\mathcal D$ such that $(h_n,f_n)\to(h,f)$ in $L^2(\Omega_T)$. We know that $\nabla h_{n_k}\rightharpoonup\nabla h$ weakly in $L^2(\Omega_T)$ and $\nabla f_{n_k}\rightharpoonup\nabla f$ weakly in $L^2(\Omega_T)$. Since the sequence $\big(\nabla h_{n_k},\nabla f_{n_k}\big)_k$ is uniformly bounded in the space $(L^r(\Omega_T))^2$ with $r>2$, then $(\nabla h,\nabla f)\in(L^r(\Omega_T))^2$, moreover $\|\nabla h\|_{(L^r(\Omega_T))^2}\le M''$ and $\|\nabla f\|_{(L^r(\Omega_T))^2}\le M'$. We proved that $\mathcal F(\widetilde W)\subset\widetilde W$. It follows from the Schauder theorem that there exist $(\tilde h,\tilde f)\in\widetilde W$ such that $\mathcal F(\tilde h,\tilde f)=(\tilde h,\tilde f)$. This fixed point of $\mathcal F$ is a weak solution of the truncated problem. The proof of the maximum principle then remains of course unchanged. $\square$

5. Uniqueness

We are now able to establish the result of uniqueness yielding the wellposedness of the problem in the space $W(0,T)$.

Theorem 5.1. *Let $(h_2, K_-, K_+, \delta, \delta_1) \in (\mathbb{R}_*^+)^5$ satisfying (3.5) and (4.2), (4.4) and (4.6) for $r = 4$. If $h_0 \in W^{1,4}(\Omega)$, $(h_D, f_D) \in L^4(0,T; W^{1,4}(\Omega))^2$ and $(Q_s, Q_f) \in L^4(\Omega_T)^2$, then the solution of the system (3.1)-(3.4) is unique in $\big(W(0,T) + h_D\big) \times \big(L^2(0,T; H^1(\Omega)) + f_D\big)$.*

Remark 5.2. Assumption (4.6) is stronger than (4.4) except when the thickness of freshwater zone inside the aquifer, δ_1, is sufficiently large.

Proof of Theorem 5.1. Let (h,f) and $(\bar{h},\bar{f})$ be two solutions of (3.1)-(3.4). Setting $u = h - \bar{h} \in W(0,T)$ and $v = f - \bar{f} \in L^2(0,T;V)$, (u,v) is a solution of the system

$$\begin{gathered}
\phi\partial_t u - \nabla\cdot(\delta + KT_s(\bar{h}))\nabla u - \nabla\cdot(K(T_s(h) - T_s(\bar{h}))\nabla h) \\
+ \nabla\cdot(K(T_s(h) - T_s(\bar{h}))\nabla f) + \nabla\cdot((KT_s(\bar{h})\nabla v) = 0, \\
-h_2\nabla\cdot(K\nabla v) + \nabla\cdot(K(T_s(h) - T_s(\bar{h}))\nabla h) + \nabla\cdot(KT_s(\bar{h})\nabla u) = 0.
\end{gathered}$$

We point out that all the estimates previously established for time T, are valid for any $t \le T$. Furthermore $h, \bar{h} \in [\delta_1, h_2]$, thus $T_s(h) - T_s(\bar{h}) = \bar{h} - h = -u$ and the previous system can be simplified as follows:

$$\begin{gathered}
\phi\partial_t u - \nabla\cdot(\delta + KT_s(\bar{h}))\nabla u + \nabla\cdot(Ku\nabla h) - \nabla\cdot(Ku\nabla f) + \nabla\cdot((KT_s(\bar{h})\nabla v) = 0, \\
-h_2\nabla\cdot(K\nabla v) - \nabla\cdot(Ku\nabla h) + \nabla\cdot(KT_s(\bar{h})\nabla u) = 0.
\end{gathered}$$

Let $t \in [0,T]$. Using the variational formulation of two latter equations in $\Omega_t := (0,t)\times\Omega$, we obtain, for any $(w_1, w_2) \in (W(0,T))^2$:

$$\begin{aligned}
&\phi\int_{\Omega_t} \partial_t u w_1\,dx\,ds + \int_{\Omega_t}\Big((\delta + KT_s(\bar{h}))\nabla u\cdot\nabla w_1 - Ku\nabla h\cdot\nabla w_1 + Ku\nabla f\cdot\nabla w_1 \\
&- KT_s(\bar{h})\nabla v\cdot\nabla w_1\Big)\,dx\,ds = 0, \\
&h_2\int_{\Omega_t} K\nabla v\cdot\nabla w_2\,dx\,ds + \int_{\Omega_t} Ku\nabla f\cdot\nabla w_2\,dx\,ds - \int_{\Omega_t} KT_s(\bar{h})\nabla v\cdot\nabla w_2\,dx\,ds = 0.
\end{aligned}$$

Taking $w_1 = u$ and $w_2 = v$, since $u(t=0,.) = 0$ a.e. on Ω, we obtain after summation of the two previous equations

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega u^2(t,x)\,dx + \int_{\Omega_t}(\delta + KT_s(\bar{h}))\nabla u\cdot\nabla u\,dx\,ds + h_2\int_{\Omega_t} K\nabla v\cdot\nabla v\,dx\,ds \\
&- 2\int_{\Omega_t} KT_s(\bar{h})\nabla v\cdot\nabla u\,dx\,ds + \int_{\Omega_t} Ku\nabla f\cdot\nabla u\,dx\,ds \\
&+ \int_{\Omega_t} Ku\nabla h\cdot\nabla(v - u)\,dx\,ds = 0;
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{\phi}{2}\int_\Omega u^2(t,x)\,dx + \int_{\Omega_t}\delta\nabla u^2\,dx\,ds + \int_{\Omega_t} KT_s(\bar{h})\nabla(u - v)\cdot\nabla(u - v)\,dx\,ds \\
&+ \int_{\Omega_t}\bar{h}K\nabla v\cdot\nabla v\,dx\,ds + \int_{\Omega_t} Ku\nabla(f - h)\cdot\nabla u\,dx\,ds + \int_{\Omega_t} Ku\nabla h\cdot\nabla v\,dx\,ds = 0.
\end{aligned}$$

Thanks to the definition of $T_s(\bar{h})$, we obtain

$$0 \leq \int_{\Omega_t} KT_s(\bar{h})\nabla(u-v)\cdot\nabla(u-v)\,dx\,ds$$

and furthermore, since $\bar{h} \in [\delta_1, h_2]$,

$$\delta_1 K_- \int_{\Omega_t} |\nabla v|^2\,dx\,ds \leq \int_{\Omega_t} \bar{h}K\nabla v\cdot\nabla v\,dx\,ds.$$

We have also

$$\begin{aligned} &\Big|\int_0^t \int_\Omega Ku\nabla(f-h)\cdot\nabla u\,dx\,ds\Big| \\ &\leq \int_0^t K_+\big(\int_\Omega u^4\,dx\big)^{1/4}\big(\int_\Omega |\nabla(f-h)|^4\,dx\big)^{1/4}\big(\int_\Omega |\nabla u|^2\,dx\big)^{1/2}\,ds. \end{aligned} \tag{5.1}$$

Using Proposition 4.1 for $r=4$ leads to

$$\Big(\int_{\Omega_T} |\nabla h|^4\,dx\,dt\Big)^{1/4} \leq C_{4,1} \quad \text{and} \quad \Big(\int_{\Omega_T} |\nabla f|^4\,dx\,dt\Big)^{1/4} \leq C_{4,2},$$

hence

$$\Big(\int_{\Omega_T} |\nabla(f-h)|^4\,dx\,dt\Big)^{1/4} \leq C_{4,1}+C_{4,2} := C_4.$$

Also the Gagliardo-Nirenberg inequality for $r=4$ can be written as

$$\Big(\int_\Omega |u|^4\,dx\Big)^{1/4} \leq C_G\|u\|_{L^2(\Omega)}^{1/2}\|\nabla u\|_{(L^2(\Omega))^2}^{1/2}.$$

Combining Gagliardo-Nirenberg and Young inequalities applied to (5.1), we obtain

$$\begin{aligned} &\Big|\int_{\Omega_t} Ku\nabla(f-h)\cdot\nabla u\,dx\,ds\Big| \\ &\leq K_+\Big(\int_0^t \|u\|_{L^2(\Omega)}^2\|\nabla u\|_{(L^2(\Omega))^2}^2 dt\Big)^{1/4}\Big(\int_{\Omega_T} |\nabla(f-h)|^4\,dx\,ds\Big)^{1/4} \\ &\quad\times\Big(\int_{\Omega_t} |\nabla u|^2\,dx\,ds\Big)^{1/2} \\ &\leq K_+C_GC_4 \max_{t\in(0,t)} \|u\|_{L^2(\Omega)}^{1/2}\Big(\int_{\Omega_t} |\nabla u|^2\,dx\,ds\Big)^{3/4} \\ &\leq K_+C_GC_4\Big(\frac{1}{8}\epsilon_1^{-3}\max_{t\in(0,t)} \|u\|_{L^2(\Omega)}^2 + 2\epsilon_1\int_{\Omega_t} |\nabla u|^2\Big), \quad \epsilon_1>0. \end{aligned}$$

As the same time,

$$\begin{aligned} &\Big|\int_0^t \int_{\Omega_t} Ku\nabla h\cdot\nabla v\,dx\,ds\Big| \\ &\leq \int_0^t K_+\big(\int_\Omega u^4\,dx\big)^{1/4}\Big(\int_\Omega |\nabla h|^4\,dx\Big)^{1/4}\Big(\int_\Omega |\nabla v|^2\,dx\Big)^{1/2}dt \\ &\leq K_+C_G\int_0^t \|u\|_{L^2(\Omega)}^{1/2}\|\nabla u\|_{(L^2(\Omega))^2}^{1/2}\Big(\int_\Omega |\nabla h|^4\Big)^{1/4}\Big(\int_\Omega |\nabla v|^2\,dx\Big)^{1/2}ds \\ &\leq K_+C_G\Big(\int_0^t \|u\|_{L^2(\Omega)}^2\|\nabla u\|_{(L^2(\Omega))^2}^2\,ds\Big)^{1/4}\Big(\int_{\Omega_T} |\nabla h|^4\,dx\,ds\Big)^{1/4} \end{aligned}$$

$$
\begin{aligned}
&\quad \times \Big(\int_{\Omega_t} |\nabla v|^2\,dx\,ds\Big)^{1/2}\\
&\le K_+ C_G C_{4,1} \max_{t\in(0,t)} \|u\|^{1/2}_{L^2(\Omega)} \Big(\int_{\Omega_t} \|\nabla u\|^2\,dx\,ds\Big)^{1/4}\Big(\int_{\Omega_t} |\nabla v|^2\,dx\,ds\Big)^{1/2}\\
&\le K_+ C_G C_{4,1}\Big(\frac{1}{2\epsilon_1}\max_{t\in(0,t)} \|u\|_{L^2(\Omega)}\Big(\int_{\Omega_t} |\nabla u|^2\,dx\,ds\Big)^{1/2} + \frac{\epsilon_1}{2}\int_{\Omega_t} |\nabla v|^2\,dx\,ds\Big)\\
&\le K_+ C_G C_{4,1}\Big(\frac{1}{16\epsilon_1^3}\max_{t\in(0,t)} \|u\|^2_{L^2(\Omega)} + \epsilon_1 \int_{\Omega_t} |\nabla u|^2\,dx\,ds\Big)\\
&\quad + \frac{K_+ C_G C_{4,1}\epsilon_1}{2}\int_{\Omega_t} |\nabla v|^2\,dx\,ds.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&\frac{\phi}{2}\int_\Omega u^2(t,x)\,dx + (\delta - K_+ C_G \epsilon_1(2C_4 + C_{4,1}))\int_{\Omega_t} |\nabla u|^2\,dx\,ds\\
&+ (\delta_1 K_- - \frac{K_+ C_G C_{4,1}}{2}\epsilon_1)\int_{\Omega_t} |\nabla v|^2\,dx\,ds\\
&\le \frac{K_+}{8\epsilon_1^3} C_G (C_4 + \frac{C_{4,1}}{2}) \max_{t\in(0,T)} \int_\Omega u^2(t,x)\,dx.
\end{aligned} \tag{5.2}
$$

Fix $\epsilon_1 > 0$ such that

$$
\delta - K_+\epsilon_1 C_G(2C_4 + C_{4,1}) > 0 \quad \text{and} \quad \delta_1 K_- - \frac{K_+ C_{4,1} C_G}{2}\epsilon_1 > 0.
$$

Hence, passing to the maximum for $t \in (0,T)$ on the left-hand side of (5.2), we obtain

$$
\frac{\phi}{2}\max_{t\in(0,T)} \int_\Omega u^2(t,x)\,dx \le \frac{K_+}{8\epsilon_1^3} C_G (C_4 + \frac{C_{4,1}}{2}) \max_{t\in(0,T)} \int_\Omega u^2(t,x)\,dx;
$$

that is

$$
(\frac{\phi}{2} - \frac{K_+}{8\epsilon_1^3} C_G (C_4 + \frac{C_{4,1}}{2})) \max_{t\in(0,T)} \int_\Omega u^2(t,x)\,dx \le 0. \tag{5.3}
$$

If ϕ satisfies

$$
\frac{\phi}{2} - \frac{K_+ C_G}{16\epsilon_1^3}(2C_4 + C_{4,1}) > 0, \quad \epsilon_1 < \inf\Big(\frac{2\delta_1 K_-}{K_+ C_{4,1} C_G}, \frac{\delta}{K_+(2C_4 + C_{4,1}), C_G)}\Big), \tag{5.4}
$$

the relation (5.3) implies that $\max_{t\in(0,T)}\int_\Omega u^2(t,x)\,dx = 0$ and so $u = 0$ a.e. in Ω_T.

This information combined with the inequality (5.2) yields $\int_{\Omega_T} |\nabla v|^2 dx\,dt = 0$. Since $v \in L^2(0,T; H_0^1(\Omega))$, we conclude that $v = 0$ a.e. in Ω_T.

Conditions (5.4) may look very restrictive. However, we can pick the coefficient ϕ arbitrary large (by introducing an appropriate time scaling), so that the conditions (5.4) can indeed be satisfied. Setting

$$
t_0 = \frac{T \times 8\epsilon_1^3}{K_+ C_G (2C_4 + C_{4,1})},
$$

we proved the uniqueness for the short time $t \in [0, t_0]$. But taking $t = t_0$ as new initial time, the uniqueness is obtained for all $t_0 \le t \le 2t_0$. Using this observation inductively, we derive the uniqueness on the whole range of study $[0,T]$. The proof of Theorem 5.1 is complete. $\square$

References

[1] M. Alfaro, P. Alifrangis; *Convergence of a mass conserving Allen-Cahn equation whose Lagrange multiplier is nonlocal and local*, Interfaces Free Bound., to appear.

[2] M. Alfaro, D. Hilhorst, M. Hiroshi; *Optimal interface width for the Allen-Cahn equation*, RIMS Kokyuroku, 1416 9 (2005), 148–160.

[3] A. Bensoussan, J. L. Lion, G. Papanicoulou; *Asymptotic analysis for periodic structure*, North-Holland, Amsterdam, 1978.

[4] J. Bear, A. H. D. Cheng, S. Sorek, D. Ouazar, I. Herrera; *Seawater intrusion in coastal aquifers: Concepts, Methods and Practices*, Kluwer Academic Pub, 1999.

[5] J. W. Cahn, J. E. Hilliard; *Free energy of non-uniform systems. I. Interfacial free energy*, J. Chem. Phys., 28 (1958), 258–267.

[6] C. Choquet, M. M. Diédhiou, C. Rosier; *Derivation of a Sharp-Diffuse Interfaces Model for Seawater Intrusion in a Free Aquifer. Numerical Simulations*, SIAM J. Appl. Math. 76 (2016), no. 1, 138-158.

[7] C. Choquet, J. Li, C. Rosier; *Global existence for seawater intrusion models: Comparison between sharp interface and sharp-diffuse interface approaches*, Electrn. J. Differential Equ., Vol. 2015 (2015), No. 126, pp. 1-27.

[8] G. Gagneux, M. Madaune-Tort; *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*. Mathématiques & Applications, 22, Springer, 1996.

[9] J. L. Lions, E. Magenes; *Problèmes aux limites non homogènes*, Vol. 1, Dunod, 1968.

[10] N. G. Meyers; *An Lp-estimate for the gradient of solution of second order elliptic divergence equations*, Ann. Sc. Norm. Sup. Pisa, Vol. 17 (1963), pp. 189-206.

[11] P. Ranjan, S. Kazama, M. Sawamoto; *Modeling of the dynamics of saltwater freshwater interface in coastal aquifers*, http://www.wrrc.dpri.kyotou.ac.jp/aphw/APHW2004/proceedings/OHS/56-OHS-A333/56-OHS-A333.pdf, (Mar. 10, 2006).

[12] J. Simon; *Compact sets in the space* $L^p(0,T,B)$, Ann. Mat. Pura Appl., vol. 146 (4) (1987), 65–96.

[13] E. Zeidler; *Nonlinear functional analysis and its applications*, Part 1, Springer Verlag, 1986.

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