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Asymptotic Analysis of a Radionuclide Transport Model With Unbounded Viscosity

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Abstract

We consider a model describing compressible nuclear waste disposal contamination in porous media. The transport of brine and radionuclides is described by a nonlinear coupled degenerate parabolic system. The viscosity of the fluid is unbounded and concentrations and temperature dependent. We study the asymptotic behavior of the model for little Peclet numbers.

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1 Introduction

We consider here a waste-disposal problem in which high-level radioactive waste is burried in a salt dome. The salt then dissolves to create a brine and N radionuclides are transported by a miscible compressible flow. We aim to study the asymptotic behavior of the displacement for the expected regimes with a low Darcy rate of flow. It corresponds to little associate Peclet numbers, lower than 1, when the permeability of the rock is small. The dispersions effects are then neglectible with regard to the diffusions ones (see [5]). We include in the model the mechanisms of sorption, of radioactive filiation and decay and the important thermal effects that they induce. The far-field repository is represented by a domain Ω of $\mathbb{R}^2$ with smooth boundary Γ. The unit normal pointing outward Ω is denoted by ν. The time interval of interest is $(0,T)$, $\Omega_T = \Omega \times (0,T)$, and $\Gamma_T = \Gamma \times (0,T)$. We denote by p the pressure, by $\hat{c}$ the salt concentration, by $(c_i)_{i=1}^{N-2}$ the concentration of $(N-2)$ radionuclides and by θ the absolute temperature. The Darcy velocity is designated by $\underline{u}$. Due to the mass and energy conservation (*cf* [12, 5, 11]), the flow is governed in Ω_T by the system (1.1)-(1.5) below. The parameter ε describes the order of the dispersions effects. Our aim is to study the asymptotic behavior of this system as ε tends to zero and to justify the existence of physically relevant solutions for the limit model

$$\phi_1 F\,\partial_t p_\varepsilon + \phi\,\partial_t F + \operatorname{div}(\underline{u}_\varepsilon) = -q - \sum_{j=1}^{N} q_j + R_s'(\hat{c}_\varepsilon), \tag{1.1}$$

$$\underline{u}_\varepsilon = -\frac{k}{\mu(c_\varepsilon,\theta_\varepsilon)}\nabla p_\varepsilon, \tag{1.2}$$

$$\begin{aligned}
&\partial_t\theta_\varepsilon - \operatorname{div}\Big(\frac{1}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\big(\phi c_p(\theta_\varepsilon)(K_m(p_\varepsilon)+\varepsilon|\underline{u}_\varepsilon|)\nabla\theta_\varepsilon - H_\varepsilon\underline{u}_\varepsilon\big)\Big)\\
&\quad -\frac{\theta_\varepsilon-\theta_0}{F}\operatorname{div}\Big(\frac{c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\underline{u}_\varepsilon\Big) + \mathcal{C}(\rho(p_\varepsilon),\theta_\varepsilon)\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon - \frac{U_\varepsilon}{F d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\Big(\\
&\quad \phi\,\partial_t F + q + \sum_{j=1}^{N} q_j - R_s'(\hat{c}_\varepsilon)\Big) = \frac{-1}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(q_H + qH_\varepsilon),
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
&\phi\,\partial_t\hat{c}_\varepsilon + \underline{u}_\varepsilon\cdot\nabla\hat{c}_\varepsilon - \operatorname{div}(\phi(D_m+\varepsilon|\underline{u}_\varepsilon|)\nabla\hat{c}_\varepsilon) + d_3(\hat{c}_\varepsilon)\,\partial_t p_\varepsilon - \phi\,\partial_t F\,\hat{c}_\varepsilon\\
&\quad = \hat{c}_\varepsilon\sum_{j=1}^{N} q_j + (1-\hat{c}_\varepsilon)\,R_s'(\hat{c}_\varepsilon),
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
&\phi\,K_i\,\partial_t c_{i,\varepsilon} + \underline{u}_\varepsilon\cdot\nabla c_{i,\varepsilon} - \operatorname{div}(\phi(D_m+\varepsilon|\underline{u}_\varepsilon|)\nabla c_{i,\varepsilon}) + d_{3i}(c_{i,\varepsilon})\,\partial_t p_\varepsilon - \phi\,\partial_t F\,c_{i,\varepsilon}\\
&\quad = -c_{i,\varepsilon}\,R_s'(\hat{c}_\varepsilon) - q_i + c_{i,\varepsilon}\sum_{j=1}^{N} q_j - \lambda_i\,K_i\,\phi\,c_{i,\varepsilon}\\
&\quad + \sum_{j=1,j\neq i}^{N-2} k_{ij}\,\lambda_j\,K_j\,\phi\,c_{j,\varepsilon} + k_{iN}\,\lambda_N\,K_N\,\phi\,\widetilde{c}_\varepsilon,
\end{aligned} \tag{1.5}$$

with

$$c = (\hat{c}, c_1, .., c_{N-2}), \quad \widetilde{c} = 1 - \sum_{j=1}^{N-2} c_j - \hat{c}.$$

The diffusions effects are characterized by K_m which is the heat conductivity of rock and fluid, and by D_m which is the molecular diffusion. We assume

$$K_m = K_m(p) = \frac{k_m}{\rho(p)},\ \ k_m > 0, \quad D_m > 0. \tag{1.6}$$

The specifical internal energy and the enthalpy are defined by

$$U = U(\theta) = c_p(\theta)\,(\theta-\theta_0), \quad H = H(\theta,p) = U_0 + U + \frac{p}{\rho}.$$

The density ρ satisfies

$$\rho = \rho(p) = \rho_0 \exp(c_w(p - p_0)). \tag{1.7}$$

The real number c_w is the compressibility of the fluid. The reference energy, temperature, pressure and density U_0, θ_0, p_0, ρ_0, are given real numbers. A strong coupling is induced in the system by the viscosity μ which is concentrations and temperature dependent. There is no explicit model of the viscosity valid for any nonnegative temperature. But $\mu \to \infty$ as the temperature goes to the temperature of transition liquid/solid (*cf* [10]). Denoting by $\theta_- > 0$ this temperature, we thus assume that $1/\mu(c,\theta)$ is a given nonlinear and nonnegative function, defined in $(0,1)^{N-1} \times [\theta_-, +\infty)$, satisfying

$$\begin{cases} 1/\sqrt{\mu} \in W^{1,\infty}((0,1)^{N-1} \times [\theta_-, +\infty)), \\ \Big((c,\theta) \in (0,1)^{N-1} \times [\theta_-, +\infty) \text{ and } \mu(c,\theta)^{-1} = 0\Big) \Longleftrightarrow \theta = \theta_-. \end{cases} \tag{1.8}$$

The viscosity is then unbounded for temperatures in a neighborhood of θ_-. The crucial consequence is the degenerating of the parabolic pressure equation (1.1). We denote by μ_- the real number such that

$$0 < \mu_- \leq \mu(c,\theta) \quad \forall (c,\theta) \in (0,1)^{N-1} \times (\theta_-, +\infty). \tag{1.9}$$

We have

$$d_2(\rho,\theta) = \phi(x)\, c_p(\theta) + (1-\phi(x))\, \frac{\rho_R}{\rho}\, c_{pR}(\theta),$$

$$\mathcal{C}(\rho,\theta) = (\theta - \theta_0)\, D_\theta\Big(\frac{c_p(\theta)}{d_2(\rho,\theta)}\Big).$$

The positive function c_p (resp. c_{pR}) is the specific heat of the fluid (resp. of the rock), and ρ_R is the rock density constant. For these functions we have once again no explicit model valid for a wide range of temperature. For a nuclear repository site one generally adopts the following form of specific heat (see [9] for Yucca Mountain and the references therein).

$$c_{p(R)}(\theta) = A_0 + A_1\theta + A_2\theta^2 + A_3\theta^3,$$

with $(A_0, A_1, A_2, A_3) \in \mathbb{R}^4$. This type of relation remains true during phase transformations. Indeed these transitions absorb heat and then increase the specific heat capacities of the constituents as a function of temperature. Consistently with the latter relation, we assume that the heat capacities are defined in $(\theta_-, +\infty)$ and that there exists some real numbers $\Big(\frac{c_p}{d_2}\Big)_-$ and $\Big(\frac{1}{d_2}\Big)_+$ such that

$$\begin{cases} 0 < \Big(\frac{c_p}{d_2}\Big)_- \leq \frac{c_p(\theta)}{d_2(\rho,\theta)} \leq \Big(\frac{c_p}{d_2}\Big)_-^{-1}, \quad \frac{1}{d_2(\rho,\theta)} \leq \Big(\frac{1}{d_2}\Big)_+, \\ \qquad \forall \theta \in (\theta_-, +\infty),\ \forall p \in (m, M), \\ \Big(\frac{c_p(\theta)\,\theta}{d_2(\rho,\theta)}, D_\theta\Big(\frac{c_p(\theta)}{d_2(\rho,\theta)}\Big)\,\theta\Big) \in (L^\infty((\theta_-,\infty) \times (m,M)))^2. \end{cases} \tag{1.10}$$

the real numbers m and M being defined in (1.20) below. The functions d_3 and d_{3i} are defined for $c \in (0,1)$ by

$$d_3(c) = \phi_1(x)\, c\,(1 - F(x,t))\,, \quad d_{3i}(c) = \phi_1(x)\, c\,(K_i - F(x,t))\,.$$

The porosity ϕ (and $\phi_1 = c_w\phi$), the components of the permeability tensor k and the retardation factors K_i are in $L^\infty(\Omega)$, while $F(x,t)$ belongs to $W^{1,\infty}(\Omega_T)$. Moreover, for some real numbers $0 < F_- \le F_+$, $0 < \phi_- \le \phi_+$, $0 < k_- \le k_+$ and $1 \le K_- \le K_+$, we have

$$\begin{cases} F_- \le F(x,t) \le F_+ \quad \text{a.e. in } \Omega_T, \\ \phi_- \le \phi(x) \le \phi_+ \quad \text{a.e. in } \Omega, \\ k_-|\xi|^2 \le k(x)\xi\cdot\xi, \quad |k(x)\xi| \le k_+|\xi| \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^2, \\ K_- \le K_i(x) \le K_+ \quad \text{a.e. in } \Omega,\ i = 1,..N. \end{cases} \tag{1.11}$$

The real number k_{ji} is the mass rate between the parent radionuclide j and the daughter one i. Since the processes of radioactive decay and filiation compensate themselves, the real numbers k_{ji} are such that $\sum_{j=1,j\neq i}^{N} k_{ji} = 1$, $1 \le i \le N$. Note also that $K_N\widetilde{c} = F(x,t) - \sum_{j=1}^{N-2} K_j(x)c_j - \hat{c}$, where $F(x,t)$ describes the total sorption capacity of the rock. The real number $\lambda_i^{-1} > 0$ is the half-life of the radionuclide i. The salt source term is defined for $c \in (0,1)$ by

$$R_s'(c) = \frac{c_s\,\phi\,K_s\,f_s}{1+c_s}\,(1-c).$$

The real numbers c_s, f_s and K_s are rate constants characterizing salt dissolution in the reservoir. The other source terms q, q_i, q_H belong to $L^\infty(\Omega_T)$, and $q_i(x,t) \ge 0$ a.e. in Ω_T. We assume besides

$$\begin{aligned} &\frac{c_p}{F}\Big(-q\,(1-F) - \sum_{j=1}^{N} q_j + R_s'(C) - \phi\,\partial_t F\Big)(\theta_- - \theta_0) \\ &\qquad + q\left(U_0 + (c_w\,\rho(1/c_w))^{-1}\right) + q_H \le 0 \qquad \forall C \in (0,1), \end{aligned} \tag{1.12}$$

where $\theta_- > 0$. It means that the reaction is always exothermic when θ is in a neighborhood of θ_- (*cf* [14]). This latter assumption is all the more reasonable because we can choose a convenient reference temperature θ_0.

We consider the following initial and boundary conditions.

$$\underline{u}_\varepsilon\cdot\nu = 0 \quad \text{on } \Gamma_T, \qquad p_\varepsilon(x,0) = p_{init}(x) \quad \text{in } \Omega, \tag{1.13}$$

$$(K_m(p_\varepsilon) + \varepsilon|\underline{u}_\varepsilon|)\nabla\theta_\varepsilon\cdot\nu = 0 \quad \text{on } \Gamma_T, \qquad \theta_\varepsilon(x,0) = \theta_{init}(x) \quad \text{in } \Omega, \tag{1.14}$$

$$(D_m + \varepsilon|\underline{u}_\varepsilon|)\nabla\hat{c}_\varepsilon\cdot\nu = 0 \quad \text{on } \Gamma_T, \qquad \hat{c}_\varepsilon(x,0) = \hat{c}_{init}(x) \quad \text{in } \Omega, \tag{1.15}$$

$$(D_m + \varepsilon|\underline{u}_\varepsilon|)\,\nabla c_{i,\varepsilon}\cdot\nu = 0 \quad \text{on } \Gamma_T, \qquad c_{i,\varepsilon}(x,0) = c_{i,init}(x) \quad \text{in } \Omega, \tag{1.16}$$

with $p_{init} \in H^1(\Omega)$ and $(\theta_{init}, \hat{c}_{init}, (c_{i,init})_{i=1}^{N-2}) \in (L^\infty(\Omega))^N$ satisfying

$$m_0 \le p_{init}(x) \le M_0 \quad \text{a.e. in } \Omega, \tag{1.17}$$

$$0 < \theta_- \leq \theta_{init}(x) \quad \text{a.e. in } \Omega, \tag{1.18}$$

$$0 \leq \hat{c}_{init}(x), c_{i,init}(x), \quad \sum_{i=1}^{N-2} c_{i,init}(x) + \hat{c}_{init}(x) \leq 1 \quad \text{a.e. in } \Omega. \tag{1.19}$$

We define two real numbers m and M by

$$m = -(\|q\|_\infty + \phi_+ \|\partial_t F\|_\infty)\, T + m_0, \; M = \Big(\frac{c_s\, \phi\, K_s\, f_s}{1 + c_s} + \phi_+ \|\partial_t F\|_\infty\Big)\, T + M_0, \tag{1.20}$$

where M_0 is such that $\phi_-(\inf_{(\theta_-,+\infty)} c_p) + (1-\phi_+)(\inf_{(\theta_-,+\infty)} c_{pR})\, \rho_R/\rho(M) > 0$. Then $d_2(\rho,\theta) > 0$ if $\rho \leq \rho(M)$.

For any fixed real $\varepsilon > 0$, the following existence result is deduced from [2].

Theorem 1.1 *Under the aforementioned hypotheses and for any fixed $\varepsilon > 0$, there exists a weak solution $(p_\varepsilon, \theta_\varepsilon, \hat{c}_\varepsilon, (c_{i,\varepsilon})_{i=1}^{N-2})$ of Problem (1.1)-(1.5), (1.13)-(1.16) satisfying*

i) the function $p_\varepsilon \in L^\infty(\Omega_T)$, with $m \leq p_\varepsilon(x,t) \leq M$ a.e. in Ω_T, is solution of (1.1)-(1.2), (1.13) verified in $L^2(0,T;H^{-1}(\Omega))$; the velocity $\underline{u}_\varepsilon = -\dfrac{k}{\mu(c_\varepsilon,\theta_\varepsilon)}\nabla p_\varepsilon$ belongs to $(L^2(\Omega_T))^2$ and $\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}\underline{u}_\varepsilon$ is in $(L^2(\Omega_T))^2$;

ii) the function $(\hat{c}_\varepsilon, c_{1,\varepsilon}, .., c_{N-2,\varepsilon}) \in \big(L^\infty(\Omega_T) \cap L^2(0,T;H^1(\Omega))\big)^{N-1}$ is such that $0 \leq \hat{c}_\varepsilon(x,t), c_{i,\varepsilon}(x,t)$ and $\sum_{i=1}^{N-2} c_{i,\varepsilon}(x,t) + \hat{c}_\varepsilon(x,t) \leq 1$ a.e. in Ω_T, while $\theta_\varepsilon \in L^\infty(\Omega_T) \cap L^2(0,T;H^1(\Omega))$ and $\theta_\varepsilon(x,t) \geq \theta_- > 0$ a.e. in Ω_T.

In the present paper, we aim to prove the following convergence result.

Theorem 1.2 *For extracted subsequences, the solution $(p_\varepsilon, \theta_\varepsilon, \hat{c}_\varepsilon, (c_{i,\varepsilon})_{i=1}^{N-2})$ of Pb. (1.1)-(1.5), (1.13)-(1.16) converges in a weak sense to $(p, \theta, \hat{c}, (c_i)_{i=1}^{N-2})$ which is a weak solution of the following problem where the dispersions effects are completely neglected.*

$$\phi_1 F\, \partial_t p + \phi\, \partial_t F + \mathrm{div}(\underline{u}) = -q - \sum_{j=1}^{N} q_j + R_s'(\hat{c}), \tag{1.21}$$

$$\underline{u} = -\frac{k}{\mu(c,\theta)}\nabla p, \tag{1.22}$$

$$\begin{aligned}
&\partial_t\theta - \mathrm{div}\Big(\frac{1}{d_2(\rho(p),\theta)}\big(\phi c_p(\theta) K_m(p)\nabla\theta - H\,\underline{u}\big)\Big) - \frac{\theta-\theta_0}{F}\,\mathrm{div}\Big(\frac{c_p(\theta)}{d_2(\rho(p),\theta)}\,\underline{u}\Big)\\
&\quad + \mathcal{C}(\rho(p),\theta)\underline{u}\cdot\nabla\theta - \frac{U}{F d_2(\rho(p),\theta)}\Big(\phi\,\partial_t F + q + \sum_{j=1}^{N} q_j - R_s'(\hat{c})\Big)\\
&\quad = \frac{-1}{d_2(\rho(p),\theta)}\,(q_H + qH),
\end{aligned} \tag{1.23}$$

$$\phi\,\partial_t\hat{c}+\underline{u}\cdot\nabla\hat{c}-\mathrm{div}(\phi D_m\nabla\hat{c})+d_3(\hat{c})\,\partial_t p-\phi\,\partial_t F\,\hat{c}$$
$$=\hat{c}\sum_{j=1}^{N}q_j+(1-\hat{c})\,R_s'(\hat{c}), \tag{1.24}$$
$$\phi\,K_i\,\partial_t c_i+\underline{u}\cdot\nabla c_i-\mathrm{div}(\phi D_m\nabla c_i)+d_{3i}(c_i)\,\partial_t p-\phi\,\partial_t F\,c_i=-c_i\,R_s'(\hat{c})-q_i$$
$$+c_i\sum_{j=1}^{N}q_j-\lambda_i\,K_i\,\phi\,c_i+\sum_{j=1,j\neq i}^{N-2}k_{ij}\,\lambda_j\,K_j\,\phi\,c_j+k_{iN}\,\lambda_N\,K_N\,\phi\,\widetilde{c}, \tag{1.25}$$

provided by the following initial and boundary conditions

$$\underline{u}\cdot\nu=0\quad\text{on }\Gamma_T,\qquad p(x,0)=p_{init}(x)\quad\text{in }\Omega, \tag{1.26}$$
$$K_m(p)\nabla\theta\cdot\nu=0\quad\text{on }\Gamma_T,\qquad \theta(x,0)=\theta_{init}(x)\quad\text{in }\Omega, \tag{1.27}$$
$$D_m\nabla\hat{c}\cdot\nu=0\quad\text{on }\Gamma_T,\qquad \hat{c}(x,0)=\hat{c}_{init}(x)\quad\text{in }\Omega, \tag{1.28}$$
$$D_m\nabla c_i\cdot\nu=0\quad\text{on }\Gamma_T,\qquad c_i(x,0)=c_{i,init}(x)\quad\text{in }\Omega. \tag{1.29}$$

Furthermore, $(p,\theta,\hat{c},(c_i)_{i=1}^{N-2})$ *has the same regularity properties as those of the solution of the original problem described in (i) and (ii) of Theorem 1.*

The simulation (see [11]) and the numerical analysis (see for instance [7, 4, 6]) of such problems where the dispersion effects are neglected has been extensively studied in the past decade. But the rigorous justification of the model with neglected dispersion is not addressed. We can only cite [1] who treat a simplified non radioactive model with constant viscosity. And [3] considers the one-dimensional case for a mixture of two species with different compressibilities (this adds a nonlinearity in the problem) but with a bounded viscosity. More generally, there are very few mathematical results about fluid problems with unbounded viscosity (see for instance the recent work [8] and the references therein).

The paper is organized as follows. Classical energy estimates are performed in Section 2. We also state the compactness results which can be obtained with arguments of Aubin's type. Section 3 is devoted to the convergence analysis. We use astute tools to get sufficient informations about the pressure behavior in spite of the degenerating of Eq. (1.1).

2 Energy estimates and first compactness results

In what follows, the letter C denotes a generic quantity, independent of ε. We begin with some properties of the pressure p_ε solution of (1.1)-(1.2), (1.13).

Lemma 2.1 *The function* $p_\varepsilon\in L^\infty(\Omega_T)\cap L^2(0,T;H^1(\Omega))$ *solution of (1.1)-(1.2), (1.13) satisfies*

$$m\le p_\varepsilon(x,t)\le M\quad\text{a.e. in }\Omega_T,$$

$$\|p_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}+\Big\|\Big(\frac{k_-}{\mu(c_\varepsilon,\theta_\varepsilon)}\Big)^{1/2}\nabla p_\varepsilon\Big\|_{(L^2(\Omega_T))^2}\le C,\quad \|\underline{u}_\varepsilon\|_{(L^2(\Omega_T))^2}\le C_1,$$

where C (resp. C_1) is a constant which only depends on T (resp. on T and μ_- defined in (1.9)). Furthermore, the function $\phi_1 F \partial_t p_\varepsilon$ is uniformly bounded in $L^2(0,T;(H^1(\Omega))')$.

Proof. The first estimate is a direct consequence of the construction of p_ε in [2]. One also can check that since the right hand-side terms of (1.1) belong to $L^\infty(\Omega_T)$, a maximum principle with Hypothese (1.17) leads to $m \le p_\varepsilon(x,t) \le M$ a.e. in Ω_T. We now multiply Eq. (1.1) by p_ε and integrate by parts over Ω. We obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi_1 F\,|p_\varepsilon|^2\,dx + \int_\Omega \left(\phi p_\varepsilon - \frac{1}{2}\phi_1|p_\varepsilon|^2\right)\partial_t F\,dx$$
$$+\int_\Omega \frac{k}{\mu(c_\varepsilon,\theta_\varepsilon)}\nabla p_\varepsilon\cdot\nabla p_\varepsilon\,dx = -\int_\Omega \big(q+\sum_{j=1}^N q_j - R_s'(\hat{c}_\varepsilon)\big)\,p_\varepsilon\,dx.$$

We estimate the second term of the left hand-side and the right hand-side of the latter relation using the assumptions $\phi_1,\phi \in L^\infty(\Omega)$, $\partial_t F \in L^\infty(\Omega_T)$, $q, q_j \in L^\infty(\Omega_T)$ and $\hat{c}_\varepsilon \in L^\infty(\Omega_T)$ (see the maximum principle *(ii)* of Theorem 1). With Assumption (1.11) for k, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi_1 F\,|p_\varepsilon|^2\,dx + \int_\Omega \frac{k_-}{\mu(c_\varepsilon,\theta_\varepsilon)}|\nabla p_\varepsilon|^2\,dx \le C\int_\Omega |p_\varepsilon|^2\,dx + C.$$

We then use the Gronwall lemma and the assumption $\phi_1 F \ge c_w\phi_- F_- > 0$ to obtain the estimates for p_ε in $L^\infty(0,T;L^2(\Omega))$ and for ∇p_ε in $(L^2(\Omega_T))^2$. The estimate for $\underline{u}_\varepsilon$ in $(L^2(\Omega_T))^2$ then follows from Assumption (1.9). Finally, multiplying Eq. (1.1) by any test function $\psi \in L^2(0,T;H^1(\Omega))$, one checks that

$$|\langle \phi_1 F\,\partial_t p_\varepsilon, \psi\rangle_{L^2(0,T;(H^1(\Omega))')\times L^2(0,T;H^1(\Omega))}| \le C(\|\psi\|_{L^1(\Omega_T)} + \|\nabla\psi\|_{(L^2(\Omega_T))^2}).$$

This ends the proof of Lemma 1. ∎

We now study the temperature problem (1.3)-(1.14). We claim the following result.

Lemma 2.2 *For any $\varepsilon > 0$, the function θ_ε solution of (1.3)-(1.14) is in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and it satisfies:*

i) $\theta_\varepsilon(x,t) \ge \theta_- > 0$ almost everywhere in Ω_T;

ii) the sequence (θ_ε) is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, and $((\varepsilon|\underline{u}_\varepsilon|)^{\frac{1}{2}}\nabla\theta_\varepsilon)$ is uniformly bounded in the space $(L^2(\Omega_T))^2$;

iii) the sequence (θ_ε) is sequentially compact in $L^2(\Omega_T)$.

Proof. Assuming (1.6)-(1.12), the existence of a solution θ_ε to the parabolic problem (1.3), (1.14) with $\theta_- \le \theta_\varepsilon(x,t)$ a.e. in Ω_T is given by the construction in [2]. To prove *ii)*, we multiply Eq. (1.3) by θ_ε and integrate over Ω. Integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\theta_\varepsilon(\cdot,t)|^2\,dx + \int_\Omega \frac{\phi c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(K_m+\varepsilon|\underline{u}_\varepsilon|)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon\,dx\\
&\quad -\int_\Omega \frac{H_\varepsilon}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,dx + \int_\Omega \mathcal{C}(\rho(p_\varepsilon),\theta_\varepsilon)\,\theta_\varepsilon\,\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon\,dx\\
&\quad -\int_\Omega \frac{\theta_\varepsilon-\theta_0}{F}\,\mathrm{div}\Big(\frac{c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,\underline{u}_\varepsilon\Big)\,\theta_\varepsilon\,dx\\
&\quad -\int_\Omega \frac{U_\varepsilon}{Fd_2(\rho(p_\varepsilon),\theta_\varepsilon)}\big(\phi\,\partial_t F + q + \sum_{j=1}^N q_j - R_s'(\hat{c}_\varepsilon)\big)\,\theta_\varepsilon\,dx\\
&= -\int_\Omega \frac{q_H}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,\theta_\varepsilon\,dx - \int_\Omega \frac{q}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\big(U_0+U_\varepsilon+\frac{p_\varepsilon}{\rho(p_\varepsilon)}\big)\,\theta_\varepsilon\,dx. \qquad (2.1)
\end{aligned}$$

We begin by recalling that the functions p_ε and then $\rho(p_\varepsilon)$ are bounded in $L^\infty(\Omega_T)$, uniformly in ε. We also bear in mind the bounds for c_p/d_2, $1/d_2$ and $\mathcal{C}$ given by Assumption (1.10). Due to the definition of H_ε, we have

$$\begin{aligned}
&\Big|\int_\Omega \frac{1}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,H_\varepsilon(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,dx\Big|\\
&\qquad = \Big|\int_\Omega \frac{1}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\big(U_0+c_p(\theta_\varepsilon)(\theta_\varepsilon-\theta_0)+\frac{p_\varepsilon}{\rho(p_\varepsilon)}\big)\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,dx\Big|.
\end{aligned}$$

Thanks to the Cauchy-Schwarz and Young inequalities, we write with Assumption (1.10)

$$\begin{aligned}
&\Big|\int_\Omega \frac{1}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(U_0-c_p(\theta_\varepsilon)\,\theta_0)\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,dx\Big| \le \Big(\big(\frac{c_p}{d_2}\big)_-^{-1}+\big(\frac{1}{d_2}\big)_+\Big)\\
&\quad\times\Big(\frac{C}{4\delta}\int_\Omega |\underline{u}_\varepsilon|^2\,dx+\delta\int_\Omega |\nabla\theta_\varepsilon|^2\,dx\Big) \le \frac{C}{\delta}\int_\Omega |\underline{u}_\varepsilon|^2\,dx+\delta\int_\Omega|\nabla\theta_\varepsilon|^2\,dx,\\
&\Big|\int_\Omega \frac{c_p(\theta_\varepsilon)\,\theta_\varepsilon}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,dx\Big| \le \|\frac{c_p(\theta)\theta}{d_2(\rho,\theta)}\|_\infty\Big(\delta\int_\Omega|\nabla\theta_\varepsilon|^2\,dx\\
&\quad+\frac{C}{4\delta}\int_\Omega|\underline{u}_\varepsilon|^2\,dx\Big),\\
&\Big|\int_\Omega \mathcal{C}(\rho(p_\varepsilon),\theta_\varepsilon)\,\theta_\varepsilon\,\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,dx\Big| \le \|\mathcal{C}(\rho(p),\theta)\,\theta\|_\infty\Big(\delta\int_\Omega|\nabla\theta_\varepsilon|^2\,dx\\
&\quad+\frac{C}{4\delta}\int_\Omega|\underline{u}_\varepsilon|^2\,dx\Big)
\end{aligned}$$

and

$$\Big|\int_\Omega \frac{1}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,\frac{p_\varepsilon}{\rho(p_\varepsilon)}\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,dx\Big| \le \delta\int_\Omega|\nabla\theta_\varepsilon|^2\,dx+\frac{C}{4\delta}\int_\Omega|\underline{u}_\varepsilon|^2\,dx,$$

for any $\delta>0$. The function $(p_\varepsilon/\rho(p_\varepsilon))$ is indeed uniformly bounded in $L^\infty(\Omega_T)$ since $m\le p_\varepsilon(x,t)\le M$ a.e. in Ω_T. Now we transform the fourth term of Relation

(2.1). It follows by integration by parts that

$$\int_\Omega \frac{\theta_\varepsilon - \theta_0}{F}\,\mathrm{div}\Big(\frac{c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,\underline{u}_\varepsilon\Big)\,\theta_\varepsilon\,dx$$
$$= -\int_\Omega \frac{c_p(\theta_\varepsilon)\,\theta_\varepsilon}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(\theta_\varepsilon-\theta_0)\,\underline{u}_\varepsilon\cdot\nabla(F^{-1})\,dx$$
$$-\int_\Omega \frac{c_p(\theta_\varepsilon)\,\theta_\varepsilon}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,\frac{2}{F}\,\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon\,dx + \int_\Omega \frac{c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,\frac{\theta_0}{F}\,\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon\,dx.$$

We have replaced the duality product $\langle\cdot,\cdot\rangle_{(H^1(\Omega))'\times H^1(\Omega)}$ by three terms in the form $\langle\cdot,\cdot\rangle_{L^2(\Omega)\times L^2(\Omega)}$. We recall that $F^{-1}\in W^{1,\infty}(\Omega_T)$. Thus, similar tools as previously allow to estimate the terms in the right hand-side of the latter description of $\langle \mathrm{div}\big(\frac{c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\underline{u}_\varepsilon\big), \frac{\theta_\varepsilon-\theta_0}{F}\theta_\varepsilon\rangle_{(H^1(\Omega))'\times H^1(\Omega)}$. And the other terms of (2.1) do not bring additional difficulty. Bearing in mind that

$$\int_\Omega \frac{\phi c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(K_m+\varepsilon|\underline{u}_\varepsilon|)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon\,dx$$
$$\geq \int_\Omega \Big(\frac{c_p}{d_2}\Big)_-\phi_-\Big(\frac{k_m}{\rho(M)}+\varepsilon|\underline{u}_\varepsilon|\Big)|\nabla\theta_\varepsilon|^2\,dx,$$

the previous estimates finally yield in (2.1) to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\theta_\varepsilon(\cdot,t)|^2\,dx + \int_\Omega \bigg(\Big(\frac{c_p}{d_2}\Big)_-\phi_-\Big(\frac{k_m}{\rho(M)}+\varepsilon|\underline{u}_\varepsilon|\Big)-\delta\bigg)|\nabla\theta_\varepsilon|^2\,dx$$
$$\leq \frac{C}{\delta}\big(1+\|\underline{u}_\varepsilon\|^2_{(L^2(\Omega))^2}\big) + C\int_\Omega |\theta_\varepsilon|^2\,dx, \tag{2.2}$$

for any $\delta>0$. We choose δ such that $\Big(\big(\frac{c_p}{d_2}\big)_-\phi_-k_m/\rho(M)-\delta\Big)>0$. The Gronwall lemma then yields the result *ii)*.

It remains to prove *iii)*. Let $\psi\in L^\infty(0,T;W^{1,4}(\Omega))$. We multiply Eq. (1.3) by ψ and integrate over Ω_T. Integrating by parts, we get

$$\big|\langle\partial_t\theta_\varepsilon,\psi\rangle_{L^1(0,T;(W^{1,4}(\Omega))'),L^\infty(0,T;W^{1,4}(\Omega))}\big|$$
$$\leq \Big|\int_{\Omega_T} \frac{\phi c_p(\theta_\varepsilon)}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\Big(\frac{k_m}{\rho(p_\varepsilon)}+\varepsilon|\underline{u}_\varepsilon|\Big)\nabla\theta_\varepsilon\cdot\nabla\psi\Big|$$
$$+\Big|\int_{\Omega_T}\frac{H_\varepsilon}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(\underline{u}_\varepsilon\cdot\nabla\psi)\Big| + \Big|\int_{\Omega_T}\frac{U_\varepsilon}{Fd_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(\underline{u}_\varepsilon\cdot\nabla\psi)\Big|$$
$$+\Big|\int_{\Omega_T}\mathcal{C}(\rho(p_\varepsilon),\theta_\varepsilon)\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,\psi\Big| + \Big|\int_{\Omega_T}\frac{c_p(\theta_\varepsilon)}{Fd_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(\underline{u}_\varepsilon\cdot\nabla\theta_\varepsilon)\,\psi\Big|$$
$$+\Big|\int_{\Omega_T}\frac{U_\varepsilon}{F^2d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(\underline{u}_\varepsilon\cdot\nabla F)\,\psi\Big| + \Big|\int_{\Omega_T}\frac{1}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,(q_H+qH_\varepsilon)\,\psi\Big|$$
$$+\Big|\int_{\Omega_T}\frac{U_\varepsilon}{Fd_2(\rho(p_\varepsilon),\theta_\varepsilon)}\,\Big(\phi\partial_tF+q+\sum_{j=1}^N q_j - R_s'(c_\varepsilon)\Big)\,\psi\Big|.$$

Each term of the right hand-side of the latter relation can be estimated as follows.

$$
\begin{aligned}
&\big|\langle \partial_t\theta_\varepsilon,\psi\rangle_{L^1(0,T;(W^{1,4}(\Omega))'),L^\infty(0,T;W^{1,4}(\Omega))}\big|\\
&\le C\|\nabla\theta_\varepsilon\|_{(L^2(\Omega_T))^2}\|\nabla\psi\|_{(L^2(\Omega_T))^2}+C\|\varepsilon|\underline{u}_\varepsilon|\nabla\theta_\varepsilon\|_{(L^{4/3}(\Omega_T))^2}\|\nabla\psi\|_{(L^4(\Omega_T))^2}\\
&+\Big\|\frac{H_\varepsilon+U_\varepsilon}{d_2(\rho(p_\varepsilon),\theta_\varepsilon)}\Big\|_{L^\infty(\Omega_T)}\|\underline{u}_\varepsilon\|_{(L^2(\Omega_T))^2}\|\nabla\psi\|_{(L^2(\Omega_T))^2}\\
&+C((c_p/d_2)_-,(1/d_2)_+,m,M,\|F\|_{W^{1,\infty}(\Omega_T)},F_-,\|q\|_{L^\infty},\|q_j\|_{L^\infty})\|\psi\|_{L^1(\Omega_T)}\\
&+C\|\underline{u}_\varepsilon\|_{(L^2(\Omega_T))^2}\|\nabla\theta_\varepsilon\|_{(L^2(\Omega_T))^2}\|\psi\|_{L^\infty(\Omega_T)}\le C\|\psi\|_{L^\infty(0,T;W^{1,4}(\Omega))}.
\end{aligned}
$$

So the sequence $(\partial_t\theta_\varepsilon)$ is uniformly bounded in $L^1(0,T;(W^{1,4}(\Omega))')$. We then conclude with arguments of Aubin's type [13]. The proof is achieved. ■

Next we consider the concentrations problem. We claim the following lemma.

Lemma 2.3 *For any fixed $\varepsilon>0$, the solution $c_\varepsilon=(\hat c_\varepsilon,c_{1,\varepsilon},..,c_{N-2,\varepsilon})$ of Problem (1.4)-(1.5), (1.15)-(1.16) belongs to $\big(L^\infty(\Omega_T)\cap L^2(0,T;H^1(\Omega))\big)^{N-1}$ and satisfies*

- *i) $0\le c_{i,\varepsilon}(x,t),\hat c_\varepsilon(x,t)$ and $\sum_{i=1}^{N-2}c_{i,\varepsilon}(x,t)+\hat c_\varepsilon(x,t)\le 1$ a.e. in Ω_T;*
- *ii) the sequences $(\hat c_\varepsilon)$ and $(c_{i,\varepsilon})$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$, while $((\varepsilon|\underline{u}_\varepsilon|)^{\frac12}\nabla\hat c_\varepsilon)$ and $((\varepsilon|\underline{u}_\varepsilon|)^{\frac12}\nabla c_{i,\varepsilon})$ are bounded in $(L^2(\Omega_T))^2$;*
- *iii) the sequences $(\hat c_\varepsilon)$ and $(c_{i,\varepsilon})$ are sequentially compact in $L^2(\Omega_T)$.*

Proof. The maximum principle in item *i)* is due to the construction of the solution c_ε in [2]. The proof of item *ii)* follows the lines of the proof of *ii)* in Lemma 2. The only difference lies in the terms containing $\partial_t p_\varepsilon$. Let us consider for instance the one appearing when we multiply the salt equation (1.4) by $\hat c_\varepsilon$ and then integrate over Ω. It writes $\int_\Omega d_3(\hat c_\varepsilon)\partial_t p_\varepsilon\hat c_\varepsilon dx$. We recall that by Lemma 1, we can only assert that $\partial_t p_\varepsilon$ is uniformly bounded in $L^2(0,T;(H^1(\Omega))')$. So the zero order estimate for $\hat c_\varepsilon$ given by the maximum principle *i)* is not sufficient for getting an estimate of $\int_\Omega d_3(\hat c_\varepsilon)\partial_t p_\varepsilon\hat c_\varepsilon dx$. We thus note that the former expression is defined by the duality product $\langle\phi_1F\partial_t p_\varepsilon,\frac{d_3(\hat c_\varepsilon)\hat c_\varepsilon}{\phi_1 F}\rangle_{(H^1(\Omega))'\times H^1(\Omega)}$. Using the Cauchy-Schwarz and the Young inequalities, we then estimate it as follows.

$$
\begin{aligned}
&\Big|\langle\phi_1F\,\partial_t p_\varepsilon,\frac{d_3(\hat c_\varepsilon)\hat c_\varepsilon}{\phi_1F}\rangle_{(H^1(\Omega))'\times H^1(\Omega)}\Big|\le\|\phi_1F\,\partial_t p_\varepsilon\|_{(H^1(\Omega))'}\Big\|\frac{d_3(\hat c_\varepsilon)\hat c_\varepsilon}{\phi_1F}\Big\|_{H^1(\Omega)}\\
&\le C\|\phi_1F\,\partial_t p_\varepsilon\|_{(H^1(\Omega))'}\|\nabla F\|_{(L^\infty(\Omega_T))^2}+\frac{C\|\hat c_\varepsilon\|^2_{L^\infty(\Omega_T)}}{4\delta}\|\phi_1F\,\partial_t p_\varepsilon\|^2_{(H^1(\Omega))'}\\
&+\delta\int_\Omega|\nabla\hat c_\varepsilon|^2\,dx,
\end{aligned}
$$

for any $\delta>0$. The terms of the right hand-side of the latter relation can be included in a relation for $\hat c_\varepsilon$ similar to (2.2). We recall that we have stated in

Lemma 1 that $\phi_1 F \partial_t p_\varepsilon$ is uniformly bounded in $L^2(0,T;(H^1(\Omega))')$. The Gronwall lemma then gives *ii)*. Similar adaptations of the proof of Lemma 2 *iii)* allow to claim that $\phi\partial_t \hat{c}_\varepsilon$ and $\phi\partial_t c_{i,\varepsilon}$ are uniformly bounded in $L^1(0,T;(W^{1,4}(\Omega))')$. The sequential compactness of $(\phi\hat{c}_\varepsilon)$ and $(\phi c_{i,\varepsilon})$ in $L^2(0,T;(H^1(\Omega))')$ follows from Aubin's arguments type. Since the functions $\hat{c}_\varepsilon$ and $c_{i,\varepsilon}$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$, we can pass to the limit in the products $\langle \phi\hat{c}_\varepsilon, \hat{c}_\varepsilon\rangle$ and $\langle \phi c_{i,\varepsilon}, c_{i,\varepsilon}\rangle$ of $L^2(0,T;(H^1(\Omega))') \times L^2(0,T;H^1(\Omega))$. Since $\phi(x) \geq \phi_- > 0$ a.e. in Ω, we conclude that $(\hat{c}_\varepsilon)$ and $(c_{i,\varepsilon})$ are sequentially compact in $L^2(\Omega_T)$. ■

3 Convergence results and proof of Theorem 2

The estimates of the previous section prove the existence of limit functions $p \in L^\infty(\Omega_T)$, $\underline{u} \in \left(L^2(\Omega_T)\right)^2$, $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, $c = (\hat{c}, c_1, .., c_{N-2}) \in \left(L^\infty(\Omega_T) \cap L^2(0,T;H^1(\Omega))\right)^{N-1}$, satisfying for extracted subsequences

$$\begin{aligned}
&p_\varepsilon \rightharpoonup p \quad * -\text{weakly in } L^\infty(\Omega_T), \quad \underline{u}_\varepsilon \rightharpoonup \underline{u} \quad \text{weakly in } (L^2(\Omega_T))^2,\\
&\theta_\varepsilon \rightharpoonup \theta \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ and a.e. in } \Omega_T,\\
&c_\varepsilon \rightharpoonup c \quad \text{weakly in } (L^2(0,T;H^1(\Omega)))^{N-1} \text{ and a.e. in } \Omega_T.
\end{aligned}$$

Furthermore, the functions θ and c are physically relevant. Indeed, the maximum principles of Lemmas 2 and 3 give at the limit

$$\theta(x,t) \geq \theta_- > 0 \quad \text{a.e. in } \Omega_T,$$

$$0 \leq \hat{c}(x,t), c_i(x,t) \text{ and } \sum_{i=1}^{N-2} c_i(x,t) + \hat{c}(x,t) \leq 1 \quad \text{a.e. in } \Omega_T.$$

Letting $\varepsilon \to 0$ in (1.1), we get in a first step

$$\phi_1 F\, \partial_t p + \phi\, \partial_t F + \operatorname{div}(\underline{u}) = -q - \sum_{j=1}^{N} q_j + R_s'(\hat{c}) \quad \text{in } \Omega_T. \tag{3.1}$$

We now study the limit behavior of the Darcy law (1.2). The main difficulty is the degenerating of Eq. (1.1). The lack of estimate on ∇p_ε does not allow to pass directly to the limit in (1.2).

Lemma 3.1 *The limit Darcy law is*

$$\underline{u} = -\frac{k}{\mu(c,\theta)} \nabla p \quad \text{in } \Omega_T.$$

Proof. Let us prove that the structure of the Darcy law is conserved by the asymptotic process. We have to compute the limit in $(L^2(\Omega_T))^2$ of $(1/\mu(c_\varepsilon,\theta_\varepsilon))\nabla p_\varepsilon$. We consider in a first step the sequence $(1/\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)})\nabla p_\varepsilon$. It writes

$$\frac{1}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\nabla p_\varepsilon = \operatorname{div}\Big(\frac{1}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\,p_\varepsilon\Big) - p_\varepsilon\,\nabla\Big(\frac{1}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\Big).$$

We now aim to pass to the limit in the terms of the right hand-side. Since $\mu(c_\varepsilon,\theta_\varepsilon) \to \mu(c,\theta)$ almost everywhere in Ω_T, the convergence of the sequence $\operatorname{div}(p_\varepsilon/\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)})$ to $\operatorname{div}(p/\sqrt{\mu(c,\theta)})$ is clear. Using Lemma 1, we note that $\partial_t(\phi_1 F p_\varepsilon) = \phi_1 F\partial_t p_\varepsilon + \phi_1 p_\varepsilon \partial_t F$ (respectively $\phi_1 F p_\varepsilon$) is uniformly bounded in $L^2(0,T;(H^1(\Omega))')$ (respectively in $L^\infty(\Omega_T)$). Aubin's arguments [13] then lead to the sequential compactness of $(\phi_1 F p_\varepsilon)$ in $L^2(0,T;(H^1(\Omega))')$. By Lemmas 1, 2 and 3, the function $(1/\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)})p_\varepsilon$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. Thus, we can compute

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T}\frac{\phi_1 F}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\,p_\varepsilon^2 dxdt &= \lim_{\varepsilon\to 0}\int_0^T\langle \phi_1 F p_\varepsilon, \frac{p_\varepsilon}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\rangle_{(H^1(\Omega))'\times H^1(\Omega)}dt\\ &= \int_{\Omega_T}\frac{\phi_1 F}{\sqrt{\mu(c,\theta)}}\,p^2\,dxdt.\end{aligned}$$

Since $\phi_1(x)F(x,t) \ge c_w\phi_- F_- > 0$ a.e. in Ω_T, we conclude that

$$\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/4}}p_\varepsilon \to \frac{1}{\mu(c,\theta)^{1/4}}p \quad \text{a.e. in } \Omega_T \text{ and in } L^p(\Omega_T),\ \forall p\in(1,+\infty). \tag{3.2}$$

With the a.e. convergence in Ω_T of the sequence $(\mu'(c_\varepsilon,\theta_\varepsilon)/\mu(c_\varepsilon,\theta_\varepsilon)^{5/4})$ to its limit $\mu'(c,\theta)/\mu(c,\theta)^{5/4}$, it leads to

$$\begin{aligned}p_\varepsilon\,\nabla\Big(\frac{1}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\Big) &= -\frac{\mu'(c_\varepsilon,\theta_\varepsilon)}{2\mu(c_\varepsilon,\theta_\varepsilon)^{3/2}}\,p_\varepsilon\,\Big(\nabla\hat{c}_\varepsilon + \sum_{i=1}^{N-2}\nabla c_{i,\varepsilon} + \nabla\theta_\varepsilon\Big)\\ &\rightharpoonup p\,\nabla\Big(\frac{1}{\sqrt{\mu(c,\theta)}}\Big)\end{aligned}$$

in $(L^2(\Omega_T))^2$, and then

$$\frac{1}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\,\nabla p_\varepsilon \rightharpoonup \frac{1}{\sqrt{\mu(c,\theta)}}\,\nabla p \quad \text{in } (L^2(\Omega_T))^2.$$

Besides, the Darcy law (1.2) gives at the limit

$$\underline{u} = -\frac{k}{\mu(c,\theta)}\nabla p \quad \text{in } \Omega_T. \tag{3.3}$$

■

To treat the nonlinearities in Eqs. (1.3)-(1.5), we need an additional compactness result on (p_ε).

Lemma 3.2 *i) The sequence (p_ε) converges to p almost everywhere in Ω_T.*

ii) Furthermore, as $\varepsilon \to 0$, we have the following strong convergence.

$$\frac{1}{\sqrt{\mu(c_\varepsilon,\theta_\varepsilon)}}\,\nabla p_\varepsilon \to \frac{1}{\sqrt{\mu(c,\theta)}}\,\nabla p \quad \text{strongly in } (L^2(\Omega_T))^2 \text{ and a.e. in } \Omega_T.$$

Proof. Since $\mu(c_\varepsilon,\theta_\varepsilon) \to \mu(c,\theta)$ a.e. in Ω_T with $\mu(c_\varepsilon,\theta_\varepsilon)(x,t), \mu(c,\theta)(x,t) \geq \mu_- > 0$, the first part of the lemma is proved by (3.2). We then multiply Eq. (1.1) by p_ε, Eq. (3.1) by p and integrate over $\Omega_t = \Omega \times (0,t)$, $t \in (0,T)$. We obtain almost everywhere in $(0,T)$

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \phi_1 F(\cdot,t)\,|p_\varepsilon(\cdot,t)|^2 dx - \frac{1}{2}\int_\Omega \phi_1 F(\cdot,0)\,|p_{init}|^2 dx - \int_{\Omega_t} \phi_1\,\partial_t F\,|p_\varepsilon|^2 dxds \\
&+ \int_{\Omega_t} \frac{k}{\mu(c_\varepsilon,\theta_\varepsilon)}\,\nabla p_\varepsilon\cdot\nabla p_\varepsilon dxds = \int_{\Omega_t} \big(-\phi\,\partial_t F - q - \sum_{j=1}^N q_j + R_s'(c_\varepsilon)\big)p_\varepsilon dxds,\\
&\frac{1}{2}\int_\Omega \phi_1 F(\cdot,t)\,|p(\cdot,t)|^2 dx - \frac{1}{2}\int_\Omega \phi_1 F(\cdot,0)\,|p_{init}|^2 dx - \int_{\Omega_t} \phi_1\,\partial_t F\,|p|^2 dxds \\
&+ \int_{\Omega_t} \frac{k}{\mu(c,\theta)}\,\nabla p\cdot\nabla p\,dxds = \int_{\Omega_t} \big(-\phi\,\partial_t F - q - \sum_{j=1}^N q_j + R_s'(\hat{c})\big)p\,dxds.
\end{aligned}$$

We substract the two relations, and let $\varepsilon \to 0$. Using the result *i)* of Lemma 5, we get for almost any $t \in (0,T)$

$$\lim_{\varepsilon\to 0}\int_{\Omega_t} \frac{k}{\mu(c_\varepsilon,\theta_\varepsilon)}\nabla p_\varepsilon\cdot\nabla p_\varepsilon\,dxds = \int_{\Omega_t} \frac{k}{\mu(c,\theta)}\,\nabla p\cdot\nabla p\,dxds.$$

It allows to compute the following limit.

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_t} k\Big(&\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla p_\varepsilon - \frac{1}{\mu(c,\theta)^{1/2}}\nabla p\Big)\\
&\cdot\Big(\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla p_\varepsilon - \frac{1}{\mu(c,\theta)^{1/2}}\nabla p\Big)\,dxds = 0.
\end{aligned}$$

With Hypothese (1.11) for k, we conclude that

$$\begin{aligned}
0 \leq \lim_{\varepsilon\to 0}\int_{\Omega_t}&\Big(\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla p_\varepsilon - \frac{1}{\mu(c,\theta)^{1/2}}\nabla p\Big)\cdot\Big(\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla p_\varepsilon\\
&-\frac{1}{\mu(c,\theta)^{1/2}}\nabla p\Big)\,dxds \leq \frac{1}{k_-}\lim_{\varepsilon\to 0}\int_{\Omega_t} k\Big(\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla p_\varepsilon - \frac{1}{\mu(c,\theta)^{1/2}}\nabla p\Big)\\
&\cdot\Big(\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla p_\varepsilon - \frac{1}{\mu(c,\theta)^{1/2}}\nabla p\Big)\,dxds = 0.
\end{aligned}$$

The proof is achieved. ■

We now have the tools to pass to the limit in (1.3)-(1.5) as $\varepsilon \to 0$. We only give some details for the salt equation (1.4). We write the nonlinear terms under the following forms:

$$\begin{aligned}
&\underline{u}_\varepsilon \cdot \nabla \hat{c}_\varepsilon = -k\Big(\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla p_\varepsilon\Big) \cdot \Big(\frac{1}{\mu(c_\varepsilon,\theta_\varepsilon)^{1/2}}\nabla \hat{c}_\varepsilon\Big),\\
&\phi_1 F\,\hat{c}_\varepsilon\,\partial_t p_\varepsilon = -\hat{c}_\varepsilon\,\mathrm{div}(\underline{u}_\varepsilon) - \phi\,\partial_t F\,\hat{c}_\varepsilon - q\,\hat{c}_\varepsilon - \hat{c}_\varepsilon \sum_{j=1}^{N} q_j + R_s'(\hat{c}_\varepsilon)\,\hat{c}_\varepsilon\\
&\qquad = \mathrm{div}(\hat{c}_\varepsilon\,\underline{u}_\varepsilon) - \underline{u}_\varepsilon \cdot \nabla \hat{c}_\varepsilon - \phi\,\partial_t F\,\hat{c}_\varepsilon - q\,\hat{c}_\varepsilon - \hat{c}_\varepsilon \sum_{j=1}^{N} q_j + R_s'(\hat{c}_\varepsilon)\,\hat{c}_\varepsilon.
\end{aligned}$$

Our compactness results are sufficient to pass to the limit in these expressions. Eqs. (1.3) and (1.5) can be treated in the same way. Theorem 2 is proved.

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