Asymptotic Analysis of a Radionuclide Transport Model With Unbounded Viscosity

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Received 18 December 2007
Communicated by Herbert Amann

Abstract

We consider a model describing compressible nuclear waste disposal contamination in porous media. The transport of brine and radionuclides is described by a nonlinear coupled degenerate parabolic system. The viscosity of the fluid is unbounded and concentrations and temperature dependent. We study the asymptotic behavior of the model for little Peclet numbers.

2000 Mathematics Subject Classification. 35K55, 35K65, 35B40, 76S05, 76V05.
Key words. Nuclear waste, miscible compressible displacement, porous media, nonlinear coupled degenerate parabolic system, asymptotic analysis.

1 Introduction

We consider here a waste-disposal problem in which high-level radioactive waste is buried in a salt dome. The salt then dissolves to create a brine and $N$ radionuclides are transported by a miscible compressible flow. We aim to study the asymptotic behavior of the displacement for the expected regimes with a low Darcy rate of flow. It corresponds to little associate Peclet numbers, lower than 1, when the permeability of the rock is small. The dispersions effects are then negligible with regard to the diffusions ones (see [5]). We include in the model the mechanisms of sorption, of radioactive filiation and decay and the important thermal effects that they induce. The far-field repository is represented by a domain $\Omega$ of $\mathbb{R}^2$ with
The specific internal energy and the enthalpy are defined by 
rock and fluid, and by Limit model $\varepsilon$ tends to zero and to justify the existence of physically relevant solutions for the dispersions effects. Our aim is to study the asymptotic behavior of this system as in $\Omega$ the pressure, by $\hat{\nu}$ smooth boundary $\Gamma$. The unit normal pointing outward $\Omega$ is denoted by $\nu$. Due to the mass and energy conservation (cf [12, 5, 11]), the flow is governed in $\Omega_T$ by the system (1.1)-(1.5) below. The parameter $\varepsilon$ describes the order of the dispersions effects. Our aim is to study the asymptotic behavior of this system as $\varepsilon$ tends to zero and to justify the existence of physically relevant solutions for the limit model

\begin{align}
\phi F \partial_t p + \phi \partial_t F + \text{div}(\nu \varepsilon) &= -q - \sum_{j=1}^{N} q_j + R'_s(\varepsilon), \quad (1.1) \\
\nu \varepsilon &= -\frac{k}{\mu(c,\varepsilon,\theta)} \nabla p, \quad (1.2) \\
\partial_t \varepsilon - \text{div}\left( \frac{1}{d_2(\rho(p),\varepsilon)} (\phi c_p(\varepsilon)(K_m(p) + \varepsilon |\varepsilon|) \nabla \theta - H_{\varepsilon \varepsilon}) \right) \\
- \theta_0 - F \text{div}\left( \frac{c_p(\varepsilon)}{d_2(\rho(p),\varepsilon)} \varepsilon \right) + C(\rho(p),\varepsilon) \nu \varepsilon \cdot \nabla \varepsilon = \frac{U_\varepsilon}{d_2(\rho(p),\varepsilon)} \left( q_H + q_{H\varepsilon} \right), \quad (1.3) \\
\phi \partial_t \varepsilon + \nu \varepsilon \cdot \nabla \varepsilon - \text{div}(\phi(D_m + \varepsilon |\varepsilon|) \nabla \varepsilon) + d_3(\varepsilon) \partial_t p - \phi \partial_t F \varepsilon \\
= \hat{\varepsilon} + \sum_{j=1}^{N} q_j + (1 - \hat{\varepsilon}) R'_s(\hat{\varepsilon}), \quad (1.4) \\
\phi \partial_t \varepsilon + \nu \varepsilon \cdot \nabla \varepsilon - \text{div}(\phi(D_m + \varepsilon |\varepsilon|) \nabla c_{i,\varepsilon}) + d_3(c_{i,\varepsilon}) \partial_t p - \phi \partial_t F c_{i,\varepsilon} \\
= -c_{i,\varepsilon} R'_s(\hat{\varepsilon}) - q_i + c_{i,\varepsilon} \sum_{j=1}^{N} q_j - \lambda_i K_i \phi c_{i,\varepsilon} \\
+ \sum_{j=1,j\neq i}^{N-2} k_{i,j} \lambda_j K_j \phi c_{j,\varepsilon} + k_{i,N} \lambda_N K_N \phi \varepsilon, \quad (1.5)
\end{align}

with

$$c = (\hat{\varepsilon}, c_1, ..., c_{N-2}), \quad \hat{\varepsilon} = 1 - \sum_{j=1}^{N-2} c_j - \hat{\varepsilon}.$$ 

The diffusions effects are characterized by $K_m$ which is the heat conductivity of rock and fluid, and by $D_m$ which is the molecular diffusion. We assume

$$K_m = K_m(p) = \frac{k_m}{\rho(p)}, \quad k_m > 0, \quad D_m > 0. \quad (1.6)$$

The specific internal energy and the enthalpy are defined by

$$U = U(\theta) = \phi(\theta)(\theta - \theta_0), \quad H = H(\theta, p) = U_0 + U + \frac{p}{\rho}.$$
Radionuclide transport model with unbounded viscosity

The density $\rho$ satisfies

$$\rho = \rho(p) = \rho_0 \exp(c_w(p - p_0)).$$  \hspace{1cm} (1.7)

The real number $c_w$ is the compressibility of the fluid. The reference energy, temperature, pressure and density $U_0$, $\theta_0$, $p_0$, $\rho_0$, are given real numbers. A strong coupling is induced in the system by the viscosity $\mu$ which is concentrations and temperature dependent. There is no explicit model of the viscosity valid for any nonnegative temperature. But $\mu \to \infty$ as the temperature goes to the temperature of transition liquid/solid (cf. [10]). Denoting by $\theta_- > 0$ this temperature, we thus assume that the heat capacities are defined in $(0, 1)^{N-1} \times [\theta_-, +\infty)$, satisfying

$$\begin{cases}
1/\sqrt{\mu} \in W^{1,\infty}((0, 1)^{N-1} \times [\theta_-, +\infty)), \\
(c, \theta) \in (0, 1)^{N-1} \times [\theta_-, +\infty) \text{ and } \mu(c, \theta)^{-1} = 0 \iff \theta = \theta_-. \hspace{1cm} (1.8)
\end{cases}$$

The viscosity is then unbounded for temperatures in a neighborhood of $\theta_-$. The crucial consequence is the degenerating of the parabolic pressure equation (1.1). We denote by $\mu_-$ the real number such that

$$0 < \mu_- \leq \mu(c, \theta) \ \forall (c, \theta) \in (0, 1)^{N-1} \times (\theta_-, +\infty). \hspace{1cm} (1.9)$$

We have

$$d_2(\rho, \theta) = \phi(x) c_p(\theta) + (1 - \phi(x)) \frac{\rho R}{\rho} c_{pR}(\theta),$$

$$C(\rho, \theta) = (\theta - \theta_0) D_\theta \left( \frac{c_p(\theta)}{d_2(\rho, \theta)} \right).$$

The positive function $c_p$ (resp. $c_{pR}$) is the specific heat of the fluid (resp. of the rock), and $\rho_R$ is the rock density constant. For these functions we have once again explicit model valid for a wide range of temperature. For a nuclear repository site one generally adopts the following form of specific heat (see [9] for Yucca Mountain and the references therein).

$$c_{p(R)}(\theta) = A_0 + A_1 \theta + A_2 \theta^2 + A_3 \theta^3,$$

with $(A_0, A_1, A_2, A_3) \in \mathbb{R}^4$. This type of relation remains true during phase transformations. Indeed these transitions absorb heat and then increase the specific heat capacities of the constituents as a function of temperature. Consistently with the latter relation, we assume that the heat capacities are defined in $(\theta_-, +\infty)$ and that there exists some real numbers $\left( \frac{c_p}{d_2} \right)_- \text{ and } \left( \frac{1}{d_2} \right)_+$ such that

$$\begin{cases}
0 < \left( \frac{c_p}{d_2} \right)_- \leq \frac{c_p(\theta)}{d_2(\rho, \theta)} \leq \left( \frac{c_p}{d_2} \right)_+^{-1}, \\
\forall \theta \in (\theta_-, +\infty), \forall p \in (m, M),
\end{cases}$$

$$\begin{cases}
\left( \frac{c_p(\theta)}{d_2(\rho, \theta)} \right)_- D_\theta \left( \frac{c_p(\theta)}{d_2(\rho, \theta)} \theta \right) \in (L^\infty((\theta_-, \infty) \times (m, M))^2. \hspace{1cm} (1.10)
\end{cases}$$
the real numbers $m$ and $M$ being defined in (1.20) below. The functions $d_3$ and $d_{3i}$ are defined for $c \in (0,1)$ by

$$d_3(c) = \phi_1(x) c (1 - F(x,t)),$$  
$$d_{3i}(c) = \phi_1(x) c (K_i - F(x,t)).$$

The porosity $\phi$ (and $\phi_1 = c_w \phi$), the components of the permeability tensor $k$ and the retardation factors $K_i$ are in $L^\infty(\Omega)$, while $F(x,t)$ belongs to $W^{1,\infty}(\Omega_T)$. Moreover, for some real numbers $0 < F_- \leq F_+ < 0 < k_- \leq k_+$ and $1 \leq K_- \leq K_+$, we have

$$\begin{cases}
F_- \leq F(x,t) \leq F_+ \quad \text{a.e. in } \Omega_T, \\
\phi_- \leq \phi(x) \leq \phi_+ \quad \text{a.e. in } \Omega, \\
 k_- |\xi|^2 \leq k(x) \xi \cdot \xi, \quad |k(x)\xi| \leq k_+ |\xi| \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^2, \\
K_- \leq K_i(x) \leq K_+ \quad \text{a.e. in } \Omega, \quad i = 1, \ldots, N.
\end{cases} (1.11)$$

The real number $k_{ji}$ is the mass rate between the parent radionuclide $j$ and the daughter one $i$. Since the processes of radioactive decay and filiation compensate themselves, the real numbers $k_{ji}$ are such that $\sum_{j=1, j \neq i}^{N} k_{ji} = 1$, $1 \leq i \leq N$. Note also that $K_{Ni} \tilde{c} = F(x,t) - \sum_{j=1}^{N-1} K_j(x)c_j - \hat{c}$, where $F(x,t)$ describes the total sorption capacity of the rock. The real number $\lambda^{-1} > 0$ is the half-life of the radionuclide $i$. The salt source term is defined for $c \in (0,1)$ by

$$R'_s(c) = \frac{c_s \phi}{1 + c_s} K_s f_s (1 - c).$$

The real numbers $c_s$, $f_s$ and $K_s$ are rate constants characterizing salt dissolution in the reservoir. The other source terms $q_s$, $q_l$, $q_H$ belong to $L^\infty(\Omega_T)$, and $q_i(x,t) \geq 0$ a.e. in $\Omega_T$. We assume besides

$$\frac{c_p}{F} \left( -q (1 - F) - \sum_{j=1}^{N} q_j + R'_s(C) - \phi \partial_t F \right) (\theta - \theta_0)$$  
$$+ q \left( U_0 + (c_w \rho (1/c_w))^{-1} \right) + q_H \leq 0 \quad \forall C \in (0,1), (1.12)$$

where $\theta_+ > 0$. It means that the reaction is always exothermic when $\theta$ is in a neighborhood of $\theta_+$ (cf [14]). This latter assumption is all the more reasonable because we can choose a convenient reference temperature $\theta_0$.

We consider the following initial and boundary conditions.

$$\begin{aligned}
\frac{u}{\varepsilon} \cdot \nu &= 0 \quad \text{on } \Gamma_T, \\
p_c(x,0) &= p_{\text{init}}(x) \quad \text{in } \Omega, \\
(K_m(p_c) + \varepsilon |u_c|) \nabla \theta_c \cdot \nu &= 0 \quad \text{on } \Gamma_T, \\
\theta_c(x,0) &= \theta_{\text{init}}(x) \quad \text{in } \Omega, \\
(D_m + \varepsilon |u_c|) \nabla c_c \cdot \nu &= 0 \quad \text{on } \Gamma_T, \\
c_c(x,0) &= c_{\text{init}}(x) \quad \text{in } \Omega, \\
(D_m + \varepsilon |u_c|) \nabla c_{\text{in}} \cdot \nu &= 0 \quad \text{on } \Gamma_T, \\
c_{\text{in}}(x,0) &= c_{\text{init}}(x) \quad \text{in } \Omega, (1.16)
\end{aligned}$$

with $p_{\text{init}} \in H^1(\Omega)$ and $(\theta_{\text{init}}, c_{\text{init}}, (c_{i,\text{init}})_{i=1}^{N-1}) \in (L^\infty(\Omega))^N$ satisfying

$$m_0 \leq p_{\text{init}}(x) \leq M_0 \quad \text{a.e. in } \Omega, (1.17)$$
0 < \theta_- \leq \theta_{init}(x) \quad \text{a.e. in } \Omega, \quad (1.18)

0 \leq \dot{c}_{init}(x), c_{i,init}(x), \quad \sum_{i=1}^{N-2} c_{i,init}(x) + \dot{c}_{init}(x) \leq 1 \quad \text{a.e. in } \Omega. \quad (1.19)

We define two real numbers \( m \) and \( M \) by

\[ m = -\left(\|q\|_\infty + \phi_+ \|\partial_t F\|_\infty\right) T + m_0, \quad M = \left(\frac{c_p \phi K_{\xi} f_\xi}{1 + c_p} + \phi_+ \|\partial_t F\|_\infty\right) T + M_0, \quad (1.20) \]

where \( M_0 \) is such that \( \phi_-(\inf_{(\theta_-^-, \infty)} c_p) + (1 - \phi_+(\inf_{(\theta_-^-, \infty)} c_p R)) \rho_R / \rho(M) > 0 \). Then \( d_2(\rho, \theta) > 0 \) if \( \rho \leq \rho(M) \).

For any fixed real \( \varepsilon > 0 \), the following existence result is deduced from [2].

**Theorem 1.1** Under the aforementioned hypotheses and for any fixed \( \varepsilon > 0 \), there exists a weak solution \( (p_\varepsilon, \theta_\varepsilon, \dot{c}_\varepsilon, (c_{i,\varepsilon})_{i=1}^{N-2}) \) of Problem (1.1)-(1.5), (1.13)-(1.16) satisfying

i) the function \( p_\varepsilon \in L^\infty(\Omega_T) \), with \( m \leq p_\varepsilon(x, t) \leq M \) a.e. in \( \Omega_T \), is solution of (1.1)-(1.2), (1.13) verified in \( L^2(0, T; H^{-1}(\Omega)) \); the velocity \( u_\varepsilon = -\frac{k}{\mu(c_\varepsilon, \theta_\varepsilon)} \nabla p_\varepsilon \) belongs to \( L^2(\Omega_T, \Omega) \) and \( \sqrt{\mu(c_\varepsilon, \theta_\varepsilon)} u_\varepsilon \) is in \( L^2(\Omega_T) \);

ii) the function \( (\dot{c}_\varepsilon, c_{1,\varepsilon}, ..., c_{N-2,\varepsilon}) \in (L^\infty(\Omega_T) \cap L^2(0, T; H^1(\Omega)))^{N-1} \) is such that \( 0 \leq \dot{c}_\varepsilon(x, t), c_{i,\varepsilon}(x, t) \) and \( \sum_{i=1}^{N-2} c_{i,\varepsilon}(x, t) + \dot{c}_\varepsilon(x, t) \leq 1 \) a.e. in \( \Omega_T \), while \( \theta_\varepsilon \in L^\infty(\Omega_T) \cap L^2(0, T; H^1(\Omega)) \) and \( \dot{c}_\varepsilon(x, t) \geq \theta_- > 0 \) a.e. in \( \Omega_T \).

In the present paper, we aim to prove the following convergence result.

**Theorem 1.2** For extracted subsequences, the solution \( (p_\varepsilon, \theta_\varepsilon, \dot{c}_\varepsilon, (c_{i,\varepsilon})_{i=1}^{N-2}) \) of Pb. (1.1)-(1.5), (1.13)-(1.16) converges in a weak sense to \( (p, \theta, \dot{c}, (c_{i})_{i=1}^{N-2}) \) which is a weak solution of the following problem where the dispersions effects are completely neglected.

\[
\phi_1 F \partial_t p + \phi_+ \partial_t F + \text{div}(u) = -q - \sum_{j=1}^{N} q_j + R_s' (\dot{c}), \quad (1.21)
\]

\[
u = -\frac{k}{\mu(c, \theta)} \nabla p, \quad (1.22)
\]

\[
\partial_t \theta - \text{div}\left(\frac{1}{d_2(p(p), \theta)} \left(\phi c_p(\theta) K_m(p) \nabla \theta - H_u \frac{u}{F d_2(p(p), \theta)}\right) - \frac{\theta - \theta_0}{F} \text{div}\left(\frac{c_p(\theta)}{d_2(p_p, \theta)} \frac{u}{F d_2(p(p), \theta)}\right) + C(p(p), \theta) \nabla \theta - \frac{U}{F d_2(p(p), \theta)} \left(\phi \partial_t F + q + \sum_{j=1}^{N} q_j - R_s' (\dot{c})\right)\right) = \frac{1}{d_2(p(p), \theta)} (q_H + q_H), \quad (1.23)
\]
\[ \phi \partial_t \hat{c} + \mathbf{u} \cdot \nabla \hat{c} - \text{div}(\phi D_m \nabla \hat{c}) + d_3(\hat{c}) \partial_t p - \phi \partial_t F \hat{c} = \hat{c} \sum_{j=1}^N q_j + (1 - \hat{c}) R'_s(\hat{c}), \]  

(1.24)

\[ \phi K_i \partial_t c_i + \mathbf{u} \cdot \nabla c_i - \text{div}(\phi D_m \nabla c_i) + d_{3i}(c_i) \partial_t p - \phi \partial_t F c_i = -c_i R'_s(\hat{c}) - q_i \]

(1.25)

provided by the following initial and boundary conditions

\[ \mathbf{u} \cdot \nu = 0 \quad \text{on} \quad \Gamma_T, \quad p(x,0) = p_{\text{init}}(x) \quad \text{in} \quad \Omega, \]

(1.26)

\[ \bar{K}_m(p) \nabla \theta \cdot \nu = 0 \quad \text{on} \quad \Gamma_T, \quad \theta(x,0) = \theta_{\text{init}}(x) \quad \text{in} \quad \Omega, \]

(1.27)

\[ D_m \nabla \hat{c} \cdot \nu = 0 \quad \text{on} \quad \Gamma_T, \quad \hat{c}(x,0) = \hat{c}_{\text{init}}(x) \quad \text{in} \quad \Omega, \]

(1.28)

\[ D_m \nabla c_i \cdot \nu = 0 \quad \text{on} \quad \Gamma_T, \quad c_i(x,0) = c_{i,\text{init}}(x) \quad \text{in} \quad \Omega. \]

(1.29)

Furthermore, \((p, \theta, \hat{c}, (c_i)_{i=1}^{N-2})\) has the same regularity properties as those of the solution of the original problem described in (i) and (ii) of Theorem 1.

The simulation (see [11]) and the numerical analysis (see for instance [7, 4, 6]) of such problems where the dispersion effects are neglected has been extensively studied in the past decade. But the rigorous justification of the model with neglected dispersion is not addressed. We can only cite [1] who treat a simplified non radioactive model with constant viscosity. And [3] considers the one-dimensional case for a mixture of two species with different compressibilities (this adds a non-linearity in the problem) but with a bounded viscosity. More generally, there are very few mathematical results about fluid problems with unbounded viscosity (see for instance the recent work [8] and the references therein).

The paper is organized as follows. Classical energy estimates are performed in Section 2. We also state the compactness results which can be obtained with arguments of Aubin’s type. Section 3 is devoted to the convergence analysis. We use astute tools to get sufficient informations about the pressure behavior in spite of the degenerating of Eq. (1.1).

## 2 Energy estimates and first compactness results

In what follows, the letter \( C \) denotes a generic quantity, independent of \( \varepsilon \). We begin with some properties of the pressure \( p_\varepsilon \) solution of (1.1)-(1.2), (1.13).

**Lemma 2.1** The function \( p_\varepsilon \in L^\infty(\Omega_T) \cap L^2(0,T; H^1(\Omega)) \) solution of (1.1)-(1.2), (1.13) satisfies

\[ m \leq p_\varepsilon(x,t) \leq M \quad \text{a.e. in} \quad \Omega_T, \]

\[ \|p_\varepsilon\|_{L^\infty(0,T; L^2(\Omega))} + \left\| \left( \frac{k_{\varepsilon}}{\mu(\varepsilon, c_\varepsilon)} \right)^{1/2} \nabla p_\varepsilon \right\|_{(L^2(\Omega_T))^2} \leq C, \quad \|\mathbf{u}_\varepsilon\|_{(L^2(\Omega_T))^2} \leq C_1, \]
where $C$ (resp. $C_1$) is a constant which only depends on $T$ (resp. on $T$ and $\mu_-$ defined in (1.9)). Furthermore, the function $\phi_1 F \partial_t p_\varepsilon$ is uniformly bounded in $L^2(0, T; (H^1(\Omega))^*)$.

Proof. The first estimate is a direct consequence of the construction of $p_\varepsilon$ in [2]. One also can check that since the right hand-side terms of (1.1) belong to $L^\infty(\Omega_T)$, a maximum principle with Hypothese (1.17) leads to $m \leq p_\varepsilon(x, t) \leq M$ a.e. in $\Omega_T$.

We now multiply Eq. (1.1) by $p_\varepsilon$ and integrate by parts over $\Omega$. We obtain

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \phi_1 F |p_\varepsilon|^2 dx + \int_\Omega (\phi_1 p_\varepsilon - \frac{1}{2} \phi_1 |p_\varepsilon|^2) \partial_t F dx 
+ \int_\Omega \frac{\mu(c_\varepsilon, \theta_\varepsilon)}{\mu(c_\varepsilon, \theta_\varepsilon)} \nabla p_\varepsilon \cdot \nabla p_\varepsilon dx = - \int_\Omega (q + \sum_{j=1}^N q_j - R_\varepsilon'(\varepsilon) ) p_\varepsilon dx.
$$

We estimate the second term of the left hand-side and the right hand-side of the latter relation using the assumptions $\phi_1, \phi \in L^\infty(\Omega), \partial_t F \in L^\infty(\Omega_T)$, $q, q_j \in L^\infty(\Omega_T)$ and $\varepsilon c_\varepsilon \in L^\infty(\Omega_T)$ (see the maximum principle (ii) of Theorem 1). With Assumption (1.11) for $k$, we get

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \phi_1 F |p_\varepsilon|^2 dx + \int_\Omega \frac{k_\varepsilon}{\mu(c_\varepsilon, \theta_\varepsilon)} |\nabla p_\varepsilon|^2 dx \leq C \int_\Omega |p_\varepsilon|^2 dx + C.
$$

We then use the Gronwall lemma and the assumption $\phi_1 F \geq c_w \phi$. $\mu_-$ > 0 to obtain the estimates for $p_\varepsilon$ in $L^\infty(0, T; L^2(\Omega))$ and for $\nabla p_\varepsilon$ in $(L^2(\Omega_T))^2$. The estimate for $u_\varepsilon$ in $(L^2(\Omega_T))^2$ then follows from Assumption (1.9). Finally, multiplying Eq. (1.1) by any test function $\psi \in L^2(0, T; H^1(\Omega))$, one checks that

$$
|\int_\Omega (\phi_1 F \partial_t p_\varepsilon, \psi)_{L^2(0, T; (H^1(\Omega)))^\prime \times L^2(0, T; H^1(\Omega))}| \leq C(\|\psi\|_{L^1(\Omega_T)} + \|\nabla \psi\|_{(L^2(\Omega_T))^2}).
$$

This ends the proof of Lemma 1.

We now study the temperature problem (1.3)-(1.14). We claim the following result.

**Lemma 2.2** For any $\varepsilon > 0$, the function $\theta_\varepsilon$ solution of (1.3)-(1.14) is in the space $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ and it satisfies:

i) $\theta_\varepsilon(x, t) \geq \theta_- > 0$ almost everywhere in $\Omega_T$;

ii) the sequence $(\theta_\varepsilon)$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, and $(\varepsilon |u_\varepsilon|)^{\frac{1}{2}} \nabla \theta_\varepsilon$ is uniformly bounded in the space $(L^2(\Omega_T))^2$;

iii) the sequence $(\theta_\varepsilon)$ is sequentially compact in $L^2(\Omega_T)$.

Proof. Assuming (1.6)-(1.12), the existence of a solution $\theta_\varepsilon$ to the parabolic problem (1.3), (1.14) with $\theta_- \leq \theta_\varepsilon(x, t)$ a.e. in $\Omega_T$ is given by the construction in [2]. To
prove ii), we multiply Eq. (1.3) by $\theta_\varepsilon$ and integrate over $\Omega$. Integrating by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\theta_\varepsilon(\cdot, t)|^2 \, dx + \int_\Omega \frac{\phi_{p_\varepsilon}(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} (K_m + \varepsilon |\omega_\varepsilon|) \nabla \theta_\varepsilon \cdot \nabla \theta_\varepsilon \, dx \\
- \int_\Omega \frac{H_\varepsilon}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} (\omega_\varepsilon \cdot \nabla \theta_\varepsilon) \, dx + \int_\Omega C(\rho(p_\varepsilon), \theta_\varepsilon) \theta_\varepsilon \omega_\varepsilon \cdot \nabla \theta_\varepsilon \, dx \\
- \int_\Omega \frac{\theta_\varepsilon - \theta_0}{F} \text{div} \left( \frac{c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} \omega_\varepsilon \right) \theta_\varepsilon \, dx \\
- \int_\Omega \frac{U_\varepsilon}{F} \theta_\varepsilon \omega_\varepsilon \, dx + \sum_{j=1}^N q_j - R'_\varepsilon(\hat{c}_\varepsilon)) \theta_\varepsilon \, dx \\
= - \int_\Omega \frac{q_{H_\varepsilon}}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} \theta_\varepsilon \, dx - \int_\Omega \frac{q}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} (U_0 + U_\varepsilon + \frac{p_\varepsilon}{\rho(p_\varepsilon)}) \theta_\varepsilon \, dx. \quad (2.1)
\]

We begin by recalling that the functions $p_\varepsilon$ and then $\rho(p_\varepsilon)$ are bounded in $L^\infty(\Omega_T)$, uniformly in $\varepsilon$. We also bear in mind the bounds for $c_p/d_2$, $1/d_2$ and $C$ given by Assumption (1.10). Due to the definition of $H_\varepsilon$, we have
\[
\left| \int_\Omega \frac{1}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} H_\varepsilon (\omega_\varepsilon \cdot \nabla \theta_\varepsilon) \, dx \right| \\
= \left| \int_\Omega \frac{1}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} (U_0 + c_p(\theta_\varepsilon) (\theta_\varepsilon - \theta_0) + \frac{p_\varepsilon}{\rho(p_\varepsilon)}) (\omega_\varepsilon \cdot \nabla \theta_\varepsilon) \, dx \right|.
\]

Thanks to the Cauchy-Schwarz and Young inequalities, we write with Assumption (1.10)
\[
\left| \int_\Omega \frac{1}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} (U_0 - c_p(\theta_\varepsilon) \theta_0) (\omega_\varepsilon \cdot \nabla \theta_\varepsilon) \, dx \right| \leq \left( \frac{c_p}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} \right)^{-1} + \left( \frac{1}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} \right)^{-1} \\
\times C \int_\Omega |\omega_\varepsilon|^2 \, dx + \delta \int_\Omega |\nabla \theta_\varepsilon|^2 \, dx \leq C \frac{1}{\delta} \int_\Omega |\omega_\varepsilon|^2 \, dx + \delta \int_\Omega |\nabla \theta_\varepsilon|^2 \, dx,
\]
\[
\left| \int_\Omega \frac{c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon), \theta_\varepsilon)} (\omega_\varepsilon \cdot \nabla \theta_\varepsilon) \, dx \right| \leq \|c_p(\theta_\varepsilon)\|_\infty \left( \delta \int_\Omega |\nabla \theta_\varepsilon|^2 \, dx + \frac{C}{4\delta} \int_\Omega |\omega_\varepsilon|^2 \, dx \right),
\]
\[
\left| \int_\Omega C(\rho(p_\varepsilon), \theta_\varepsilon) \theta_\varepsilon (\omega_\varepsilon \cdot \nabla \theta_\varepsilon) \, dx \right| \leq \|C(\rho(p_\varepsilon), \theta_\varepsilon)\|_\infty \left( \delta \int_\Omega |\nabla \theta_\varepsilon|^2 \, dx + \frac{C}{4\delta} \int_\Omega |\omega_\varepsilon|^2 \, dx \right)
\]
and
\[
\left| \int_\Omega \frac{p_\varepsilon}{\rho(p_\varepsilon)} (\omega_\varepsilon \cdot \nabla \theta_\varepsilon) \, dx \right| \leq \delta \int_\Omega |\nabla \theta_\varepsilon|^2 \, dx + \frac{C}{4\delta} \int_\Omega |\omega_\varepsilon|^2 \, dx,
\]
for any $\delta > 0$. The function $(p_\varepsilon/\rho(p_\varepsilon))$ is indeed uniformly bounded in $L^\infty(\Omega_T)$ since $m \leq p_\varepsilon(x,t) \leq M$ a.e. in $\Omega_T$. Now we transform the fourth term of Relation
(2.1). It follows by integration by parts that
\[
\int_{\Omega} \frac{\theta_e - \theta_0}{F} \text{div} \left( \frac{c_p(\theta_e)}{d_2(\rho(p_x), \theta)} u_e \right) \theta_e \, dx
\]
\[= - \int_{\Omega} \frac{c_p(\theta_e)}{d_2(\rho(p_x), \theta)} (\theta_e - \theta_0) \cdot \nabla (F^{-1}) \, dx \]
\[- \int_{\Omega} \frac{c_p(\theta_e)}{d_2(\rho(p_x), \theta)} 2 \frac{\partial}{\partial t} \cdot \nabla \theta_e \, dx + \int_{\Omega} \frac{c_p(\theta_e)}{d_2(\rho(p_x), \theta)} \theta_0 \cdot \nabla \theta_e \, dx.
\]

We have replaced the duality product \langle \cdot, \cdot \rangle_{H^1(\Omega)^* \times H^1(\Omega)} by three terms in the form \langle \cdot, \cdot \rangle_{L^2(\Omega)^* \times L^2(\Omega)}. We recall that \(F^{-1} \in W^{1,\infty}(\Omega_T)\). Thus, similar tools as previously allow to estimate the terms in the right hand-side of the latter description of \langle \text{div} (\frac{c_p(\theta)}{d_2(\rho(p_x), \theta)} u_e), \frac{\partial}{\partial t} \theta \rangle_{H^1(\Omega)^* \times H^1(\Omega)} and the other terms of (2.1) do not bring additional difficulty. Bearing in mind that
\[
\int_{\Omega} \frac{\phi c_p(\theta_e)}{d_2(\rho(p_x), \theta)} (K_m + \varepsilon |u_e|) \nabla \theta_e \cdot \nabla \theta_e \, dx
\]
\[\ge \int_{\Omega} \left( \frac{c_p}{d_2} \right) \phi - \left( \frac{k_m}{\rho(M)} + \varepsilon |u_e| \right) |\nabla \theta_e|^2 \, dx,
\]
the previous estimates finally yield in (2.1) to
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta_e(t)|^2 \, dx + \int_{\Omega} \left( \left( \frac{c_p}{d_2} \right) - \phi - \left( \frac{k_m}{\rho(M)} + \varepsilon |u_e| \right) \right) |\nabla \theta_e|^2 \, dx
\]
\[\le \frac{C}{\delta} (1 + \|u_e\|_{L^2(\Omega)}^2) + C \int_{\Omega} |\theta_e|^2 \, dx,
\]
for any \(\delta > 0\). We choose \(\delta \) such that \(\left( \left( \frac{c_p}{d_2} \right) - \phi - \frac{k_m}{\rho(M)} - \delta \right) > 0\). The Gronwall lemma then yields the result \(ii\).

It remains to prove \(iii\). Let \(\psi \in L^\infty(0, T; W^{1,4}(\Omega))\). We multiply Eq. (1.3) by \(\psi\) and integrate over \(\Omega_T\). Integrating by parts, we get
\[
|\langle \partial_t \theta_e, \psi \rangle_{L^1(0, T; (W^{1,4}(\Omega))^*)}, L^\infty(0, T; W^{1,4}(\Omega))| \leq \]
\[\int_{\Omega_T} \frac{\phi c_p(\theta_e)}{d_2(\rho(p_x), \theta)} \left( \frac{k_m}{\rho(p_x)} + \varepsilon |u_e| \right) \nabla \theta_e \cdot \nabla \psi \]
\[+ \int_{\Omega_T} \frac{c_p(\theta_e)}{d_2(\rho(p_x), \theta)} |u_e \cdot \nabla \psi| + \int_{\Omega_T} \frac{U_e}{F d_2(\rho(p_x), \theta)} (u_e \cdot \nabla \psi)
\]
\[+ \int_{\Omega_T} C(\rho(p_x), \theta) (u_e \cdot \nabla \theta_e) \psi + \int_{\Omega_T} \frac{c_p(\theta_e)}{d_2(\rho(p_x), \theta)} (u_e \cdot \nabla \theta_e) \psi
\]
\[+ \int_{\Omega_T} \frac{U_e}{F^2 d_2(\rho(p_x), \theta)} (u_e \cdot \nabla F) \psi + \int_{\Omega_T} \frac{1}{d_2(\rho(p_x), \theta)} (q u + q H_e) \psi
\]
\[+ \int_{\Omega_T} \frac{U_e}{F d_2(\rho(p_x), \theta)} (\phi \partial_t F + q + \sum_{j=1}^N q_j - R'(c_e)) \psi.
\]
Each term of the right hand-side of the latter relation can be estimated as follows.

\[
\left| \langle \partial_t \theta, \psi \rangle_{L^1([0,T];(W^{1,4}(\Omega)))}, L^\infty([0,T];W^{1,4}(\Omega)) \right| \\
\leq C \| \nabla \theta \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)}^2 + C \| c \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)}^2
\]

\[
+ \frac{d_2(\rho(p_x), \theta_x)}{d_2(\rho(p_x), \theta_x)} \| u_x \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)}^2
\]

\[
+ C((c_p/d_2) - (1/d_2)_+, m, M, F \| \nabla \psi \|_{L^2(\Omega)}, F_- \| \nabla \psi \|_{L^2(\Omega)}, q_j \| \nabla \psi \|_{L^2(\Omega)}
\]

\[
+ C \| u_x \|_{L^2(\Omega)} \| \nabla \theta_x \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)} \leq C \| \nabla \psi \|_{L^\infty([0,T];W^{1,4}(\Omega))}.
\]

So the sequence \((\partial_t \theta_x)\) is uniformly bounded in \(L^1(0,T; (W^{1,4}(\Omega))').\) We then conclude with arguments of Aubin’s type \([13]\). The proof is achieved.

Next we consider the concentrations problem. We claim the following lemma.

**Lemma 2.3** For any fixed \(\varepsilon > 0\), the solution \(c_\varepsilon = (\hat{c}_\varepsilon, c_{1,\varepsilon}, \ldots, c_{N-2,\varepsilon})\) of Problem (1.4)-(1.5), (1.15)-(1.16) belongs to \((L^\infty(\Omega_T) \cap L^2(0,T; H^1(\Omega)))^{N-1}\) and satisfies

i) \(0 \leq c_{1,\varepsilon}(x,t), \hat{c}_\varepsilon(x,t)\) and \(\sum_{i=1}^{N-2} c_{i,\varepsilon}(x,t) + \hat{c}_\varepsilon(x,t) \leq 1\) a.e. in \(\Omega_T;\)

ii) the sequences \((\hat{c}_\varepsilon)\) and \((c_{1,\varepsilon})\) are uniformly bounded in \(L^2(0,T; H^1(\Omega))\), while

\((\| c_{1,\varepsilon} \|) \frac{1}{\varepsilon} \nabla \hat{c}_\varepsilon\) and \((\| c_{1,\varepsilon} \|) \frac{1}{\varepsilon} \nabla \hat{c}_\varepsilon\) are bounded in \(L^2(\Omega_T)^2\);

iii) the sequences \((\hat{c}_\varepsilon)\) and \((c_{1,\varepsilon})\) are sequentially compact in \(L^2(\Omega_T)\).

**Proof.** The maximum principle in item i) is due to the construction of the solution \(c_\varepsilon\) in \([2]\). The proof of item ii) follows the lines of the proof of ii) in Lemma 2. The only difference lies in the terms containing \(\partial_t p_\varepsilon\). Let us consider for instance the one appearing when we multiply the salt equation (1.4) by \(\hat{c}_\varepsilon\) and then integrate over \(\Omega\). It writes \(\int_{\Omega} d_3(\hat{c}_\varepsilon) \partial_t p_\varepsilon \hat{c}_\varepsilon dx\). We recall that by Lemma 1, we can only assert that \(\partial_t p_\varepsilon\) is uniformly bounded in \(L^2(0,T; (H^1(\Omega))').\) So the zero order estimate for \(\hat{c}_\varepsilon\) given by the maximum principle i) is not sufficient for getting an estimate of \(\int_{\Omega} d_3(\hat{c}_\varepsilon) \partial_t p_\varepsilon \hat{c}_\varepsilon dx\). We thus note that the former expression is defined by the duality product \(\langle \phi_1 F \partial_t p_\varepsilon, \frac{d_3(\hat{c}_\varepsilon)}{\phi_1 F} \rangle_{(H^1(\Omega))^* \times H^1(\Omega)}\). Using the Cauchy-Schwarz and the Young inequalities, we then estimate it as follows.

\[
\leq C \| \phi_1 F \partial_t p_\varepsilon \|_{(H^1(\Omega))^*} \| \nabla F \|_{(L^\infty(\Omega_T))^2}^2 + C \| \frac{\| \hat{c}_\varepsilon \|_{L^\infty(\Omega_T)}^2}{\phi_1 F} \|_{(H^1(\Omega))^*}^2
\]

\[
+ \delta \int_{\Omega} \| \nabla \hat{c}_\varepsilon \|^2 dx,
\]

for any \(\delta > 0\). The terms of the right hand-side of the latter relation can be included in a relation for \(\hat{c}_\varepsilon\) similar to (2.2). We recall that we have stated in
Lemma 1 that \( \phi_1 F \partial_t p_\varepsilon \) is uniformly bounded in \( L^2(0, T; (H^1(\Omega))') \). The Gronwall lemma then gives \( ii \). Similar adaptations of the proof of Lemma 2 \( iii \) allow to claim that \( \phi \partial_t \hat{c}_\varepsilon \) and \( \phi \partial_t c_{i,\varepsilon} \) are uniformly bounded in \( L^1(0, T; (W^{1,4}(\Omega))') \). The sequential compactness of \( (\phi \hat{c}_\varepsilon) \) and \( (\phi c_{i,\varepsilon}) \) in \( L^2(0, T; (H^1(\Omega))') \) follows from Aubin’s arguments type. Since the functions \( \hat{c}_\varepsilon \) and \( c_{i,\varepsilon} \) are uniformly bounded in \( L^2(0, T; H^1(\Omega)) \), we can pass to the limit in the products \( \langle \phi \hat{c}_\varepsilon, \hat{c}_\varepsilon \rangle \) and \( \langle \phi c_{i,\varepsilon}, c_{i,\varepsilon} \rangle \) of \( L^2(0, T; (H^1(\Omega))') \times L^2(0, T; H^1(\Omega)) \). Since \( \phi(x) \geq \phi_\varepsilon > 0 \) a.e. in \( \Omega \), we conclude that \( (\hat{c}_\varepsilon) \) and \( (c_{i,\varepsilon}) \) are sequentially compact in \( L^2(\Omega_T) \).

3 Convergence results and proof of Theorem 2

The estimates of the previous section prove the existence of limit functions \( p \in L^\infty(\Omega_T), u \in (L^2(\Omega))^2, \theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), c = (\hat{c}, c_1, ..., c_{N-2}) \in (L^\infty(\Omega_T) \cap L^2(0, T; H^1(\Omega)))^{N-1} \), satisfying for extracted subsequences

\[
\begin{align*}
    p_\varepsilon & \rightharpoonup p \quad \text{weakly in} \ L^\infty(\Omega_T), \quad \frac{\partial u}{\partial t} \rightharpoonup u \quad \text{weakly in} \ (L^2(\Omega))^2, \\
    \theta_\varepsilon & \rightharpoonup \theta \quad \text{weakly in} \ L^2(0, T; H^1(\Omega)) \text{a.e. in} \ \Omega_T, \\
    c_\varepsilon & \rightharpoonup c \quad \text{weakly in} \ (L^2(0, T; H^1(\Omega)))^{N-1} \text{a.e. in} \ \Omega_T.
\end{align*}
\]

Furthermore, the functions \( \theta \) and \( c \) are physically relevant. Indeed, the maximum principles of Lemmas 2 and 3 give at the limit

\[
\theta(x, t) \geq \theta_\varepsilon > 0 \quad \text{a.e. in} \ \Omega_T,
\]

\[
0 \leq \hat{c}(x, t), c_i(x, t) \text{ and } \sum_{i=1}^{N-2} c_i(x, t) + \hat{c}(x, t) \leq 1 \quad \text{a.e. in} \ \Omega_T.
\]

Letting \( \varepsilon \to 0 \) in (1.1), we get in a first step

\[
\phi_1 F \partial_t p + \phi \partial_t F + \text{div}(u) = -q - \sum_{j=1}^{N} q_j + R'_s(\hat{c}) \quad \text{in} \ \Omega_T. \tag{3.1}
\]

We now study the limit behavior of the Darcy law (1.2). The main difficulty is the degenerating of Eq. (1.1). The lack of estimate on \( \nabla p_\varepsilon \) does not allow to pass directly to the limit in (1.2).

Lemma 3.1 The limit Darcy law is

\[
u = -\frac{k}{\mu(c, \theta)} \nabla p \quad \text{in} \ \Omega_T.
\]
Besides, the Darcy law (1.2) gives at the limit
\[
\frac{1}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} \nabla p_\varepsilon = \text{div} \left( \frac{1}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} p_\varepsilon \right) - p_\varepsilon \nabla \left( \frac{1}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} \right).
\]

We now aim to pass to the limit in the terms of the right-hand side. Since 
\[\mu(c_\varepsilon, \theta_\varepsilon) \rightharpoonup \mu(c, \theta)\] almost everywhere in \(\Omega_T\), the convergence of the sequence 
\[\text{div}(p_\varepsilon/\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)})\] to \(\text{div}(p/\sqrt{\mu(c, \theta)})\) is clear. Using Lemma 1, we note that 
\[
\partial_t(\phi_1Fp_\varepsilon) = \phi_1Fp_\varepsilon + \phi_1 p_\varepsilon \partial_t F
\]
(respectively \(\phi_1Fp_\varepsilon\) is uniformly bounded in \(L^2(0,T; (H^1(\Omega))')\)) (respectively in \(L^\infty(\Omega_T)\)). Aubin’s arguments \[13\] then lead to the sequential compactness of \((\phi_1Fp_\varepsilon)\) in \(L^2(0,T; (H^1(\Omega))')\). By Lemmas 1, 2 and 3, the function \((1/\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)})p_\varepsilon\) is uniformly bounded in \(L^2(0,T; H^1(\Omega))\). Thus, we can compute

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \frac{\phi_1F}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} p_\varepsilon^2 \, dx \, dt = \lim_{\varepsilon \to 0} \int_0^T \langle \phi_1Fp_\varepsilon, \frac{p_\varepsilon}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} \rangle_{(H^1(\Omega))' \times H^1(\Omega)} \, dt
\]

\[
= \int_{\Omega_T} \frac{\phi_1F}{\sqrt{\mu(c, \theta)}} p^2 \, dx \, dt.
\]

Since \(\phi_1(x)F(x,t) \geq c_w \phi_1 F_\varepsilon > 0\) a.e. in \(\Omega_T\), we conclude that

\[\frac{1}{\mu(c_\varepsilon, \theta_\varepsilon)^{1/4}} p_\varepsilon \rightharpoonup \frac{1}{\mu(c, \theta)^{1/4}} p\] a.e. in \(\Omega_T\) and in \(L^p(\Omega_T)\), \(\forall p \in (1, +\infty)\). (3.2)

With the a.e. convergence in \(\Omega_T\) of the sequence \((\mu'(c_\varepsilon, \theta_\varepsilon)/\mu(c_\varepsilon, \theta_\varepsilon)^{5/4})\) to its limit \(\mu'(c, \theta)/\mu(c, \theta)^{5/4}\), it leads to

\[
\frac{1}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} \nabla p_\varepsilon \left( \frac{1}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} \right) = - \frac{\mu'(c_\varepsilon, \theta_\varepsilon)}{2\mu(c_\varepsilon, \theta_\varepsilon)^{3/2}} p_\varepsilon \left( \nabla c_\varepsilon + \sum_{i=1}^{N-2} \nabla c_{i,\varepsilon} + \nabla \theta_\varepsilon \right)
\]

\[
\to p \left( \frac{1}{\sqrt{\mu(c, \theta)}} \right) \nabla \left( \frac{1}{\sqrt{\mu(c, \theta)}} \right)
\]

in \((L^2(\Omega_T))^2\), and then

\[
\frac{1}{\sqrt{\mu(c_\varepsilon, \theta_\varepsilon)}} \nabla p_\varepsilon \rightharpoonup \frac{1}{\sqrt{\mu(c, \theta)}} \nabla p \quad \text{in} \quad (L^2(\Omega_T))^2.
\]

Besides, the Darcy law (1.2) gives at the limit

\[
u = - \frac{k}{\mu(c, \theta)} \nabla p \quad \text{in} \quad \Omega_T.
\]
To treat the nonlinearities in Eqs. (1.3)-(1.5), we need an additional compactness result on \((p_\varepsilon)\).

**Lemma 3.2**

1. The sequence \((p_\varepsilon)\) converges to \(p\) almost everywhere in \(\Omega_T\).

2. Furthermore, as \(\varepsilon \to 0\), we have the following strong convergence.

\[
\frac{1}{\sqrt{\mu(c, \theta)}} \nabla p_\varepsilon \to \frac{1}{\sqrt{\mu(c, \theta)}} \nabla p \quad \text{strongly in } (L^2(\Omega_T))^2 \text{ and a.e. in } \Omega_T.
\]

**Proof.** Since \(\mu(c, \theta) \to \mu(c, \theta)\) a.e. in \(\Omega_T\) with \(\mu(c, \theta)(x, t), \mu(c, \theta)(x, t) \geq \mu_- > 0\), the first part of the lemma is proved by (3.2). We then multiply Eq. (1.1) by \(p_\varepsilon\), Eq. (3.1) by \(p\) and integrate over \(\Omega_t = \Omega \times (0, t), \ t \in (0, T)\). We obtain almost everywhere in \((0, T)\)

\[
\frac{1}{2} \int_{\Omega_t} \phi_1 F(\cdot, t) |p_\varepsilon(\cdot, t)|^2 dx - \frac{1}{2} \int_{\Omega_t} \phi_1 F(\cdot, 0) |p_{\text{init}}|^2 dx - \int_{\Omega_t} \phi_1 \partial_t F |p_\varepsilon|^2 dx ds
\]

\[
+ \int_{\Omega_t} \frac{k}{\mu(c, \theta)} \nabla p_\varepsilon \cdot \nabla p dx ds = \int_{\Omega_t} (\partial_1 F - q - \sum_{j=1}^N \delta_{j} + R'(c_\varepsilon))p_\varepsilon dx ds,
\]

\[
\frac{1}{2} \int_{\Omega_t} \phi_1 F(\cdot, t) |p(\cdot, t)|^2 dx - \frac{1}{2} \int_{\Omega_t} \phi_1 F(\cdot, 0) |p_{\text{init}}|^2 dx - \int_{\Omega_t} \phi_1 \partial_t F |p|^2 dx ds
\]

\[
+ \int_{\Omega_t} \frac{k}{\mu(c, \theta)} \nabla p \cdot \nabla p dx ds = \int_{\Omega_t} (\partial_1 F - q - \sum_{j=1}^N \delta_{j} + R'(c))p dx ds.
\]

We substract the two relations, and let \(\varepsilon \to 0\). Using the result \(i)\) of Lemma 5, we get for almost any \(t \in (0, T)\)

\[
\lim_{\varepsilon \to 0} \int_{\Omega_t} \frac{k}{\mu(c, \theta)} \nabla p_\varepsilon \cdot \nabla p_\varepsilon dx ds = \int_{\Omega_t} \frac{k}{\mu(c, \theta)} \nabla p \cdot \nabla p dx ds.
\]

It allows to compute the following limit.

\[
\lim_{\varepsilon \to 0} \int_{\Omega_t} \frac{k}{\mu(c, \theta)^{1/2}} \nabla p_\varepsilon - \frac{1}{\mu(c, \theta)^{1/2}} \nabla p \nabla p_\varepsilon - \frac{1}{\mu(c, \theta)^{1/2}} \nabla p \nabla p dx ds = 0.
\]

With Hypothesis (1.11) for \(k\), we conclude that

\[
0 \leq \lim_{\varepsilon \to 0} \int_{\Omega_t} \frac{1}{\mu(c, \theta)^{1/2}} \nabla p_\varepsilon - \frac{1}{\mu(c, \theta)^{1/2}} \nabla p \nabla p_\varepsilon - \frac{1}{\mu(c, \theta)^{1/2}} \nabla p \nabla p dx ds \leq \frac{1}{k} \lim_{\varepsilon \to 0} \int_{\Omega_t} k \left( \frac{1}{\mu(c, \theta)^{1/2}} \nabla p_\varepsilon - \frac{1}{\mu(c, \theta)^{1/2}} \nabla p \right) dx ds = 0.
\]
The proof is achieved.

We now have the tools to pass to the limit in (1.3)-(1.5) as $\varepsilon \to 0$. We only give some details for the salt equation (1.4). We write the nonlinear terms under the following forms:

\[
\begin{align*}
\mathbf{u}_\varepsilon \cdot \nabla \hat{c}_\varepsilon &= -k \left( \frac{1}{\mu(c_\varepsilon, \theta_\varepsilon)^{1/2}} \nabla p_\varepsilon \right) \cdot \left( \frac{1}{\mu(c_\varepsilon, \theta_\varepsilon)^{1/2}} \nabla \hat{c}_\varepsilon \right), \\
\phi_1 F\hat{c}_\varepsilon \partial_t p_\varepsilon &= -\hat{c}_\varepsilon \text{div}(\mathbf{u}_\varepsilon) - \phi \partial_t F\hat{c}_\varepsilon - q \hat{c}_\varepsilon - \hat{c}_\varepsilon \sum_{j=1}^{N} q_j + R'(\hat{c}_\varepsilon) \hat{c}_\varepsilon \\
&= \text{div}(\mathbf{u}_\varepsilon) - \mathbf{u}_\varepsilon \cdot \nabla \hat{c}_\varepsilon - \phi \partial_t F\hat{c}_\varepsilon - q \hat{c}_\varepsilon - \hat{c}_\varepsilon \sum_{j=1}^{N} q_j + R'(\hat{c}_\varepsilon) \hat{c}_\varepsilon.
\end{align*}
\]

Our compactness results are sufficient to pass to the limit in these expressions. Eqs. (1.3) and (1.5) can be treated in the same way. Theorem 2 is proved.

References


