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HOMOGENIZATION OF THE LANDAU-LIFSHITZ-GILBERT EQUATION IN A CONTRASTED COMPOSITE MEDIUM

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Abstract. We study the Landau-Lifshitz-Gilbert equation in a composite ferromagnetic medium made of two different materials with highly contrasted properties. Over the so-called matrix domain, the effective field, the demagnetizing field and the bulk anisotropy field are scaled with regard to a parameter ϵ representing the size of the matrix blocks. This scaling preserves the physics of the magnetization as ϵ tends to zero. Using homogenization theory, we derive the corresponding effective model. To this aim we use the concept of two-scale convergence together with a new homogenization procedure for handling with the nonlinear terms. More precisely, an appropriate dilation operator is applied in a embedded cells network, the network being constrained by the microscopic geometry. We prove that the less magnetic part of the medium contributes through additional memory terms in the effective field.

1. **Setting of the problem.** Heterogeneous media are commonly adopted for electromagnetic applications in many branches of industry and science due to their ability to be tailored to meet specific requirements. For the reduction of eddy current loss, medium to high frequency components for electrical and electronic devices are frequently composed of heterogeneous soft magnets. Examples are the Mn-Zn ferrites, widely used in power electronics for transformers and inductor cores, or the soft magnetic composites, very promising for high speed electrical machines (see for instance part 1 in [3]). All these materials are designed for having both good magnetic properties and a quite high macroscopic resistivity. Nevertheless the interpretation of the experimental data is very difficult because of their sensitivity to many error sources. The development of analytical models for the determination of the

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effective properties of heterogeneous materials has a long tradition. More recently, thanks to the progress in computing power, the modeling of the electromagnetic behavior of heterogeneous media has been also faced by numerical approaches. As a drawback, the numerical implementation leads to exorbitant computational burden when fine spatial discretization have to coexist with the macroscopic sample size.

In the general context of problems described by differential equations in finely periodic structures, homogenization techniques have been widely applied to determine effective properties. Anyway, such asymptotic processes are performant and quite straightforward when dealing with global physical phenomena, while they show some limits when local effects are not negligible, even if local correctors are considered.

The main objective of this paper is to perform a rigorous derivation of the homogenized Landau-Lifshitz-Gilbert (LLG) equation associated to a highly contrasted composite ferromagnetic material. This is a typical example where a nonlinear and multiscale problem leads to difficulties for the justification of the effective model. We develop a new method, based on the use of an appropriate sequence of embedded cells together with dilation operators. The homogenization of the LLG equation is seldom addressed. A layered ferromagnetic medium was considered in [14]. The effective behavior of the demagnetization field operator in periodically perforated domains is studied in [23] using the classical two-scale convergence method. Some nonlinear terms in the LLG model have the same structure than ones of the Ginzburg-Landau functional. We thus also mention [18], [21], [7] and the references therein.

The ferromagnetic medium is assumed to have two distinct components. The matrix part consists of disjoint blocks where the dynamics are slow, surrounded by a thin layer of another material with better magnetic properties. More precisely, the medium occupies the set $\Omega \subset \mathbb{R}^3$ which is assumed to be a bounded, two-connected domain with a periodic structure controlled by a parameter $\epsilon > 0$ which represents the size of each block of the matrix (see also Figure 1). The standard period is a cell Q consisting of a two-connected matrix block Q_m with external smooth boundary ∂Q_m, surrounded by a two-connected domain Q_f. The ϵ-composite medium consists of copies ϵQ covering Ω. We denote by $\partial\Omega$ the external (Lipschitz) boundary of Ω, by γ^ϵ the matrix boundary and by n and ν^ϵ the corresponding exterior normals. The exterior normal to Q_m will be denoted by ν. For any $\epsilon > 0$, we denote by Ω_m^ϵ the matrix part of the domain and by Ω_f^ϵ the other part, so that

$$\Omega_m^\epsilon = \Omega \cap \Big\{ \bigcup_{\xi \in \mathcal{A}} \epsilon(Q_m + \xi) \Big\}, \quad \Omega_f^\epsilon = \Omega \setminus \overline{\Omega_m^\epsilon}, \quad \Gamma^\epsilon = \Omega \cap \Big\{ \bigcup_{\xi \in \mathcal{A}} \epsilon(\partial Q_m + \xi) \Big\}$$

where $\mathcal{A}$ is an appropriate infinite lattice. We denote by $J = (0, T)$ the time interval of interest, $T > 0$. For the sake of the simplicity we assume $|Q| = 1$, more precisely

$$\overline{Q} = [-1/2; 1/2]^3.$$

Let us now describe the PDEs system modeling the behavior of the magnetization in such a medium. The magnetization vector $\mathbf{M}^\epsilon \in \mathbb{R}^3$ is in the form

$$\mathbf{M}^\epsilon = \chi_m^\epsilon m^\epsilon + \chi_f^\epsilon M^\epsilon$$

where χ_m^ϵ (resp. χ_f^ϵ) is the characteristic function of Ω_m^ϵ (resp. Ω_f^ϵ). The magnetization is associated with the nonconvex constraint

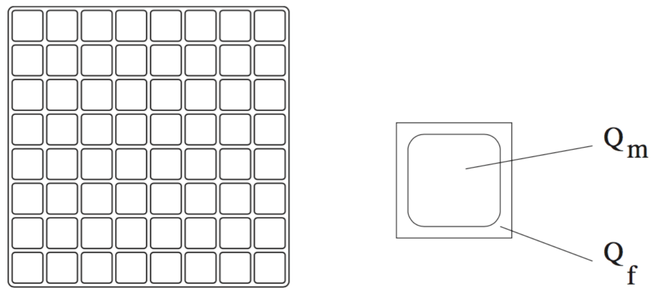

FIGURE 1. An example of periodic structure for the domain and the standard cell

$$|\mathbf{M}^\epsilon| = \begin{cases} 1 \text{ in } \Omega, \\ 0 \text{ in } \mathbb{R}^3 \setminus \Omega. \end{cases} \tag{1}$$

The time evolution of the magnetization vector may be described by the LLG equation ([2], [15]):

$$\frac{1}{1+\alpha^2}\Big(\partial_t \mathbf{M}^\epsilon - \alpha \mathbf{M}^\epsilon \times \partial_t \mathbf{M}^\epsilon\Big) = -\mathbf{M}^\epsilon \times \mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon) \quad \text{in } \Omega \times J. \tag{2}$$

The term parameterized by a factor α describes Gilbert damping torque and the right-hand side accounts for torque by the effective field $\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon)$ which is given by

$$\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon) = \operatorname{div}\big(A^\epsilon \nabla \mathbf{M}^\epsilon\big) + \phi_{va}^\epsilon(\mathbf{M}^\epsilon) + \mathcal{H}_d^\epsilon(\mathbf{M}^\epsilon). \tag{3}$$

Tensor A^ϵ satisfies

$$\chi_m^\epsilon(x) A^\epsilon(x) = \epsilon^2 \chi_m^\epsilon(x) A_m(x, x/\epsilon), \quad \chi_f^\epsilon(x) A^\epsilon(x) = \chi_f^\epsilon(x) A_f(x)$$

where $A_k = (A_{kij})_{1\le i,j\le 3}$, $k = m, f$, is a 3×3 symmetric, positive-definite matrix with coefficients valued in $\mathbb{R}^3$ and of class $\mathcal{C}^\infty(\overline{\Omega}) \otimes \mathcal{C}^\infty_\#(Q)$[1]. We assume that A_k, $k = m, f$, is uniformly coercive, *i.e.*, there exists a constant $A_- > 0$ such that, for any (x, y) in $\Omega \times Q$, for any $(\zeta_1, \zeta_2, \zeta_3)$ in $\mathbb{R}^3$,

$$\sum_{i,j=1}^{3} A_{kij}(x,y)\zeta_i\zeta_j \ge A_-(\sum_{i=1}^{3} \zeta_i^2) = A_-|\xi|^2.$$

We also assume that A_m is an admissible test function for the two-scale convergence (in the sense of [25]). The term ϕ_{va}^ϵ expresses the effects of the volume anisotropy energy. It reads

$$\phi_{va}^\epsilon(\mathbf{M}^\epsilon) = K_v^\epsilon\big(\mathbf{M}^\epsilon - (\mathbf{M}^\epsilon \cdot u)u\big)$$

where $K_v^\epsilon > 0$ is a scalar bounded function and the constant vector u is the direction of the easy magnetization axis. In what follows we thus simply assume that $\phi_{va}^\epsilon(\mathbf{M}^\epsilon) = \nabla\Lambda(\mathbf{M}^\epsilon)$ is a continuous gradient function such that $0 \le \Lambda(u) \le \Lambda_\infty$, $\Lambda_\infty \in \mathbb{R}_+$, for any $u \in \mathcal{S}^2$ and

$$\phi_{va}^\epsilon(\mathbf{M}^\epsilon) = \chi_m^\epsilon(x)\phi_{va,m}(x, x/\epsilon, m^\epsilon) + \chi_f^\epsilon(x)\phi_{va,f}(x, M^\epsilon),$$

[1]All along the paper we use the subscript $\#$ to specify that we deal with Q-periodic functions.

function $\phi_{va,m}$ being moreover periodic with regard to its second variable. In the magnetostatic approximation context ([5]), the demagnetizing field $\mathcal{H}_d^\epsilon(\mathbf{M}^\epsilon)$ satisfies, in $J \times \mathbb{R}^3$, the equation $\mathrm{curl}\mathcal{H}_d^\epsilon = 0$ and the stray field equation

$$\mathrm{div}\big(\mu^\epsilon \mathcal{H}_d^\epsilon + \chi_\Omega \mathbf{M}^\epsilon\big) = 0$$

where μ^ϵ is the permeability. Classical models keeps the latter equations or simply assume that $\mathcal{H}_d^\epsilon$ is some potential depending on $\mathbf{M}^\epsilon$. We consider both of these modelings by assuming

$$\begin{aligned}&\mathcal{H}_d^\epsilon = \nabla\Xi^\epsilon(\mathbf{M}^\epsilon) + \mathbf{H}^\epsilon\\ &\nabla\Xi^\epsilon(\mathbf{M}^\epsilon) = \epsilon\chi_m^\epsilon \nabla\Xi_m(m^\epsilon) + \chi_f^\epsilon \nabla\Xi_f(M^\epsilon)\end{aligned} \tag{4}$$

where each continuous gradient function satisfies $0 \leq \Xi_k(u) \leq \Xi_\infty$, $\Xi_\infty \in \mathbb{R}_+$, for any $u \in \mathcal{S}^2$, $k = f, m$, and where $\mathbf{H}^\epsilon = \epsilon\chi_m^\epsilon h^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega})H^\epsilon$ satisfies

$$\mathrm{curl}(\mathbf{H}^\epsilon) = 0, \tag{5}$$

$$\mathrm{div}\big(\epsilon\chi_m^\epsilon h^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega})H^\epsilon + \epsilon\chi_m^\epsilon m^\epsilon + \chi_f^\epsilon M^\epsilon\big) = 0. \tag{6}$$

For the sake of the simplicity, we have assumed a constant permeability. We shall consider a potential formulation of this problem. Indeed, due to (5), there exists scalar potentials p^ϵ and P^ϵ such that

$$\mathbf{H}^\epsilon = \epsilon\chi_m^\epsilon \nabla p^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega})\nabla P^\epsilon.$$

In view of (6), (p^ϵ, P^ϵ) is defined by

$$\mathrm{div}\big(\epsilon^2\chi_m^\epsilon \nabla p^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega})\nabla P^\epsilon + \epsilon\chi_m^\epsilon m^\epsilon + \chi_f^\epsilon M^\epsilon\big) = 0. \tag{7}$$

We complete the model with initial, boundary and transfer conditions. The initial data for the magnetization is

$$\mathbf{M}^\epsilon(0, x) = M_{init}(x), \ \ |M_{init}(x)|^2 = 1 \text{ a.e. in } \Omega. \tag{8}$$

The stay field equation (7) is completed by an initial condition

$$\epsilon^2\chi_m^\epsilon p^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega})P^\epsilon = P_{init} \text{ at } t = 0,$$

subject to the constraint

$$\Delta P_{init} + \mathrm{div}(\chi_\Omega M_{init}) = 0.$$

The external boundary condition is a no-flux type

$$\partial_{nA^\epsilon}\mathbf{M}^\epsilon = 0 \text{ on } J \times \partial\Omega. \tag{9}$$

At the interface Γ^ϵ between the two parts of the composite medium, we assume the continuity of the magnetization

$$m^\epsilon = M^\epsilon \text{ on } J \times \Gamma^\epsilon \tag{10}$$

and the conservation of the scaled fluxes across Γ^ϵ as follows

$$\epsilon^2 A_m^\epsilon \nabla m^\epsilon \cdot \nu^\epsilon = -A_f^\epsilon \nabla M^\epsilon \cdot \nu^\epsilon \quad \text{ on } J \times \Gamma^\epsilon, \tag{11}$$

$$p^\epsilon = P^\epsilon, \quad (\epsilon^2\nabla p^\epsilon + \epsilon m^\epsilon)\cdot\nu^\epsilon = -(\nabla P^\epsilon + M^\epsilon)\cdot\nu^\epsilon \quad \text{ on } J \times \Gamma^\epsilon. \tag{12}$$

Gathering all these elements, we finally get the following system

$$\left\{\begin{array}{l}
\frac{1}{1+\alpha^2}\Big(\partial_t \mathbf{M}^\epsilon - \alpha \mathbf{M}^\epsilon \times \partial_t \mathbf{M}^\epsilon\Big) = -\mathbf{M}^\epsilon \times \mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon) \quad \text{in } \Omega \times J,\\
\mathbf{M}^\epsilon = \chi_m^\epsilon m^\epsilon + \chi_f^\epsilon M^\epsilon, \ \ |\mathbf{M}^\epsilon| = 1,\\
\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon) = \operatorname{div}\big(A^\epsilon \nabla \mathbf{M}^\epsilon\big) + \phi_{va}^\epsilon(\mathbf{M}^\epsilon) + \mathcal{H}_d^\epsilon(\mathbf{M}^\epsilon),\\
\phi_{va}^\epsilon(\mathbf{M}^\epsilon) = \chi_m^\epsilon(x)\phi_{va,m}(x, x/\epsilon, m^\epsilon) + \chi_f^\epsilon(x)\phi_{va,f}(x, M^\epsilon),\\
\mathcal{H}_d^\epsilon(\mathbf{M}^\epsilon) = \nabla \Xi^\epsilon(\mathbf{M}^\epsilon) + \mathbf{H}^\epsilon,\\
\nabla \Xi^\epsilon(\mathbf{M}^\epsilon) = \epsilon \chi_m^\epsilon \nabla \Xi_m(m^\epsilon) + \chi_f^\epsilon \nabla \Xi_f(M^\epsilon),\\
\mathbf{H}^\epsilon = \epsilon \chi_m^\epsilon \nabla p^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega}) \nabla P^\epsilon,\\
\operatorname{div}\big(\epsilon^2 \chi_m^\epsilon \nabla p^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega})\nabla P^\epsilon + \epsilon \chi_m^\epsilon m^\epsilon + \chi_f^\epsilon M^\epsilon\big) = 0 \quad \text{in } \Omega \times J,\\
\mathbf{M}^\epsilon(0,x) = M_{init}(x) \text{ in } \Omega, \ \ |M_{init}(x)|^2 = 1 \text{ a.e. in } \Omega,\\
\epsilon^2 \chi_m^\epsilon p^\epsilon + (\chi_f^\epsilon + \chi_{\mathbb{R}^3\setminus\Omega}) P^\epsilon = P_{init} \text{ in } \Omega, \text{ at } t = 0,\\
\Delta P_{init} + \operatorname{div}(\chi_\Omega M_{init}) = 0,\\
\partial_{nA^\epsilon} \mathbf{M}^\epsilon = 0 \text{ on } J \times \partial\Omega,\\
m^\epsilon = M^\epsilon \text{ on } J \times \Gamma^\epsilon,\\
\epsilon^2 A_m^\epsilon \nabla m^\epsilon \cdot \nu^\epsilon = -A_f^\epsilon \nabla M^\epsilon \cdot \nu^\epsilon \quad \text{on } J \times \Gamma^\epsilon,\\
p^\epsilon = P^\epsilon, \quad (\epsilon^2 \nabla p^\epsilon + \epsilon m^\epsilon)\cdot \nu^\epsilon = -(\nabla P^\epsilon + M^\epsilon)\cdot \nu^\epsilon \quad \text{on } J \times \Gamma^\epsilon.
\end{array}\right. \tag{13}$$

The aim of the paper is to derive an effective (homogeneous) model for this composite microscopic problem, by letting $\epsilon \to 0$. We prove that it is still a Landau-Lifshitz-Gilbert equation, but with a new source term which is a memory term produced by the slow dynamics part of the microscopic model. On the contrary, the structure of the associated stray field equation is not modified by the matrix part of the microscopic model.

We use various tools of the homogenization theory. We begin by exploiting the periodic structure of the problem through two-scale convergence arguments ([20, 4]). The process let us exhibit the existence of memory terms due to the less conductive part of the domain in the effective model. But the ϵ-scaling in the matrix part of (13) clearly does not allow to get compactness results and to pass to the limit in the nonlinear terms. We thus adopt another approach. On the one hand, we introduce a dilation operator, in the spirit of the periodic unfolding method of *e.g.* [11]. The ϵ-scaling disappears, at the expense of doubling the space dimension. Classical compactness results thus remain inaccessible. On the other hand, we thus exploit the periodic structure in a new way. It is based on the intuition that the lattice of matrix blocks tends to a set of points which is dense in Ω as ϵ tends to zero. Around any of these points, we succeed in constructing a sequence of embedded grids where we restrict the dimension and pass to the limit. We finally show that the obtained information is sufficient to identify the limit problem in the whole space. The method is original, even if a so-called density argument was already mentioned (but not detailed neither used) in [8].

The outline of this work is the following. The effective model is provided in the next section. The remaining part of the paper consists in its justification. For the sake of completeness, in Section 3, we begin by checking that the effective model may be computed through formal asymptotic expansions. Section 4 is devoted to the rigorous justification of the upscaling process, namely by proving that a subsequence of solutions of Problem (13) converges in some sense to a solution of the effective model. After stating uniform estimates, we apply two-scale convergence results to some extension of the solution in the most conductive part of the domain, Ω_f^ϵ. Next we introduce the dilation operator, the embedded grids approach and the 'density' arguments for the solution in Ω_m^ϵ.

2. **Main result.** Let $(v_j)_{j=1\dots 3}$ and $(w_j)_{j=1\dots 3}$ respectively the Q-periodic solutions of the following problems

$$\begin{cases} -\mathrm{div}_y(A_f(\nabla_y v_j + e_j)) = 0 \text{ in } \Omega \times Q_f, \\ A_f(\nabla_y v_j + e_j)\cdot \nu = 0 \text{ on } \Omega \times \partial Q_f, \end{cases} \tag{14}$$

$$\begin{cases} -\mathrm{div}_y(\nabla_y w_j + e_j) = 0 \text{ in } Q_f, \\ \nabla_y w_j \cdot \nu = -e_j \cdot \nu \text{ on } \partial Q_f, \end{cases} \tag{15}$$

where the vector e_j, $1 \le j \le 3$, is the jth unit vector of the canonical orthonormal basis. We define A^H and W^H by

$$A_{ij}^H = \int_{Q_f} A_f(e_i + \nabla_y v_i)\cdot(e_j + \nabla_y v_j)\, dy, \quad 1 \le i,j \le 3, \tag{16}$$

$$W_{ij}^H = |Q_f| + \int_{Q_f} \partial_i w_j(y)\, dy, \quad 1 \le i,j \le 3, \tag{17}$$

The effective magnetization vector M and the effective demagnetizing field P satisfy

$$\begin{aligned} |Q_f|\partial_t M - \alpha |Q_f| M \times \partial_t M = -(1+\alpha^2) M \times \Big(&\mathrm{div}(A^H \nabla M) + |Q_f|\phi_{va,f}(M) \\ &+\Xi_f'(M) W^H \nabla M + W^H \nabla P - \int_{Q_m} \mathrm{div}_y(A_m(x,y)\nabla_y m_0)\, dy\Big) \\ &\text{in } \Omega \times J, \end{aligned} \tag{18}$$

$$A^H \nabla M \cdot n = 0 \quad \text{on } \partial\Omega \times J, \tag{19}$$

$$\mathrm{div}\big(\chi_{\mathbb{R}^3\setminus\Omega}\nabla P + \chi_\Omega W^H(\nabla P + M)\big) = 0 \quad \text{in } \mathbb{R}^3 \times J, \tag{20}$$

Moreover the source terms involving m_0 and p_0 are computed thanks to the following problem:

$$\begin{aligned} &\partial_t m_0 - \alpha m_0 \times \partial_t m_0 \\ &\quad = -(1+\alpha^2) m_0 \times \Big(\mathrm{div}_y(A_m(x,y)\nabla_y m_0) + \phi_{va,m}(x,y,m_0) \\ &\quad\quad + \nabla_y(\Xi_m(y,m_0)) + \nabla_y p_0\Big) \quad \text{in } \Omega \times Q_m \times J, \end{aligned} \tag{21}$$

$$\mathrm{div}_y(\nabla_y p_0 + m_0) = 0 \quad \text{in } \Omega \times Q_m \times J, \tag{22}$$

$$m_0 = M \text{ and } p_0 = P \quad \text{on } \partial Q_m. \tag{23}$$

The problem is completed by the initial conditions:

$$M(x,0) = m_0(x,y,0) = M_{init}(x),\ P(x,0) = p_0(x,y,0) = P_{init}(x) \quad \text{on } \Omega \times Q_m. \tag{24}$$

Remark 1. Inspection of the effective model reveals that the resulting homogenized problem is a LLG type model that contains a term representing memory effects which could be seen as a new magnetic excitation in the effective field. The memory term is induced by the slow dynamics part of the model, and it appears solely in the magnetization equation. The limiting stray field equation also depends on a new permeability, namely a kind of averaged permeability.

The effectiveness of the latter model is justified by a convergence result. Namely, we use the concept of two-scale convergence introduced by G. Nguetseng [20] and developed by G. Allaire [4]. We refer to [22] (see Subsection 2.5.2) for the time-dependent settings. Let Ω' be an open subset in $\mathbb{R}^3$. A sequence of functions (v_ϵ) in $L^2(\Omega' \times J)$ is said to two-scale converge to a limit v_0 belonging to $L^2(\Omega' \times J; L^2_\#(Q))$, if for any function $\Psi(x,y,t) \in \mathcal{D}(\Omega' \times J, C^\infty_\#(Q))$ we have

$$\lim_{\epsilon\to 0} \int_{\Omega'\times J} v_\epsilon(x,t)\Psi\Big(x, \frac{x}{\epsilon}, t\Big) dx\, dt = \int_{\Omega'\times J}\int_Q v_0(x,y,t)\Psi(x,y,t) dx\, dy\, dt.$$

The convergence result is denoted by $v_\epsilon \overset{2}{\rightharpoonup} v_0$.

We have the following properties (see [4]).

Proposition 1. *(i) From each bounded sequence (v_ϵ) in $L^2(\Omega' \times J)$ we can extract a subsequence which two-scale converges.*

(ii) Let (v_ϵ) be a bounded sequence in $H^1(\Omega' \times J)$ which converges weakly to v in $H^1(\Omega' \times J)$. Then (v_ϵ) two-scale converges to v and there exists a function $v_1 \in L^2(\Omega' \times J, H^1_\#(Q))$ such that, up to a subsequence, (∇v_ϵ) two-scale converges to $\nabla_x v + \nabla_y v_1$.

(iii) Let (v_ϵ) be a sequence in $L^2(\Omega' \times J)$ which two-scale converges to $v_0 \in L^2(\Omega' \times J \times Q)$. Assume that

$$\lim_{\epsilon\to 0} \|v_\epsilon\|_{L^2(\Omega'\times J)} = \|v_0\|_{L^2(\Omega'\times J\times Q)}.$$

Then for any sequence $(w_\epsilon) \subset L^2(\Omega' \times J)$ which two-scale converges to $w_0 \in L^2(\Omega' \times J \times Q)$, we have

$$v_\epsilon(x,t)w_\epsilon(x,t) \rightharpoonup \int_Q v_0(x,y,t)w_0(x,y,t)\, dy \quad \textit{in } \mathcal{D}'(\Omega' \times J).$$

Remark 2. Choosing $\Omega' = \Omega$ (resp. $\Omega' = \mathbb{R}^3$) in the definition and the properties above, we obtain the functional setting which is well suited for the study of the magnetization vector $\mathbf{M}^\epsilon$ (resp. of the demagnetizing field $\mathbf{H}^\epsilon$).

The main result of the paper is the following.

Theorem 2.1. *Let $(\mathbf{M}^\epsilon, \mathbf{H}^\epsilon)$ be a solution of Problem (13) for $\epsilon > 0$. There exists a subsequence of an appropriate extension of $\chi^\epsilon_f(\mathbf{M}^\epsilon, \mathbf{H}^\epsilon)$ on the one hand, and of $(\mathbf{M}^\epsilon, \mathbf{H}^\epsilon)$ on the other hand, which two-scale converges to a solution (M, P) and (m_0, p_0) of the effective model (18)-(24).*

3. **Formal asymptotic expansions.** In the present section, purely formal computations are developed for the guess of the effective model. These formal results are made rigorous by the limit process stated and proved in the next section.

We now use formal asymptotic expansions. It means that, setting $y = x/\epsilon$ for the fast space variable, we assume the following forms for the solutions:

$$\chi^\epsilon_f(x)M^\epsilon(x,t) = \chi_f(y)\sum_{i\geq 0}\epsilon^i M_i(x,y,t),$$

$$\chi^\epsilon_m(x)m^\epsilon(x,t) = \chi_m(y)\sum_{i\geq 0}\epsilon^i m_i(x,y,t),$$

$$(\chi^\epsilon_f + \chi_{\mathbb{R}^3\setminus\Omega})(x)P^\epsilon(x,t) = (\chi_\Omega(x)\chi_f(y) + \chi_{\mathbb{R}^3\setminus\Omega}(x))\sum_{i\geq 0}\epsilon^i P_i(x,y,t),$$

$$\chi_m^\epsilon(x)p^\epsilon(x,t) = \chi_m(y)\sum_{i\geq 0}\epsilon^i p_i(x,y,t).$$

where we have denoted by χ_f (resp. χ_m) the characteristic function of Q_f (resp. Q_m). We insert these expansions in (13). Selecting the terms according to the powers of epsilon, we obtain the following cascade of equations. First, we consider the constraint (1). Whatever $f^\epsilon = M^\epsilon$ or $f^\epsilon = m^\epsilon$, we infer from (1) that $\sum_{i=1}^3 {f_0}_i^2 = 1$ and $\sum_{i=1}^3 {f_0}_i {f_1}_i = 0$, that is

$$|M_0| = 1 \text{ and } M_0 \text{ is orthogonal to } M_1 \text{ in } \Omega_f^\epsilon \times J, \tag{25}$$

$$|m_0| = 1 \text{ and } m_0 \text{ is orthogonal to } m_1 \text{ in } \Omega_m^\epsilon \times J. \tag{26}$$

Next, terms of order ϵ^{-2}, ϵ^{-1} and ϵ^0 of (2) in $\Omega_f^\epsilon \times J$ give the following three equations:

$$0 = -(1+\alpha^2)M_0 \times \operatorname{div}_y(A_f\nabla_y M_0), \tag{27}$$

$$\begin{aligned} 0 = &-(1+\alpha^2)M_0 \times \Big(\operatorname{div}_x(A_f\nabla_y M_0) + \operatorname{div}_y(A_f\nabla_x M_0 + A_f\nabla_y M_1)\Big)\\ &-(1+\alpha^2)M_1 \times \operatorname{div}_y(A_f\nabla_y M_0) - (1+\alpha^2)M_0 \times \nabla_y \Xi_f(M_0)\\ &-(1+\alpha^2)M_0 \times \nabla_y P_0, \end{aligned} \tag{28}$$

$$\begin{aligned} &\partial_t M_0 - \alpha M_0 \times \partial_t M_0\\ &\quad = -(1+\alpha^2)M_0 \times \Big(\operatorname{div}_x(A_f(\nabla_x M_0 + \nabla_y M_1))\\ &\quad +\operatorname{div}_y(A_f(\nabla_x M_1 + \nabla_y M_2))\Big) - (1+\alpha^2)M_1 \times \Big(\operatorname{div}_x(A_f\nabla_y M_0)\\ &\quad \operatorname{div}_y(A_f(\nabla_x M_0 + \nabla_y M_1)) - (1+\alpha^2)M_2 \times \operatorname{div}_y(A_f\nabla_y M_0)\\ &\quad -(1+\alpha^2)M_0 \times \Big(\nabla_x\Xi_f(M_0) + \Xi_f'(M_0)\nabla_y M_1 + \Xi_f''(M_0)M_1\nabla_y M_0\\ &\quad +\Xi_f^{(3)}(M_0)M_1M_2\nabla_y M_0\Big) - (1+\alpha^2)M_0 \times \phi_{va,f}(M_0) - (1+\alpha^2)M_0 \times\\ &\quad (\nabla_x P_0 + \nabla_y P_1) - (1+\alpha^2)M_1 \times \nabla_y P_0. \end{aligned} \tag{29}$$

The same process in $\Omega_m^\epsilon \times J$ gives:

$$\begin{aligned} \partial_t m_0 - \alpha m_0 \times \partial_t m_0 = &-(1+\alpha^2)m_0 \times \operatorname{div}_y(A_m(x,y)\nabla_y m_0) - (1+\alpha^2)m_0\\ &\times\phi_{va,m}(x,y,m_0) - (1+\alpha^2)m_0 \times \nabla_y(\Xi_m(y,m_0)) - (1+\alpha^2)m_0 \times \nabla_y p_0. \end{aligned} \tag{30}$$

The expansion of the boundary conditions for M^ϵ and m^ϵ on $\Omega \times \partial Q_m \times J$ leads to:

$$M_i = m_i,\ i \geq 0, \tag{31}$$

$$A_f(x)\nabla_y M_0 \cdot \nu = 0, \tag{32}$$

$$A_f(x)(\nabla_x M_0 + \nabla_y M_1)\cdot \nu = 0, \tag{33}$$

$$A_f(x)(\nabla_x M_1 + \nabla_y M_2)\cdot \nu = -A_m(x,y)\nabla_y m_0 \cdot \nu. \tag{34}$$

The same work on the equations characterizing the demagnetizing field gives:

$$\Delta_y P_0 = 0, \tag{35}$$

$$\operatorname{div}_x(\nabla_y P_0) + \operatorname{div}_y(\nabla_x P_0) + \Delta_y P_1 + \operatorname{div}_y(\chi_\Omega M_0) = 0, \tag{36}$$

$$\Delta_x P_0 + \operatorname{div}_x(\nabla_y P_1) + \operatorname{div}_x(\chi_\Omega M_0) + \operatorname{div}_y(\nabla_x P_1 + \nabla_y P_2 + \chi_\Omega M_1) = 0, \tag{37}$$

$$\operatorname{div}_y(\nabla_y p_0 + m_0) = 0, \tag{38}$$

$$P_i = p_i,\ i \geq 0, \tag{39}$$

$$(\nabla_x P_1 + \nabla_y P_2 + M_1) \cdot \nu = -(\nabla_y p_0 + m_0) \cdot \nu, \tag{40}$$
$$\nabla_y P_0 \cdot \nu_{\partial_{Q_f}} = 0, \tag{41}$$
$$(\nabla_x P_0 + \nabla_y P_1 + \chi_\Omega M_0) \cdot \nu_{\partial_{Q_f}} = 0, \tag{42}$$

the first three equations being satisfied in $\big(((\mathbb{R}^3 \setminus \Omega) \times Q) \cup (\Omega \times Q_f)\big) \times J$, the fourth one in $\Omega \times Q_m \times J$, the next two ones in $\Omega \times \partial Q_m \times J$, and the last two ones in $\big((\mathbb{R}^3 \setminus \Omega) \times \partial Q \times J\big) \cup \big(\Omega \times \partial Q_f \times J\big)$.

Now we exploit the latter equations. First we infer from (27) completed with (32) that M_0 does not depend on the fast variable y. The same holds true for P_0 in view of (35) and (41):

$$M_0(x, y, t) = M_0(x, t) \text{ in } \Omega \times J, \quad P_0(x, y, t) = P_0(x, t) \text{ in } \mathbb{R}^3 \times J.$$

Then we characterize function M_1. On the one hand, the variational formulation corresponding to (28) with (33) is

$$\sum_{i,j} \int_{\Omega \times J} M_0 \times \Big(\int_{Q_f} A_{f\,ij} \big(\partial_{x_i} M_0 + \partial_{y_i} M_1\big)\ \partial_{y_j} \Phi \, dy \Big)\, dxdt = 0$$

for any test function $\Phi \in L^2(\Omega \times J; H^1(Q_f))$. On the other hand, in view of assertions (25), we also have

$$\sum_{i,j} \int_{\Omega \times J} M_0 \cdot \Big(\int_{Q_f} A_{f\,ij} \big(\partial_{x_i} M_0 + \partial_{y_i} M_1\big)\ \partial_{y_j} \Phi \, dy \Big)\, dxdt = 0.$$

Since moreover $\mathrm{div}_y(A_f \nabla_x M_0) = 0$, we can characterize M_1 by

$$\begin{aligned} &\mathrm{div}_y(A_f \nabla_y M_1) = 0 \ \text{ in } \Omega \times Q_f \times J, \\ &A_f \nabla_y M_1 \cdot \nu = -A_f \nabla_x M_0 \cdot \nu \ \text{ in } \Omega \times \partial Q_m \times J. \end{aligned}$$

Then

$$M_1(x, y, t) = \sum_{j=1}^{3} v_j(x, y) \partial_{x_j} M_0(x, t) + \alpha(x, t)$$

where functions w_j are defined in (14) and α is some function which does not depend on y. The first term on the right-hand side of equation (29) now reads

$$\mathrm{div}_x(A_f(\nabla_x M_0 + \nabla_y M_1)) = \mathrm{div}_x\big(A_f\big(\mathrm{Id} + (\partial_{y_i} v_j)\big)\nabla_x M_0\big).$$

Similarly, we infer from (36) and (42) that

$$\chi_\Omega(x)\chi_f(y) P_1(x, y, t) = \chi_\Omega(x)\chi_f(y) \sum_{j=1}^{3} w_j(y)(\partial_j P_0(x, t) + M_{0j}) + \beta(x, t),$$

functions w_j being defined by (15), and the two first terms in (37) read

$$\begin{aligned} \mathrm{div}_x(\nabla_x P_0 + \nabla_y P_1 + \chi_\Omega M_0) \;&=\; \mathrm{div}_x\big(\chi_{\mathbb{R}^3 \setminus \Omega}(\nabla_x P_0 + \nabla_y P_1)\big) \\ &+\; \mathrm{div}_x\big(\chi_\Omega \chi_f\big(\mathrm{Id} + (\partial_{y_i} w_j)\big)(\nabla_x P_0 + M_0)\big). \end{aligned}$$

Next step consists in integrating over Q the equations characterizing the main order terms of the expansions, that is (29) and (37), in view of obtaining the effective model. In view of the latter computations, we get at first glance the following two equations in $\Omega \times J$:

$$
\begin{aligned}
&|Q_f|\partial_t M_0 - \alpha |Q_f| M_0 \times \partial_t M_0 = -(1+\alpha^2) M_0 \times \Big(\operatorname{div}_x (A^H \nabla_x M_0) \\
&\quad + \int_{Q_f} \operatorname{div}_y (A_f (\nabla_x M_1 + \nabla_y M_2))\, dy \Big) - (1+\alpha^2) \Xi_f'(M_0) M_0 \times W^H \nabla_x M_0 \\
&\quad - (1+\alpha^2)|Q_f| M_0 \times \phi_{va,f}(M_0) - (1+\alpha^2) M_0 \times W^H \nabla_x P_0,
\end{aligned} \tag{43}
$$

and

$$
\operatorname{div}_x \big(\chi_{\mathbb{R}^3 \setminus \Omega} \nabla_x P_0 + \chi_\Omega W^H (\nabla_x P_0 + M_0) \big) + \int_{Q_f} \operatorname{div}_y (\nabla_x P_1 + \nabla_y P_2 + M_1)\, dy = 0. \tag{44}
$$

We also have, in view of (34),

$$
\begin{aligned}
&\int_{Q_f} \operatorname{div}_y (A_f (\nabla_x M_1 + \nabla_y M_2))\, dy = - \int_{\partial Q_m} A_f (\nabla_x M_1 + \nabla_y M_2) \cdot \nu \, d\sigma(y) \\
&\quad = - \int_{\partial Q_m} A_m(x,y) \nabla_y m_0 \cdot \nu \, d\sigma(y) = - \int_{Q_m} \operatorname{div}_y (A_m(x,y) \nabla_y m_0)\, dy
\end{aligned} \tag{45}
$$

and, in view of (40),

$$
\begin{aligned}
&\int_{Q_f} \operatorname{div}_y (\nabla_x P_1 + \nabla_y P_2 + M_1)\, dy = - \int_{\partial Q_m} (\nabla_x P_1 + \nabla_y P_2 + M_1) \cdot \nu \, d\sigma(y) \\
&\quad = \int_{\partial Q_m} (\nabla_y p_0 + m_0) \cdot \nu \, d\sigma(y) = \int_{Q_m} \operatorname{div}_y (\nabla_y p_0 + m_0)\, dy = 0.
\end{aligned} \tag{46}
$$

Now, setting $M_0 = M$ and $P_0 = P$, we notice that the effective model corresponds to (43)-(46) and (30), (38).

4. Rigorous derivation of the effective model.

4.1. Uniform estimates on the microscopic model.

The existence of a weak solution for (13) may be stated using classical arguments for this type of problem (see *e.g.* [5] and the references therein). Moreover we have the following estimates[2] with regard to the scaling parameter ϵ.

Proposition 2. *Assume that $M_{init} \in H^1(\Omega)$. Then any weak solution of problem (13) satisfies the following estimates:*

$$
\|\chi_f^\epsilon \partial_t M^\epsilon + \chi_m^\epsilon \partial_t m^\epsilon\|_{L^2(\Omega \times J)} \le C,
$$

$$
\|\chi_f^\epsilon \nabla M^\epsilon + \epsilon \chi_m^\epsilon \nabla m^\epsilon\|_{L^\infty(J; L^2(\Omega))} \le C,
$$

$$
|\chi_f^\epsilon M^\epsilon + \chi_m^\epsilon m^\epsilon| = 1 \ \textit{a.e. in } \Omega \times J,
$$

and

$$
\|(\chi_{\mathbb{R}^3 \setminus \Omega} + \chi_f^\epsilon) P^\epsilon + \chi_m^\epsilon p^\epsilon\|_{L^\infty(J; L^2(\Omega))} \le C,
$$

$$
\|(\chi_{\mathbb{R}^3 \setminus \Omega} + \chi_f^\epsilon) \nabla P^\epsilon + \epsilon \chi_m^\epsilon \nabla p^\epsilon\|_{L^\infty(J; L^2(\Omega))} \le C.
$$

[2] All along the paper, letter C denotes some generic constant.

Proof. The techniques used in the proof are similar to those developed for instance in [15]. We rewrite the LLG equation in the form

$$\begin{aligned}\alpha\partial_t\mathbf{M}^\epsilon &- (1+\alpha^2)\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon)\\ &= \alpha\mathbf{M}^\epsilon\times\Big(\alpha\partial_t\mathbf{M}^\epsilon-(1+\alpha^2)\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon)\Big)-(1+\alpha^2)\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon).\end{aligned} \tag{47}$$

Multiplying (47) by $\big((\alpha\partial_t\mathbf{M}^\epsilon-(1+\alpha^2)\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon)\big)$, we get

$$\frac{\alpha}{1+\alpha^2}|\partial_t\mathbf{M}^\epsilon|^2=\mathcal{H}_e^\epsilon(\mathbf{M}^\epsilon)\cdot\partial_t\mathbf{M}^\epsilon. \tag{48}$$

Integrating with respect to space and then with respect to time, we obtain

$$\mathcal{E}(\mathbf{M}^\epsilon)+\frac{2\alpha}{1+\alpha^2}\int_0^t\int_\Omega|\partial_t\mathbf{M}^\epsilon|^2\,dxdt\le\mathcal{E}(M_{init})$$

for any $t\geq 0$, where the energy $\mathcal{E}(\mathbf{M}^\epsilon)$ is defined by

$$\mathcal{E}(\mathbf{M}^\epsilon)=\int_\Omega A^\epsilon|\nabla\mathbf{M}^\epsilon|^2\,dx+\int_\Omega\Lambda(\mathbf{M}^\epsilon)\,dx+\int_\Omega\Xi(\mathbf{M}^\epsilon)\,dx+\int_{\mathbb{R}^3}|\mathbf{H}^\epsilon|^2\,dx.$$

The hypothesis $M_{init}\in H^1(\Omega)$ ensures that the initial energy $\mathcal{E}(M_{init})$ is bounded (see for example [16]). Proposition 2 is proved. $\square$

In view of exploiting the *a priori* estimates obtained in Ω_f^ϵ which is an ϵ-dependent domain, we first need to extend the functions M^ϵ and $\chi_\Omega P^\epsilon$ to the whole fixed domain Ω. To this aim, we first check that the structure of $\Omega^\epsilon=\Omega_f^\epsilon\cup\Gamma^\epsilon\cup\Omega_m^\epsilon$ satisfies the assumptions in [1]. We then can claim that there exist three real numbers $k_i=k_i(Q_f)>0$, $i=1,2,3$, and a linear and continuous extension operator $\Pi^\epsilon : H^1(\Omega_f^\epsilon)\to H^1_{loc}(\Omega)$ such that

$$\Pi^\epsilon V=V\quad\text{a.e. in }\Omega_f^\epsilon,$$
$$\int_{\Omega(\epsilon k_1)}|\Pi^\epsilon V|^2\,dx\le k_2\int_{\Omega_f^\epsilon}|V|^2\,dx,$$
$$\int_{\Omega(\epsilon k_1)}|\nabla(\Pi^\epsilon V)|^2\,dx\le k_3\int_{\Omega_f^\epsilon}|\nabla V|^2\,dx$$

for any $V\in H^1(\Omega_f^\epsilon)$. Here $\Omega(\epsilon k_1)=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\epsilon k_1\}$. To avoid dealing with boundary layers, we make the following additional assumption on the structure of the domain Ω^ϵ:

$$\Omega_m^\epsilon=\Omega(\epsilon k_1)\cap\big\{\cup_{\xi\in\mathcal{A}^\epsilon}(Q_m+\xi)\big\}\quad\text{and}\quad\Omega_f^\epsilon=\Omega\setminus\overline{\Omega_m^\epsilon}.$$

It means that we assume that the blocks are removed in an ϵk_1-neighborhood of $\partial\Omega$. Therefore estimates in Proposition 2 lead to

$$\|\Pi^\epsilon M^\epsilon\|_{L^\infty(J;H^1(\Omega))}+\|\Pi^\epsilon(\chi_\Omega P^\epsilon)\|_{L^\infty(J;H^1(\Omega))}\le C. \tag{49}$$

4.2. **Exploitation of two-scale arguments.** We infer from the estimates listed in Proposition 2 and the estimates (49) the following convergences results.

Proposition 3. *There exist limit functions* $M\in H^1(J;L^2(\Omega))\cap L^\infty(J;H^1(\Omega))$, $M_1\in L^2(\Omega\times J;H^1_\#(Q_f))$, $P\in L^\infty(J;H^1(\mathbb{R}^3))$, $P_1\in L^\infty(J;L^2(\mathbb{R}^3;H^1_\#(Q_f)))$ *on the one hand, and* $m_0\in L^2(\Omega\times J;H^1_\#(Q_m))$, $p_0\in L^2(\Omega\times J;H^1_\#(Q_m))$ *on the other* *hand, such that, for some subsequence not relabelled for convenience, the following convergence results hold true:*

$$\begin{aligned}
&\Pi^\epsilon M^\epsilon \to M \text{ in } L^2(\Omega\times J) \text{ and a.e. in } \Omega\times J,\\
&\Pi^\epsilon M^\epsilon \rightharpoonup M \text{ weakly in } H^1(J;L^2(\Omega))\cap L^2(J;H^1(\Omega)),\\
&\nabla(\Pi^\epsilon M^\epsilon) \overset{2}{\rightharpoonup} \nabla M + \nabla_y M_1,\\
&\chi_f^\epsilon M^\epsilon + \chi_{\overline{\Omega}_m^\epsilon} m^\epsilon \overset{2}{\rightharpoonup} m_0, \quad \partial_t\big(\chi_f^\epsilon M^\epsilon + \chi_{\overline{\Omega}_m^\epsilon} m^\epsilon\big) \overset{2}{\rightharpoonup} \partial_t m_0,\\
&\epsilon\nabla\big(\chi_f^\epsilon M^\epsilon + \chi_{\overline{\Omega}_m^\epsilon} m^\epsilon\big) \overset{2}{\rightharpoonup} \nabla_y m_0,
\end{aligned}$$

and

$$\begin{aligned}
&\chi_{\mathbb{R}^3\setminus\Omega}P^\epsilon + \Pi^\epsilon(\chi_\Omega P^\epsilon) \to P \text{ in } L^r(J;L^2(\mathbb{R}^3)), \forall r\geq 1, \text{ and a.e. in } \Omega\times J,\\
&\chi_{\mathbb{R}^3\setminus\Omega}P^\epsilon + \Pi^\epsilon(\chi_\Omega P^\epsilon) \rightharpoonup P \text{ weakly in } L^2(J;H^1(\mathbb{R}^3)),\\
&\nabla\big(\chi_{\mathbb{R}^3\setminus\Omega}P^\epsilon + \Pi^\epsilon(\chi_\Omega P^\epsilon)\big) \overset{2}{\rightharpoonup} \nabla P + \nabla_y P_1,\\
&\chi_f^\epsilon P^\epsilon + \chi_{\overline{\Omega}_m^\epsilon} p^\epsilon \overset{2}{\rightharpoonup} p_0, \quad \epsilon\nabla\big(\chi_f^\epsilon P^\epsilon + \chi_{\overline{\Omega}_m^\epsilon} p^\epsilon\big) \overset{2}{\rightharpoonup} \nabla_y p_0.
\end{aligned}$$

We now aim using the latter convergence results to pass to the limit $\epsilon \to 0$ in (13). First of all, notice that (23) is a direct consequence of the definition of the two-scale limits (M, m_0) and (P, P_0).

Now, we exploit the constraint equation (1) in the following auxiliary lemma.

Lemma 4.1. *For any $1 \leq i \leq 3$, the vectors $\partial_{x_i}M$, $\partial_{x_i}M + \partial_{y_i}M_1$ and $\partial_{y_i}M_1$ are perpendicular to the vector M almost everywhere in $\Omega\times J\times Q_f$.*

Proof. First we look for the derivative of the limit constraint. On the one hand, due to the a.e. convergence of $\Pi^\epsilon M^\epsilon$, we know that $\chi_f^\epsilon|M^\epsilon| \overset{2}{\rightharpoonup} \chi_f(y)|M|$ and thus

$$\chi_f^\epsilon|M^\epsilon| \rightharpoonup |Q_f|\,|M| \text{ weakly in } L^2(\Omega\times J). \tag{50}$$

On the other hand, since $\chi_f^\epsilon|M^\epsilon| = \chi_f^\epsilon$, we also have $\chi_f^\epsilon|M^\epsilon| \overset{2}{\rightharpoonup} \chi_f(y)$ (see Proposition 1 *(iii)*) and thus

$$\chi_f^\epsilon|M^\epsilon| \rightharpoonup |Q_f| \text{ weakly in } L^2(\Omega\times J). \tag{51}$$

We conclude from (50)-(51) that $|Q_f||M| = |Q_f|$ and thus

$$|M| = 1 \text{ a.e. in } \Omega\times J.$$

Deriving the latter relation with regard to x_i, for any $1\leq i\leq 3$, we compute that $\partial_{x_i}M\cdot M = 0$ a.e. in $\Omega\times J$. Thus the first result announced in the lemma.

Now we look for the limit of the derivative of the constraint. Let $1\leq i\leq 3$. Due to $\chi_f^\epsilon|M^\epsilon| = 1$, we have $\chi_f^\epsilon\partial_{x_i}M^\epsilon\cdot M^\epsilon = 0$ a.e. in $\Omega\times J$. Thus

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0}\int_{\Omega\times J}\chi_f^\epsilon(x)\left(\partial_{x_i}M^\epsilon(x,t)\cdot M^\epsilon(x,t)\right)\Psi(x,t,x/\epsilon)\,dxdt\\
&= \int_{\Omega\times J}\int_{Q_f}\big((\partial_{x_i}M(x,t)+\partial_{y_i}M_1(x,y,t))\cdot M(x,t)\big)\,\Psi(x,y,t)\,dxdydt
\end{aligned}$$

for any function $\Psi(x,y,t)\in\mathcal{D}(\Omega\times J;C_\#^\infty(Q))$. It means that $\partial_{x_i}M + \partial_{y_i}M_1$ is actually perpendicular to M. Due to the first part of the proof, the same holds true for $\partial_{y_i}M_1$. $\square$

We now pass to the two-scale limit in the part of (13) that only contains linear operators. Let $\Psi \in \mathcal{D}(\mathbb{R}^3 \times J)$, $\Psi_1 \in \mathcal{D}(\mathbb{R}^3 \times J; C^\infty_\#(Q))$ and $\psi \in \mathcal{D}(\Omega \times J; C^\infty_\#(Q))$ such that $\psi(x,y,t) = 0$ if $y \in Q_f$. Notice that these functions are test admissible for the two-scale convergence ([25]). We have

$$\begin{aligned}\int_{\mathbb{R}^3\times J} \big((\chi_{\mathbb{R}^3\setminus\Omega} + \chi_f^\epsilon)\nabla P^\epsilon + \chi_f^\epsilon M^\epsilon + \epsilon^2 \chi_{\overline{\Omega}_m^\epsilon} \nabla p^\epsilon + \epsilon \chi_{\overline{\Omega}_m^\epsilon} m^\epsilon\big) \cdot \big(\nabla_x \Psi(x,t) \\ +\epsilon \nabla_x \Psi_1(x, x/\epsilon, t) + \nabla_y \Psi_1(x, x/\epsilon, t) + \nabla_x \psi(x, x/\epsilon, t) \\ +\frac{1}{\epsilon}\nabla_y \psi(x, x/\epsilon, t)\big)\, dxdt = 0.\end{aligned}$$

Letting $\epsilon \to 0$ in the latter relation we get

$$\begin{aligned}\int_{\mathbb{R}^3\times J} \chi_{\mathbb{R}^3\setminus\Omega} \int_Q (\nabla P + \nabla_y P_1) \cdot (\nabla_x \Psi + \nabla_y \Psi_1)\, dy\, dxdt \\ + \int_{\mathbb{R}^3\times J} \chi_\Omega \int_{Q_f} (\nabla P + \nabla_y P_1 + M) \cdot (\nabla_x \Psi + \nabla_y \Psi_1)\, dy\, dxdt \\ + \int_{\Omega\times J} \int_{Q_m} (\nabla_y p_0 + m_0) \cdot \nabla_y \psi\, dy\, dxdt = 0. \qquad (52)\end{aligned}$$

Thanks to classical density arguments, the latter relation holds true for any $\Psi \in L^\infty(J; H^1(\mathbb{R}^3))$, $\Psi_1 \in L^\infty(J; H^1(\mathbb{R}^3; C^\infty_\#(Q)))$ and $\psi \in L^\infty(J; H^1(\Omega; C^\infty_\#(Q)))$ such that $\psi(x,y,t) = 0$ if $y \in Q_f$. Choosing $\Psi_1 = 0$ and $\psi = 0$ in (52), bearing in mind that $\int_Q \nabla_y P_1 dy = 0$ thanks to the Q-periodicity of P_1, we recover the variational formulation of the following problem

$$\operatorname{div}_x\Big(\chi_{\mathbb{R}^3\setminus\Omega}\nabla P + \chi_\Omega \int_{Q_f} (\nabla P + \nabla_y P_1 + M)\, dy\Big) = 0 \text{ in } \mathbb{R}^3 \times J, \qquad (53)$$

where P_1 is characterized in $\Omega \times J$ by (choose $\Psi = \psi = \chi_{\mathbb{R}^3\setminus\Omega}\Psi_1 = 0$ in (52))

$$\begin{aligned}&\operatorname{div}_y(\nabla P + \nabla_y P_1 + M) = 0 \text{ in } Q_f, \\ &\nabla_y P_1 \cdot \nu = -(\nabla P + M)\cdot \nu \text{ on } \partial Q_f.\end{aligned}$$

It follows that $\chi_\Omega(x)\chi_f(y)P_1(x,y,t) = \chi_\Omega(x)\chi_f(y)\sum_{j=1}^3 w_j(y)(\partial_j P(x,t) + M_j) + \beta(x,t)$, where β is some function which does not depend on y and where w_j, $1 \le j \le 3$, is defined in (15). Therefore

$$\operatorname{div}_x\big(\chi_\Omega(\nabla P + \nabla_y P_1 + M)\big) = \operatorname{div}_x\big(\chi_\Omega\big(\mathrm{Id} + (\partial_{y_i} w_j)\big)(\nabla P + M)\big)$$

and

$$\begin{aligned}\operatorname{div}_x\Big(\chi_{\mathbb{R}^3\setminus\Omega}\nabla P + \chi_\Omega \int_{Q_f} (\nabla P + \nabla_y P_1 + M)\, dy\Big) \\ = \operatorname{div}_x\big(\chi_{\mathbb{R}^3\setminus\Omega}\nabla P + \chi_\Omega W^H(\nabla P + M)\big),\end{aligned}$$

where W^H is defined in (17). Equation (53) is thus actually (20) in the effective model. Finally, choosing $\Psi = \Psi_1 = 0$, we recover (22).

The same type of computations for the limit behavior of (2)-(4), (8)-(11) give a more frustrating result because of the numerous nonlinearities. More precisely, we get

$$\int_{\Omega\times J}\int_{Q_f}\big(\partial_t M-\alpha M\times\partial_t M\big)\cdot\Psi\,dydxdt+\int_{\Omega\times J}\int_{Q_m}\partial_t m_0\cdot(\Psi+\psi)\,dydxdt$$
$$-\alpha\lim_{\epsilon\to 0}\int_{\Omega_m^\epsilon\times J}(m^\epsilon\times\partial_t m^\epsilon)\cdot(\Psi(x,t)+\psi(x,x/\epsilon,t))\,dxdt$$
$$=(1+\alpha^2)\Big(\int_{\Omega\times J}\int_{Q_f}\sum_{i,j}\big(M\times A_{f\,ij}(\partial_{x_i}M+\partial_{y_i}M_1)\big)\cdot(\partial_{x_j}\Psi+\partial_{y_j}\Psi_1)\,dydxdt$$
$$-\int_{\Omega\times J}\int_{Q_f}\phi_{va,f}(M)\cdot\Psi\,dydxdt$$
$$-\int_{\Omega\times J}\int_{Q_f}\big(M\times(\Xi_f)'(M)(\nabla M+\nabla_y M_1)\big)\cdot\Psi\,dydxdt$$
$$-\lim_{\epsilon\to 0}\int_{\Omega_m^\epsilon\times J}\big(m^\epsilon\times(\epsilon\nabla\Xi_m^\epsilon(m^\epsilon)+\phi_{va,m}^\epsilon(m^\epsilon))\big)\cdot\big(\Psi(x,t)+\psi(x,x/\epsilon,t)\big)\,dxdt$$
$$+\lim_{\epsilon\to 0}\int_{\Omega_m^\epsilon\times J}\sum_{i,j}\big(m^\epsilon\times A_{m\,ij}^\epsilon\epsilon\partial_{x_i}m^\epsilon\big)\cdot\partial_{y_j}\psi(x,x/\epsilon,t)\,dxdt\Big) \tag{54}$$

for any test functions $\Psi\in L^\infty(J;H^1(\Omega))$, $\Psi_1\in L^\infty(J;H^1(\Omega;C^\infty_\#(Q)))$ and $\psi\in L^\infty(J;H^1(\Omega;C^\infty_\#(Q)))$ such that $\psi(x,y,t)=0$ if $y\in Q_f$. Let us define $\ell_i\in L^2(\Omega\times J;L^2_\#(Q))$, $1\le i\le 3$, by

$$m^\epsilon\times\partial_t m^\epsilon \overset{2}{\rightharpoonup}\ell_1,$$
$$m^\epsilon\times(\epsilon\nabla\Xi_m^\epsilon(m^\epsilon)+\phi_{va,m}^\epsilon(m^\epsilon))\overset{2}{\rightharpoonup}\ell_2,$$
$$m^\epsilon\times A_m^\epsilon(\epsilon\nabla m^\epsilon)\overset{2}{\rightharpoonup}\ell_3.$$

Choosing $\Psi=\Psi_1=0$ in (54), we obtain the following "characterization" for m_0 in $\Omega\times J\times Q_m$:

$$\partial_t m_0-\alpha\ell_1=-(1+\alpha^2)\ell_2-(1+\alpha^2)\mathrm{div}_y(\ell_3). \tag{55}$$

Next, choosing $\Psi_1=\psi=0$ in (54) and using Lemma 4.1 for simplifying the vectorial product in the boundary condition, we assert that

$$|Q_f|\partial_t M-\alpha|Q_f|M\times\partial_t M+\int_{Q_m}\big(\partial_t m_0-\alpha\ell_1+(1+\alpha^2)\ell_2\big)\,dy$$
$$=-(1+\alpha^2)M\times\mathrm{div}_x\Big(\int_{Q_f}A_f(\nabla M+\nabla_y M_1)\,dy\Big)$$
$$-(1+\alpha^2)M\times\phi_{va,f}(M)$$
$$-(1+\alpha^2)M\times(\Xi_f)'(M)\Big(\int_{Q_f}(\nabla M+\nabla_y M_1)\,dy\Big)\text{ in }\Omega\times J, \tag{56}$$

$$\int_{Q_f}A_f(\nabla M+\nabla_y M_1)\,dy\cdot n=0\text{ on }\partial\Omega\times J. \tag{57}$$

Using $\Psi=\psi=0$ in (54) we get moreover for M_1

$$-\mathrm{div}_y\big(A_f(\nabla M+\nabla_y M_1)\big)=0\text{ in }\Omega\times Q_f\times J,$$
$$A_f\nabla_y M_1\cdot\nu=-A_f\nabla M\cdot\nu\text{ on }(\partial Q_m\cap\partial Q_f)\times J,$$

which leads to express M_1 using $(v_i)_{1\le i\le 3}$ (see the characterization of M_1 in Section 3). Thus (56)-(57) actually reads (bearing also in mind (55) for the expression of the non-explicit term):

$$
\begin{aligned}
&|Q_f|\partial_t M - \alpha|Q_f| M \times \partial_t M \\
&\quad = -(1+\alpha^2) M \times \operatorname{div}(A^H \nabla M) - (1+\alpha^2) M \times \phi_{va,f}(M) \\
&\quad -(1+\alpha^2)(\Xi_f)'(M) M \times W^H \nabla M + (1+\alpha^2)\int_{Q_m} \operatorname{div}_y(\ell_3)\, dy \text{ in } \Omega \times J, \quad (58)\\
&A^H \nabla M \cdot n = 0 \text{ on } \partial\Omega \times J. \quad (59)
\end{aligned}
$$

In the next subsection, we introduce another strategy for computing the non-explicit terms ℓ_i, $1 \leq i \leq 3$. Nevertheless, we already can prove that the effective problem, even with this partial formulation, is well-posed.

Lemma 4.2. *Problem (58)-(59) associated with (55), (23) and (24) admits a weak solution.*

Proof. For our purpose, we look for a regularity result for the term $\int_{Q_m} \operatorname{div}_y(\ell_3) dy$. Since all the two-scale limits are defined in $L^2(\Omega \times J \times Q_m)$, we know that equation (55) is satisfied in $H^{-1}(\Omega \times J \times Q_m)$. Then we can write, for any $\varphi \in H^1_0(J)$,

$$
\begin{aligned}
\int_J \partial_t m_0 \cdot \varphi\, dt - \alpha \int_J \ell_1 \cdot \varphi\, dt &= -(1+\alpha^2)\int_J \ell_2 \cdot \varphi\, dt \\
&-(1+\alpha^2)\int_J \operatorname{div}_y(\ell_3)\cdot\varphi\, dt \quad \text{in } H^{-1}(\Omega \times Q_m).
\end{aligned}
$$

We conclude that $\int_J \operatorname{div}_y(\ell_3) dt$ has the same regularity than $-\int_J (m_0 \cdot \partial_t \varphi - \alpha \ell_1 \cdot \varphi + (1+\alpha^2)\ell_2 \cdot \varphi) dt$, that is belongs to $L^2(\Omega \times Q_m)$. We are allowed to make the following computation

$$
-(1+\alpha^2)\int_{Q_m} \operatorname{div}_y\Big(\int_J \ell_3 \cdot \varphi\, dt\Big)\, dy = -(1+\alpha^2)\int_J \int_{\partial Q_m} ((\ell_3\cdot\varphi)\cdot\nu)\, d\sigma(y) dt \in L^2(\Omega).
$$

The source term $\int_{Q_m} \operatorname{div}_y(\ell_3) dy$ in (58) thus belongs to $L^2(\Omega \times J)$. The existence of some weak solution to (58)-(59), (23)-(24) is then ensured by the classical parabolic theory. ☐

4.3. **Exploitation of an appropriate dilation operator.** It remains to pass to the limit in the nonlinear matrix terms of the problem for giving an explicit form to the terms ℓ_i, $1 \leq i \leq 3$, in (55). We thus have to use another technique than the two-scale convergence. A first idea consists in introducing a dilation operator for upscaling the fast variable x/ϵ and thus removing the ϵ-weight in the H^1 estimates. Such an operator was formally used in [6]. It is also behind the periodic unfolding method of Cioranescu et al [11]. For each $\epsilon > 0$, we define a dilation operator $\widetilde{\cdot}$ mapping measurable functions on $\Omega_m^\epsilon \times J$ to measurable functions on $\Omega \times Q_m \times J$ by

$$
\widetilde{u}(x,y,t) = u(c^\epsilon(x) + \epsilon y, t) \quad \text{for } y \in Q_m,\ (x,t) \in \Omega \times J,
$$

where $c^\epsilon(x)$ denotes the lattice translation point of the ϵ-cell domain containing x. This dilation annihilates the scaling distinction between the slow variable x and the fast variable $y = x/\epsilon$.

Assume for instance a simple but not restrictive description of the periodic structure of Ω^ϵ, more precisely $\mathcal{A} = \mathbb{Z}^3$ and

$$
\Omega^\epsilon = \Omega \cap \big(\bigcup_{k\in\mathbb{Z}^3} \epsilon(Q+k)\big).
$$

Since $\Omega_m^\epsilon = \Omega(\epsilon k_1) \cap (\cup_{k\in\mathbb{Z}^3}\epsilon(Q_m + k))$, quantity $c^\epsilon(x)$ is the center of the ϵ-copy of Q containing x and $c^\epsilon(x) = \epsilon k$ if $x \in \epsilon(Q_m + k)$. Thus the function $\widetilde{u}$ does not depend on x in each given block $\epsilon(Q_m + k)$, $k \in \mathbb{Z}^3$, of Ω. We extend this operator from Q_m to $\cup_k(Q_m + k)$ periodically.

The dilation operator has the following properties (see [6]).

Proposition 4. *Any function $u \in L^2(J; H^1(\Omega_m^\epsilon))$ satisfies*

$$\|\widetilde{u}\|_{L^2(\Omega\times J\times Q_m)} = \|u\|_{L^2(\Omega\times J)}, \quad \nabla_y\widetilde{u} = \epsilon\widetilde{\nabla_x u} \text{ a.e. in } \Omega\times J\times Q_m.$$

If $v, w \in L^2(0,T; H^1(\Omega_m^\epsilon))$, then we have

$$\begin{aligned}
&(\widetilde{v},\widetilde{w})_{L^2(\Omega\times J\times Q_m)} = (v,w)_{L^2(\Omega_m^\epsilon\times J)},\\
&\|\nabla_y\widetilde{v}\|_{(L^2(\Omega\times J\times Q_m))^3} = \epsilon\big\|\widetilde{\nabla_x v}\big\|_{(L^2(\Omega_m^\epsilon\times J))^3},\\
&(\widetilde{v},w)_{L^2(\Omega\times J\times Q)} = (v,\widetilde{w})_{L^2(\Omega\times J\times Q)}.
\end{aligned}$$

Moreover, if $g \in L^2(\Omega\times J)$ is considered to be an element of $L^2(\Omega\times J\times Q_m)$, then

$$\widetilde{g} \to g \text{ strongly in } L^2(\Omega\times J\times Q_m) \text{ as } \epsilon\to 0.$$

This subsection is not completely disconnected from the latter one. Indeed, as emphasized in the following result, the limiting process based on two-scale convergence and the one based on weak convergence of dilated sequences are equivalent (see [8]).

Proposition 5. *If (v^ϵ) is a bounded sequence of $L^2(\Omega_m^\epsilon\times J)$ such that $\widetilde{v^\epsilon}$ converges weakly to $\tilde{v}$ in $L^2(\Omega\times J; L^2_\#(Q_m))$ and $\chi_m^\epsilon v^\epsilon$ two-scale converges to v_0, then we have*

$$\tilde{v} = v_0 \quad \text{a.e. in } \Omega\times J\times Q_m.$$

It means that for computing the non-explicit terms in (55), it is "sufficient" to fully characterize the weak limit of $(\widetilde{m^\epsilon},\widetilde{p^\epsilon})$, $\widetilde{m^\epsilon}$ (resp. $\widetilde{p^\epsilon}$) being the dilated magnetization vector (resp. field potential). It is thus natural to write the equations satisfied by $(\widetilde{m^\epsilon},\widetilde{p^\epsilon})$.

Lemma 4.3. *The dilated quantities $(\widetilde{m^\epsilon},\widetilde{p^\epsilon})$ satisfy the following set of equations*

$$\begin{aligned}
\partial_t\widetilde{m^\epsilon} - \alpha\widetilde{m^\epsilon}\times\partial_t\widetilde{m^\epsilon} = &-(1+\alpha^2)\widetilde{m^\epsilon}\times\big(\mathrm{div}_y(A_m(x,y)\nabla_y\widetilde{m^\epsilon})\\
&+\phi_{va,m}(x,y,\widetilde{m^\epsilon}) + \nabla_y\Xi_m(x,y,\widetilde{m^\epsilon}) + \nabla_y\widetilde{p^\epsilon}\big), \quad (60)\\
\mathrm{div}_y(\nabla_y\widetilde{p^\epsilon} + \widetilde{m^\epsilon}) = 0, \quad (61)
\end{aligned}$$

which are satisfied in $L^2(J; H^{-1}(Q_m))$ for almost every $x \in \Omega_m^\epsilon$. The boundary and initial conditions are

$$\begin{aligned}
&\widetilde{m^\epsilon} = \widetilde{M^\epsilon} \textit{ and } \widetilde{p^\epsilon} = \widetilde{P^\epsilon} \textit{ in } H^{1/2}(Q_m) \textit{ for a.e.}(x,t)\in\Omega_m^\epsilon\times J, \quad (62)\\
&\widetilde{m^\epsilon}_{|t=0} = \widetilde{M_{init}},\ \widetilde{p^\epsilon}_{|t=0} = \widetilde{P_{init}} \textit{ in } \Omega_m^\epsilon\times Q_m. \quad (63)
\end{aligned}$$

Proof. We detail for instance the derivation of the equation (60) satisfied by $\widetilde{m^\epsilon}$. The derivation of the one for $\widetilde{p^\epsilon}$ follows the same lines. For any given $\psi \in L^2(J; H_0^1(Q_m))$, we define $\hat{\psi}$ by

$$\hat{\psi}(x,z,t) = \begin{cases} \psi\big((z-c^\epsilon(x))/\epsilon, t\big) & \text{if } z \in \epsilon Q_m + c^\epsilon(x)\\ 0 \quad \text{else.} \end{cases}$$

We multiply (2) by $\hat{\psi}$ such that $\psi_{|t=T} = 0$. We integrate over Ω_m^ϵ. We recall that $\Omega_m^\epsilon = \cup_{x\in\Omega}\big(\epsilon Q_m + c^\epsilon(x)\big)$. Moreover, $(\epsilon Q_m + c^\epsilon(x_1)) \cap (\epsilon Q_m + c^\epsilon(x_2)) = \emptyset$ if $c^\epsilon(x_1) \neq c^\epsilon(x_2)$. We thus get, for almost every $x \in \Omega_m^\epsilon$:

$$\begin{aligned}
&-\int_J \int_{\epsilon Q_m + c^\epsilon(x)} (m^\epsilon \partial_t \hat{\psi}(x,z,t) + \alpha m^\epsilon \times \partial_t m^\epsilon \hat{\psi}(x,z,t))\, dzdt \\
&\quad = -(1+\alpha^2) \int_J \int_{\epsilon Q_m + c^\epsilon(x)} m^\epsilon \times \Big(\epsilon^2 A_m^\epsilon(x,z) \nabla m^\epsilon(z,t) \cdot \nabla_z \hat{\psi}(x,z,t) \\
&\quad +\phi_{va,m}(x,z,m^\epsilon)\, \hat{\psi}(x,z,t) + \epsilon(\nabla \Xi_m^\epsilon(x,z,m^\epsilon) + \nabla p^\epsilon)\, \hat{\psi}(x,z,t)\Big)\, dzdt \\
&\quad + \int_{\epsilon Q_m + c^\epsilon(x)} M_{init}(x)\, \hat{\psi}(x,z,0)\, dz.
\end{aligned}$$

Let $x \in \Omega_m^\epsilon$. Let $k \in \mathbb{Z}^3$ be defined by $\epsilon k = c^\epsilon(x)$. We introduce the change of variable $z \mapsto \epsilon(y+k)$. We obtain

$$\begin{aligned}
-\int_J \int_{Q_m} (\widetilde{m^\epsilon} \partial_t \psi + \alpha \widetilde{m^\epsilon} \times \partial_t \widetilde{m^\epsilon} \psi)\, dydt &= -(1+\alpha^2) \int_J \int_{Q_m} \widetilde{m^\epsilon} \\
\times \Big(A_m(x,y) \nabla \widetilde{m^\epsilon}(z,t) \cdot \nabla_y \psi &+ \phi_{va,m}(x,y,\widetilde{m^\epsilon})\, \psi \\
+(\nabla_y \Xi_m(x,y,\widetilde{m^\epsilon}) + \nabla_y \widetilde{p^\epsilon})\, \psi\Big)\, dydt &+ \int_{Q_m} \widetilde{M_{init}}(y)\, \psi(y,0)\, dy.
\end{aligned}$$

The latter is the variational formulation of (60) with the initial condition $\widetilde{m^\epsilon}_{|t=0} = \widetilde{M_{init}}$. We give some precisions about the boundary condition. Of course we can enlarge the definition of the dilation operator to a subset of $\Omega(\epsilon k_1)$ strictly containing Ω_m^ϵ. This gives sense to the boundary condition $\widetilde{m^\epsilon} = \widetilde{M^\epsilon}$ on $\partial Q_m \times J$. The result has been established for almost every $x \in \epsilon(Q_m + k)$ and for all $k \in \mathbb{Z}^3$. Then it is valid almost everywhere in Ω_m^ϵ. $\square$

The good point in (60)-(63) is clearly that the ϵ-scaling does not appear anymore. The uniform estimates leading to the following convergences, possibly for subsequences not relabeled for convenience, are thus straightforward:

$$\begin{aligned}
&\widetilde{m^\epsilon} \rightharpoonup m_0,\ \widetilde{p^\epsilon} \rightharpoonup p_0 \text{ weakly in } L^2(\Omega \times J \times Q_m), \\
&\nabla_y \widetilde{m^\epsilon} \rightharpoonup \nabla_y m_0,\ \nabla_y \widetilde{p^\epsilon} \rightharpoonup \nabla_y p_0 \text{ weakly in } L^2(\Omega \times J \times Q_m).
\end{aligned}$$

Notice that we have used Proposition 5 to ensure that the limit functions (m_0, p_0) appearing here are actually the same than the ones already defined in Proposition 3. Moreover the equation satisfied by p_0 has already been derived in the latter subsection. Nevertheless, we still do not have any compactness result for $\widetilde{m^\epsilon}(x,y,t)$ because we have no information on the boundedness of its partial derivative with respect to x. This difficulty also appeared in [8] and [19]. These authors solved it either by comparing the dilated problem with their formal guess for the limit problem ([8]) or by proving that they actually deal with a Cauchy sequence ([19]). The complex structure of our equation does not allow such approaches.

We thus adopt another method and we develop rigorously an idea already present in [10]. Due to the definition of the dilation operator, one checks easily that the dilated functions restricted to a given matrix cell of Ω^ϵ do not depend on x. Let

$k \in \mathbb{Z}^3$. Let $(\widetilde{m_k^\epsilon}, \widetilde{p_k^\epsilon})$ be defined by

$$\widetilde{m_k^\epsilon}(y,t) = \begin{cases} \widetilde{m^\epsilon}(x,y,t)_{|x\in\epsilon(Q_m+k)} & \text{if } k \text{ is such that } \epsilon(Q_m+k)\cap\Omega \neq \emptyset, \\ 0 \quad \text{else,} \end{cases}$$

$$\widetilde{p_k^\epsilon}(y,t) = \begin{cases} \widetilde{p^\epsilon}(x,y,t)_{|x\in\epsilon(Q_m+k)} & \text{if } k \text{ is such that } \epsilon(Q_m+k)\cap\Omega \neq \emptyset, \\ 0 \quad \text{else.} \end{cases}$$

Notice that quantities $\widetilde{f_k^\epsilon}$, $f = m, p$, have sense even if $\widetilde{f^\epsilon}(x,\cdot,\cdot) \in L^2(Q_m \times J)$ is defined only for a.e. $x \in \Omega$. Indeed, since $|\epsilon(Q_m+k)| = \epsilon|Q_m| \neq 0$, there exists $x_m \in \epsilon(Q_m+k)\cap\Omega^\epsilon$ and $x_p \in \epsilon(Q_m+k)\cap\Omega^\epsilon$ such that $\widetilde{m^\epsilon}(x_m,\cdot,\cdot) \in L^2(Q_m \times J)$ and $\widetilde{p^\epsilon}(x_p,\cdot,\cdot) \in L^2(Q_m \times J)$ and which let us define $\widetilde{m_k^\epsilon}$ and $\widetilde{p_k^\epsilon}$.

For any $\epsilon > 0$ such that $\epsilon(Q_m+k)\cap\Omega \neq \emptyset$, $(\widetilde{m_k^\epsilon}, \widetilde{p_k^\epsilon})$ is clearly a solution of (60)-(63) in $Q_m \times J$. On the other hand, any $\widetilde{f_k}$ associated with some $f \in L^2(\Omega \times J)$ belongs to $L^2(Q_m \times J)$ with

$$\|\widetilde{f_k}\|_{L^2(Q_m\times J)} = \frac{1}{\epsilon|Q_m|}\|\widetilde{f}\|_{L^2(\epsilon(Q_m+k)\times Q_m\times J)} \leq \frac{1}{\epsilon|Q_m|}\|\widetilde{f}\|_{L^2(\Omega\times J\times Q_m)}.$$

Thus, we have enough regularity properties to get with (60)-(63) the same estimates for $(\widetilde{m_k^\epsilon}, \widetilde{p_k^\epsilon})$ than the ones obtained for $(\widetilde{m^\epsilon}, \widetilde{p^\epsilon})$. But here the estimate of $\nabla_y\widetilde{m_k^\epsilon}$ gives a uniform bound in $H^1(Q_m)$ for $\widetilde{m_k^\epsilon}$ and thus enough compactness results to pass to the limit $\epsilon \to 0$ in (60). Furthermore, notice that $k = (k_1,k_2,k_3) \in \mathbb{Z}^3$ is such that $\epsilon(Q_m+k)\cap\Omega \neq \emptyset$ if $k_i < |\Omega|_i/\epsilon$, $i = 1,..,3$ (where $|\Omega|_i$ denotes here the value of the measure of Ω in the i^{th} direction). Then, for any $k \in \mathbb{Z}^3$, there exists $\epsilon(k) > 0$ such that for any $\epsilon < \epsilon(k)$, $\epsilon(Q_m+k)\cap\Omega \neq \emptyset$. Denoting by $(\widetilde{m_k}, \widetilde{p_k})$ the limit in $L^2(Q_m \times J)$ of $(\widetilde{m_k^\epsilon}, \widetilde{p_k^\epsilon})$, we get the following system:

$$\begin{aligned} \partial_t\widetilde{m_k} - \alpha\widetilde{m_k}\times\partial_t\widetilde{m_k} = -(1+\alpha^2)\widetilde{m_k}\times\big(&\mathrm{div}_y(A_m(x,y)\nabla_y\widetilde{m_k}) \\ &+\phi_{va,m}(x,y,\widetilde{m_k}) + \nabla_y\Xi_m(x,y,\widetilde{m_k}) + \nabla_y\widetilde{p_k}\big), \quad (64)\\ \mathrm{div}_y(\nabla_y\widetilde{p_k} + \widetilde{m_k}) = 0. \quad& (65) \end{aligned}$$

Another basic idea is that the subgrid defined in Ω by $\{k \in \mathbb{Z}^3;\ \epsilon(Q_m+k)\cap\Omega \neq \emptyset\}$ seems to become dense in Ω as $\epsilon \to 0$. Let us show that this point is sufficient to pass to the limit, at least in some part of the domain. Thanks to Section 2, we already know that our aim is to show that the limit (m_0, p_0) also satisfies (64)-(65), that is (21)-(22). In brief, we are going to prove that $m_0(x,\cdot,\cdot) = m_0^*(x)(\cdot,\cdot)$ in $L^2(Q_m \times J)$ for almost every $x \in \Omega$, where $m_0^*(x)(\cdot,\cdot)$ is defined by

$$\begin{cases} \partial_t m_0^*(x) - \alpha m_0^*(x)\times\partial_t m_0^*(x) = -(1+\alpha^2)m_0^*(x)\times\Big(\mathrm{div}_y(A_m(x,y)\nabla_y m_0^*(x)) \\ \qquad +\phi_{va,m}(x,y,m_0^*(x)) + \nabla_y(\Xi_m(y,m_0^*(x))) + \nabla_y p_0(x,\cdot,\cdot)\Big) \text{ in } Q_m\times J, \\ m_0^*(x)(y,0) = M_{init}(x) \text{ in } Q_m, \\ m_0^*(x)(y,t) = M(x,t) \text{ on } \partial Q_m \times J. \end{cases} \tag{66}$$

A crucial point is that, for any given $(M_{init}(x), M(x,\cdot)) \in \mathbb{R}^3 \times L^\infty(J)$, there exists a unique solution $m_0^*(x) \in L^2(J; H^1_\#(Q_m)) \cap H^1(J; L^2_\#(Q_m))$ of problem (66) (see [17], [24], [12], [13]).

We define the set $\mathcal{C} \subset \Omega$ by

$$\mathcal{C} = \{x_0 \in \Omega;\ \exists\epsilon_0 > 0,\ \exists x \in \Omega \text{ s.t. } x_0 = c^{\epsilon_0}(x)\}.$$

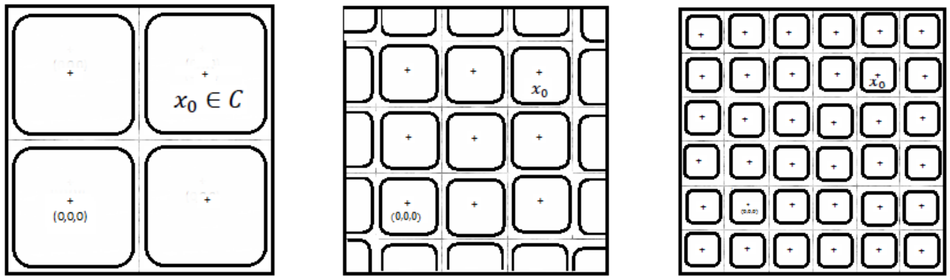

FIGURE 2. A simple setting, $\overline{\Omega} = [-1/2; 3/2]^3$. Representation of Ω^1, $\Omega^{1/2}$ and $\Omega^{1/3}$ with the corresponding points belonging to $\mathcal{C}$.

It means that $\mathcal{C}$ is the set of all points of Ω that are the center of an ϵ_0-copy of Q (and thus of Q_m) at some step, ϵ_0, of the convergence process $\epsilon \to 0$. We also define

$$\mathcal{C}^\epsilon = \{x_0 \in \Omega;\ \exists x \in \Omega \text{ s.t. } x_0 = c^\epsilon(x)\}.$$

We have $\mathcal{C} = \bigcup_{\epsilon>0} \mathcal{C}^\epsilon$.

We begin by restricting the limit process to the set $\mathcal{C} \times J$. To this aim, we develop our embedded grids approach. Let $x_0 \in \mathcal{C}$. There exists some $\epsilon_0 > 0$ such that $x_0 \in \Omega_m^{\epsilon_0}$ and x_0 is the center of an ϵ_0-copy of Q. One checks easily[3] that x_0 remains the center of an ϵ-copy of Q for any $\epsilon \le \epsilon_0$. See also Figure 2.

We then can choose a particular numbering for the description of Ω^ϵ, $\epsilon \le \epsilon_0$:

$$\Omega^\epsilon = \Omega \cap \big(x_0 + \bigcup_{k\in\mathbb{Z}^3} \epsilon(Q+k)\big). \tag{67}$$

It means that for any $\epsilon \le \epsilon_0$, x_0 is the center of the $(0,0,0)$th ϵ-copy of Q. We thus can exploit the latter remarks on the restricted functions $\widetilde{m_k^\epsilon}$ for the value $k = \mathbf{0} = (0,0,0)$. We set for $(y,t) \in Q_m \times J$

$$\widetilde{m_{\mathbf{0}x_0}^\epsilon}(y,t) = \widetilde{m_{\mathbf{0}}^\epsilon}(y,t) \text{ for the numbering (67)}. \tag{68}$$

Lemma 4.4. *Let $x_0 \in \mathcal{C}$. As ϵ tends to zero, the whole sequence $\widetilde{m_{\mathbf{0}x_0}^\epsilon}$ converges in $L^2(J; L^2_\#(Q_m))$ to the function $m_0^*(x_0)$ uniquely defined by (66).*

Proof. The proof is a particular case of the derivation of (64)-(65), namely for $k = \mathbf{0}$. We thus know that

$$\widetilde{m_{\mathbf{0}x_0}^\epsilon} \to \widetilde{m_{\mathbf{0}}} \text{ in } L^2(J; L^2_{per}(Q_m))$$

where $\widetilde{m_{\mathbf{0}}}$ satisfies

$$\begin{aligned}
&\partial_t\widetilde{m_{\mathbf{0}}} - \alpha\widetilde{m_{\mathbf{0}}} \times \partial_t\widetilde{m_{\mathbf{0}}} = -(1+\alpha^2)\widetilde{m_{\mathbf{0}}} \times \big(\mathrm{div}_y(A_m(x,y)\nabla_y\widetilde{m_{\mathbf{0}}}) \\
&\qquad\qquad + \phi_{va,m}(x,y,\widetilde{m_{\mathbf{0}}}) + \nabla_y\Xi_m(x,y,\widetilde{m_{\mathbf{0}}}) + \nabla_y\widetilde{p_{\mathbf{0}}}\big),\\
&\mathrm{div}_y(\nabla_y\widetilde{p_{\mathbf{0}}} + \widetilde{m_{\mathbf{0}}}) = 0,
\end{aligned}$$

completed by the initial and boundary conditions

$$\widetilde{m_{\mathbf{0}}}(y,0) = M_{init}(x_0) \text{ in } Q_m, \qquad \widetilde{m_{\mathbf{0}}}(y,t) = M(x_0,t) \text{ on } \partial Q_m \times J.$$

[3] Since we have mentioned for the sake of simplicity at the beginning of the subsection that $\Omega^\epsilon = \Omega \cap \big(\bigcup_{k\in\mathbb{Z}^3} \epsilon(Q+k)\big)$, it means that $(0,0,0) \in \mathcal{C}$. This assumption is of course unimportant.

Indeed, as already mentioned, we can enlarge the definition of the dilation operator to a subset of $\Omega(\epsilon k_1)$ strictly containing Ω_m^ϵ. This gives sense to the boundary condition $\widetilde{m^\epsilon} = \widetilde{M^\epsilon}$ on $\partial Q_m \times J$. In particular

$$\widetilde{m^\epsilon_{\mathbf{0}}}_{x_0}(y,t) = \widetilde{M^\epsilon_{\mathbf{0}}}_{x_0}(y,t) \text{ on } \partial Q_m \times J. \tag{69}$$

The weak L^2 limit of $\widetilde{M_{\mathbf{0}}}$ is equal to the two-scale limit of M^ϵ. Since $\Pi^\epsilon M^\epsilon$ strongly converges in $L^2(\Omega \times J)$ to M, function M (which does not depend on y) is also the two-scale limit of the restriction M^ϵ. Using the continuity of the trace operator, (69) gives at the limit $\epsilon \to 0$ the condition $\widetilde{m_{\mathbf{0}}}(y,t) = M(x_0,t)$ on $\partial Q_m \times J$. We have proved that $\widetilde{m_{\mathbf{0}}}$ satisfies (66). The solution of (66) being unique for any fixed $x = x_0 \in \Omega$, the whole sequence $\widetilde{m^\epsilon_{\mathbf{0}}}_{x_0}$ converges to the solution of (66). This ends the proof of the lemma. $\square$

Remark 3. The latter lemma means that the limit matrix magnetization m_0 is such that $m_{0|x=x_0} = m_0^*(x_0)$ for a.e. $x_0 \in \mathcal{C}$. Indeed, denoting by $\chi_{|x_0+\epsilon Q_m}$ the characteristic function of $x_0 + \epsilon Q_m$, we have

$$\widetilde{m^\epsilon_{\mathbf{0}}}(y,t) = \widetilde{m^\epsilon}(x,y,t)\chi_{|x_0+\epsilon Q_m}(x).$$

As $\epsilon \to 0$, the sequence of embedded sets $(x_0 + \epsilon Q_m)$ tends to $\{x_0\}$. As already mentioned $\widetilde{m^\epsilon} \rightharpoonup m_0$ weakly in $L^2(\Omega \times J; L^2_\#(Q_m))$ where m_0 is the two-scale limit defined in Proposition 3. It follows that, for any $\varphi \in L^2(J; L^2_\#(Q_m))$,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{Q_m\times J} \widetilde{m^\epsilon_{\mathbf{0}}}\,\varphi\,dydt &= \lim_{\epsilon\to 0}\int_{Q_m\times J} \widetilde{m^\epsilon}(x,y,t)\chi_{|x_0+\epsilon Q_m}(x)\,\varphi(y,t)\,dydt \\ &= \int_{Q_m\times J} m_0(x_0,y,t)\,\varphi(y,t)\,dydt \quad \text{a.e. } x_0 \in \mathcal{C}.\end{aligned}$$

The limit behavior of $(\widetilde{m^\epsilon}_{|x=x_0})$ of course does not depend on the choice of the numbering of the ϵ-copies of Q in Ω_m^ϵ. Problem (66) thus characterizes the limit behavior of the restriction of $\widetilde{m^\epsilon}$ in $\mathcal{C}$. This point is however not sufficient for our purpose. Indeed, on the one hand $\mathcal{C}$ is dense in Ω, but on the other hand the a.e.-convergence in $\mathcal{C}$ is not sufficiently meaningful since $|\mathcal{C}| = 0$.

The end of the paper consists in extending the result of the latter lemma from $\mathcal{C}$ to Ω. Let m_0^* be defined by $m_0^*(x,y,t) = m_0^*(x)(y,t)$ where $m_0^*(x)$ is defined by (66). Let us prove that we have actually $m_0 = m_0^*$ in $L^2(\Omega \times J; L^2_\#(Q_m))$, that is

$$\lim_{\epsilon\to 0}\int_{\Omega\times Q_m\times J} (\widetilde{m^\epsilon} - m_0^*)\,\varphi\,dxdydt = 0 \tag{70}$$

for any $\varphi \in L^2(\Omega \times J; L^2_\#(Q_m))$, or, equivalently by density, for any compactly supported test function, $\varphi \in \mathcal{C}_c(\Omega \times J; \mathcal{C}_\#(Q_m))$. For the structuration of the paper, we announce this final result in the following lemma.

Lemma 4.5. *Let $\varphi \in \mathcal{C}_c(\Omega \times J; \mathcal{C}_\#(Q_m))$. Let $\eta > 0$. There exists $\epsilon' > 0$ such that for any $\epsilon < \epsilon'$,*

$$\Big|\int_{\Omega\times Q_m\times J} (\widetilde{m^\epsilon} - m_0^*)\,\varphi\,dxdydt\Big| \le \eta.$$

Proof. Let $L_\epsilon = \int_{\Omega\times Q_m\times J}\big(\widetilde{m^\epsilon} - m_0^*\big)\,\varphi\,dxdydt$. Function $\widetilde{m^\epsilon}$ being constant on each given ϵ-cell, we write L_ϵ in the following form.

$$\begin{aligned}
L_\epsilon &= \sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\int_{(x_i^\epsilon+\epsilon Q_m)\times Q_m\times J}\big(\widetilde{m^\epsilon}(x_i^\epsilon,y,t) - m_0^*(x,y,t)\big)\,\varphi(x,y,t)\,dxdydt\\
&= \sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\int_{(x_i^\epsilon+\epsilon Q_m)\times Q_m\times J}\big(\widetilde{m^\epsilon}(x_i^\epsilon,y,t) - m_0^*(x_i^\epsilon,y,t)\big)\,\varphi(x,y,t)\,dxdydt\\
&+ \sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\int_{(x_i^\epsilon+\epsilon Q_m)\times Q_m\times J}\big(m_0^*(x_i^\epsilon,y,t) - m_0^*(x,y,t)\big)\,\varphi(x,y,t)\,dxdydt. \quad (71)
\end{aligned}$$

Let us estimate the two terms in the right-hand side of (71).

First, using the notations of Lemma 4.4 and the Cauchy-Schwarz inequality, we write

$$\begin{aligned}
&\Big|\sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\int_{(x_i^\epsilon+\epsilon Q_m)\times Q_m\times J}\big(\widetilde{m^\epsilon} - m_0^*\big)(x_i^\epsilon,y,t)\,\varphi(x,y,t)\,dxdydt\Big|\\
&\quad= \Big|\sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\int_{(x_i^\epsilon+\epsilon Q_m)\times Q_m\times J}\big(\widetilde{m^\epsilon_{\mathbf{0}x_i^\epsilon}}(y,t) - m_0^*(x_i^\epsilon)(y,t)\big)\,\varphi(x,y,t)\,dxdydt\Big|\\
&\quad\le \sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\big\|\widetilde{m^\epsilon_{\mathbf{0}x_i^\epsilon}} - m_0^*(x_i^\epsilon)\|_{L^2(Q_m\times J)}\int_{x_i^\epsilon+\epsilon Q_m}\|\varphi\|_{L^2(Q_m\times J)}\,dx\\
&\quad\le \max_{u\in\mathcal{C}^\epsilon}\big\|\widetilde{m^\epsilon_{\mathbf{0}u}} - m_0^*(u)\|_{L^2(Q_m\times J)}\|\varphi\|_{L^\infty(\Omega;L^2(J;L^2_\#(Q_m)))}\sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}|\epsilon Q_m|\\
&\quad\le C\max_{u\in\mathcal{C}^\epsilon}\big\|\widetilde{m^\epsilon_{\mathbf{0}u}} - m_0^*(u)\|_{L^2(Q_m\times J)}.
\end{aligned}$$

Indeed, $\sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}|\epsilon Q_m| \le C|\Omega| \le C$. In view of Lemma 4.4, for any $u\in\mathcal{C}^\epsilon\subset\mathcal{C}$ we have $\lim_{\epsilon\to 0}\big\|\widetilde{m^\epsilon_{\mathbf{0}u}} - m_0^*(u)\|_{L^2(Q_m\times J)} = 0$. Then, there exists $\epsilon_1 > 0$ such that for any $\epsilon < \epsilon_1$, we have

$$\Big|\sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\int_{(x_i^\epsilon+\epsilon Q_m)\times Q_m\times J}\big(\widetilde{m^\epsilon} - m_0^*\big)(x_i^\epsilon,y,t)\,\varphi(x,y,t)\,dxdydt\Big| \le \eta/3. \quad (72)$$

Next, the second term in the right-hand side of (71) reads:

$$\begin{aligned}
&\sum_{x_i^\epsilon\in\mathcal{C}^\epsilon}\int_{(x_i^\epsilon+\epsilon Q_m)\times Q_m\times J}\big(m_0^*(x_i^\epsilon,y,t) - m_0^*(x,y,t)\big)\,\varphi(x,y,t)\,dxdydt\\
&\qquad= \int_{\Omega\times Q_m\times J}\big(m_0^*(c^\epsilon(x),y,t) - m_0^*(x,y,t)\big)\,\varphi(x,y,t)\,dxdydt. \quad (73)
\end{aligned}$$

Lusin's theorem applies to function m_0^*. Namely, for any $\mu > 0$, there exists a closed set Ω_μ with $|\Omega\setminus\Omega_\mu| < \mu$ such that the restriction of m_0^* to Ω_μ is continuous. We choose μ such that

$$\begin{aligned}
&\Big|\int_{(\Omega\setminus\Omega_\mu)\times Q_m\times J}\big(m_0^*(c^\epsilon(x),y,t) - m_0^*(x,y,t)\big)\,\varphi(x,y,t)\,dxdydt\Big|\\
&\qquad\le C\|m_0^*\|_{L^2(\Omega\times Q_m\times J)}\mu^{1/2} \le \eta/3. \quad (74)
\end{aligned}$$

We recall that $c^\epsilon(x) \to x$ as $\epsilon \to 0$. Thus $\lim_{\epsilon\to 0}\int_{\Omega_\mu\times Q_m\times J}\big(m_0^*(c^\epsilon(x),y,t) - m_0^*(x,y,t)\big)\,\varphi(x,y,t)\,dxdydt = 0$ and there exists $\epsilon_2 > 0$ such that, for any $\epsilon < \epsilon_2$,

$$\Big|\int_{\Omega_\mu\times Q_m\times J}\big(m_0^*(c^\epsilon(x),y,t) - m_0^*(x,y,t)\big)\,\varphi(x,y,t)\,dxdydt\Big| \le \eta/3. \tag{75}$$

Finally, using (71)-(75), we conclude that, for any given $\varphi \in \mathcal{C}_c(\Omega\times J;\mathcal{C}_\#(Q_m))$ and $\eta > 0$, there exists $\epsilon' = \min(\epsilon_1,\epsilon_2) > 0$ such that $|L_\epsilon| < \eta$ for any $\epsilon < \epsilon'$, fulfilling the statement of the lemma. □

5. **Conclusion.** Beyond the particular application considered in the present paper, we believe that the precise description of the embedded grids approach and of the density arguments coupled with the dilation method for the homogenization of nonlinear terms will be a very useful tool for many applications. This description is, as far as we know, completely original.

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